MASTER-SEMINAR ON TORIC VARIETIES

PROF. DR. U. GÖRTZ, DR. A. PIEPER, WS 2024/2025

Toric varieties are a class of varieties which on the one hand are described quite simply by certain combinatorial data, but on the other hand provide a wealth of interesting examples. The name stems from the fact that all these varieties are provided with a group action by an "algebraic torus", i.e. by a group variety of the form $\mathbb{G}_m^r$, where $\mathbb{G}_m$ denotes the multiplicative group $\mathbb{A}_k^1 \setminus \{0\}$ (over the ground field k). The first examples of toric varieties are affine space $\mathbb{A}_k^n$ and projective space $\mathbb{P}_k^n$; we will get to know many more examples in the first lectures of the seminar.

In the seminar we will study various properties of varieties and methods from algebraic geometry using this class of examples.

Literature: We will follow the book [F] of Fulton. Below you will find further useful sources.

Requirements/prerequisites: Basic knowledge of algebraic geometry, for example to the extent of the lecture *Algebraic Geometry 1* in WS 2023/2024. Knowledge from the lecture *Algebraic Geometry 2* is helpful.

Organizational matters: The lecture should be held on the blackboard and should not last longer than 80 minutes.

Active participation is expected, also during the presentations of the others. Furthermore, we recommend that you meet with one of the organizers at least once before the presentation to ask questions.

The seminar is worth 9 ECTS in the Master's program, provided that you have given two successful presentations, at least one of which should be marked with a $*$. Alternatively, the seminar can also be credited for 6 ECTS in the Bachelor's program. In this case, one presentation (with or without $*$) is sufficient.

Program

1. **Introduction.**

2. **Convex polyhedral cones.** *Content of the talk:* [F] 1.2 This talk which does not contain any algebraic geometry, discusses some preliminaries about convex polyhedral cones. Many things can be explained graphically so that surely some of the proofs can be omitted (and because of limited time will have to be omitted). However, you should give rigorous definitions even for intuitively clear notions and eventually point out to difficulties in the proofs.

3. **Affine toric varieties.** *Content of the talk:* [F] 1.3
In this talk, *affine toric varieties* are defined as spectra of certain rings constructed from cones (in the sense of the previous lecture). Note Fulton's convention that he speaks of schemes, but by *points* he always means *closed points*. (Because he also works with complex numbers, one could also speak of prevarieties without major losses; however, we follow Fulton's language). The terms *non-singular* and *singular* can be ignored in this lecture; however, the examples on pages 17 and 18 should be dealt with. Of the exercises, at least the second exercise on page 19 should be addressed.

4. **Fans and toric varieties.** *Content of the talk:* [F] 1.4 Now we construct further *toric varieties* by gluing affine toric varieties; the gluing data are also given by combinatorial data. If possible, all exercises from this section should be addressed, especially the Hirzebruch surfaces $\mathbb{F}_a$.

5. **Local properties of toric varieties.** *Content of the talk:* [F] 2.1
We investigate which toric varieties are smooth. The notion of smoothness should be rigorously introduced/repeated. The section on the Cohen-Macaulay property should be omitted and replaced by examples.

6. **Quotients of schemes by finite groups *.** *Content of the talk:* [GW] (12.7)
As a preparation for later talks, we insert a general lecture on quotients of (affine) schemes by finite groups. Remark 12.31 can be omitted, the rest should be explained if possible.

7. **Toric surfaces, quotient singularities *.** *Content of the talk:* [F] 2.2
Now we study the local shape of two-dimensional toric varieties (in the singular case).

7 $\frac{1}{2}$. Projective toric varieties *. *Sources for this talk* [F] 1.5, [CLS] 2, [T] 3.6-3.8. Construct a toric variety X_P from a convex lattice polytope P by defining its fan as the normal fan of P, see [F] Section 1.5 or [CLS] pp. 75-77. Give the definition of a *very ample* lattice polytope [CLS] Definition 2.2.16. Present Theorem 2.3.1(a) in [CLS] and the construction preceding it, which constructs a projective variety X from a very ample lattice polytope P. Explain why this theorem implies that $X \cong X_P$ for P very ample. Solve Exercise 2.2.12 and show/sketch its solution. Conclude that toric varieties constructed from convex lattice polytopes are always projective.

In the sources you can safely ignore the discussion of projectively normal varieties and normal polytopes.

8. **Proper toric varieties *.** *Content of the talk* [F] 2.3, 2.4 up to the proposition on p. 39 (including proof).

We give a criterion for when toric varieties (or more generally: "toric" morphisms between toric varieties) are proper. The notion of limit should be reformulated algebraically (extension property of mappings similar to the valuative criteria for separated/proper morphisms). Accordingly, we never use the word "compact", but always "proper".

9. **Blow-Ups *.** *Content of the talk:* [H] I.4, [F] 2.4 from p. 39.

First introduce the concept of blow-up in general (for example as in Hartshorne, compare also [Ha] Ch. 7, [GW] (13.19)). Then discuss blow-ups of toric varieties as in Fulton and a selection of the exercises there.

10. **Smooth proper toric surfaces *.** *Content of the talk:* [F] 2.5 (including the exercises!).

In this lecture we classify the smooth proper toric surfaces according to the corresponding fans.

11. **Resolving singularities of toric varieties I *.** *Content of the talk:* [F] 2.6 to page 47 (last exercise).

First define the term *resolution of singularities* in general, see for example [GW], [H]. Then explain the resolution of singularities in the surface case, explain the Hirzebruch-Jung continued fraction development and discuss the exercises on p. 46/47.

12. **Resolving singularities of toric varieties II *.** *Content of the talk:* [F] 2.6 from page 47 below, incl. exercises.

Finally, we discuss the resolution of singularities in the higher dimensional case, including some examples.

13. **Orbits.** *Content of the talk:*[F] 3.1 and parts of 3.3 In this talk we study the action of the torus on a toric variety. Classify the orbits in terms of the fan; give the fan of an orbit closure. Emphasize in particular the orbits of codimension 1. More topics from section 3.3 can be added if time admits.

14. **Divisors on toric varieties.** *Content of the talk:* Selection of topics from [F] 3.3 and 3.4 Possible themes could be the following:

- Description of the divisor class group $\mathrm{Cl}(X)$ and/or the Picard group $\mathrm{Pic}(X)$ in terms of the fan.
- Criterion for a divisor to be (very) ample. Link back to talk 7 1/2 and/or an example of a proper non-projective toric variety.

References

[C1] D. A. Cox, *Minicourse on Toric Varieties*, `https://dacox.people.amherst.edu/lectures/toric.pdf`

[C2] D. A. Cox, *Lectures on toric varieties*, `https://dacox.people.amherst.edu/lectures/coxcimpa.pdf`

[CLS] D. A. Cox, J. B. Little, H. K. Schenck, *Toric varieties*, AMS Graduate Studies in Math. **124**.

[D] V. Danilov, *The geometry of toric varieties*, Russian Math. Surveys **33** (1978), 97–154.

[E] G. Ewald, *Combinatorial Convexity and Algebraic Geometry*, Springer Graduate Texts in Mathematics **168**, 1996.

[F] W. Fulton, *Introduction to Toric Varieties*, Princeton University Press, 1993.

[GW] U. Görtz, T. Wedhorn, *Algebraic Geometry I*, Vieweg-Teubner.

[Ha] J. Harris, *Algebraic Geometry*, Springer GTM
[H] R. Hartshorne, *Algebraic Geometry*, Springer GTM
[KKMS] G. Kempf, F. Knudsen, D. Mumford, B. Saint-Donat, *Toroidal Embeddings I*, Springer Lecture Notes in Math. **339**.
[O] T. Oda, *Convex bodies and algebraic geometry: An introduction to toric varieties*, Springer.
[T] S. Telen, *Introduction to Toric Geometry*, `https://arxiv.org/abs/2203.01690`.