Linking of motivic spheres

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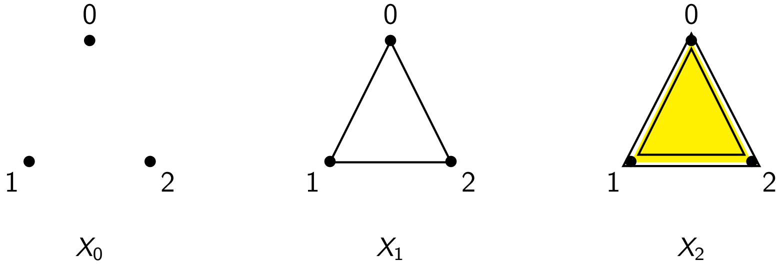
Simplicial sets

Definition

A simplicial set X is a collection of sets $(X_n)_{n\in\mathbb{N}_0}$ (X_n is called the set of n-simplices of X), of maps $(d_{n,i} : X_n \to X_{n-1})_{n\in\mathbb{N},0\leq i\leq n}$ (called faces) and of maps $(s_{n,i} : X_n \to X_{n+1})_{n\in\mathbb{N}_0,0\leq i\leq n}$ (called degeneracies) satisfying the simplicial identities. *Example: the standard triangle Δ^2.*



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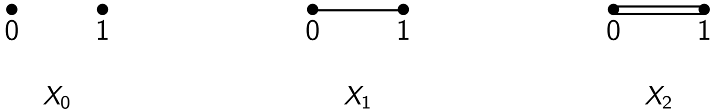
- $d_{1,0} \circ d_{2,1} = d_{1,1} \circ d_{2,2}$ (the endpoint of the last bordering line of a triangle is the starting point of the first bordering line of that triangle) and so on.

Examples: the standard line Δ^1, its boundary $S^0 := \partial\Delta^1$ and the simplicial circle $S^1 := \Delta^1/\partial\Delta^1$



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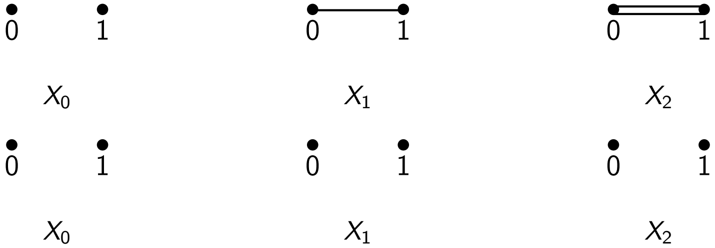

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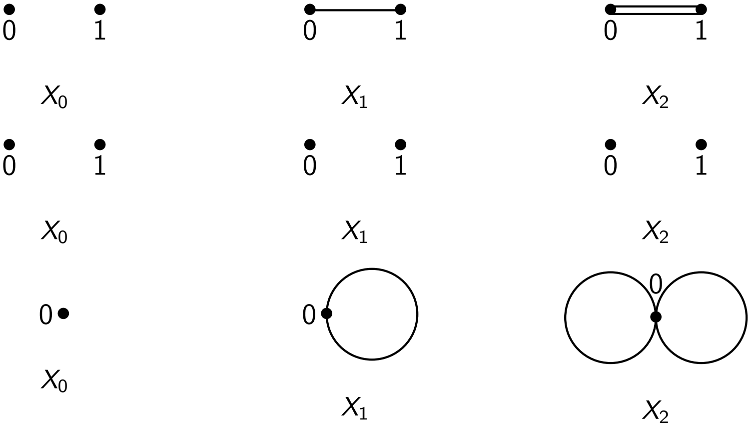

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X	$\lvert X_0\rvert$	$\lvert X_1\rvert$	$\lvert X_2\rvert$
$S^1 \times S^1$	1	4	9
$S^1 \vee S^1$	1	3	5
$S^2 := S^1 \wedge S^1$	1	2	5

X	$\lvert X_0\rvert$	$\lvert X_1\rvert$	$\lvert X_2\rvert$
Δ^2	3	6	10
$\partial\Delta^2$	3	6	9
$\Delta^2/\partial\Delta^2$	1	1	2

Motivic homotopy theory, a.k.a. $\mathbb{A}^1$-homotopy theory

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- We also have a smash product in Spc (of neutral element S^0). Beware: the smash product of two smooth F-schemes is not necessarily an F-scheme!

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- We define motivic spheres as spaces which are $\mathbb{A}^1$-homotopic to $S^i \wedge \mathbb{G}_m^{\wedge j}$ for some $i, j \in \mathbb{N}_0$, i.e. which in the $\mathbb{A}^1$-homotopy category are isomorphic to $S^i \wedge \mathbb{G}_m^{\wedge j}$ for some $i, j \in \mathbb{N}_0$.

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- In particular, the smooth k-scheme $\mathbb{P}^1$ is a motivic sphere (since $\mathbb{P}^1$ is $\mathbb{A}^1$-homotopic to $S^1 \wedge \mathbb{G}_m$), which is reassuring since when $F = \mathbb{R}$ (resp. $F = \mathbb{C}$) the space of F-points of $\mathbb{P}^1$ is homotopic (even homeomorphic) to the topological space S^1 (resp. S^2).

Smooth models of motivic spheres

Definition

Let $i, j \in \mathbb{N}_0$. A smooth model of the motivic sphere $S^i \wedge \mathbb{G}_m^{\wedge j}$ is a smooth finite-type k-scheme which is $\mathbb{A}^1$-homotopic to $S^i \wedge \mathbb{G}_m^{\wedge j}$.

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- $Q_{2n} := \mathrm{Spec}(F[x_1, \ldots, x_n, y_1, \ldots, y_n, z]/(\sum_{i=1}^n x_i y_i - z(1+z)))$ is a smooth model of $S^n \wedge \mathbb{G}_m^{\wedge n}$ for $n \in \mathbb{N}_0$ (and also $\mathbb{P}^1$ for $n = 1$).

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- $Q_{2n-1} := \mathrm{Spec}(F[x_1, \dots, x_n, y_1, \dots, y_n]/(\sum_{i=1}^n x_i y_i - 1))$ and $\mathbb{A}^n \setminus \{0\}$ are smooth models of $S^{n-1} \wedge \mathbb{G}_m^{\wedge n}$ for $n \in \mathbb{N}$.

The stable $\mathbb{A}^1$-homotopy category

- By stabilizing spaces with respect to the suspension by $\mathbb{P}^1$ (which is the smash product with $\mathbb{P}^1$), we get $\mathrm{SH}(F)$, the stable $\mathbb{A}^1$-homotopy category. Equivalently, we can stabilize with respect to the suspension by S^1 and to the suspension by $\mathbb{G}_m$.

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- Stable homotopy groups of motivic spheres $[S^i \wedge \mathbb{G}_m^{\wedge j}, S^k \wedge \mathbb{G}_m^{\wedge l}]_{\mathrm{SH}(F)} \simeq [S^{i-k}, \mathbb{G}_m^{\wedge(l-j)}]_{\mathrm{SH}(F)}$ (with $i, j, k, l \in \mathbb{Z}$) are interesting and verify that:

$$\forall i < 0, j \in \mathbb{Z},\ [S^i, \mathbb{G}_m^{\wedge j}]_{\mathrm{SH}(F)} = 0$$

$$\forall j \in \mathbb{Z},\ [S^0, \mathbb{G}_m^{\wedge j}]_{\mathrm{SH}(F)} \simeq K_j^{\mathrm{MW}}(F)$$

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- Generators of $K_*^{\mathrm{MW}}(F)$ are the $[a] \in K_1^{\mathrm{MW}}(F) \simeq [S^0, \mathbb{G}_m]_{\mathrm{SH}(F)}$ (with $a \in F^*$) which correspond to the pointed morphism associated to a and $\eta \in K_{-1}^{\mathrm{MW}}(F) \simeq [S^0, \mathbb{G}_m^{\wedge(-1)}]_{\mathrm{SH}(F)} \simeq [\mathbb{A}_F^2 \setminus \{0\}, \mathbb{P}_F^1]_{\mathrm{SH}(F)}$ which corresponds to the Hopf fibration $(x, y) \mapsto [x : y]$.

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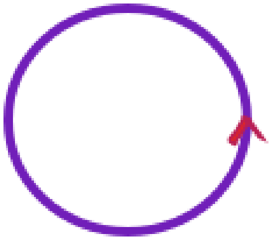
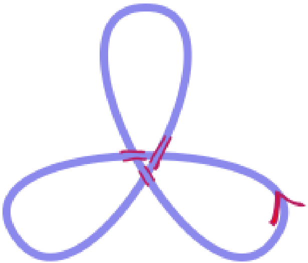
- We denote, for each $a \in F^*$, $\langle a \rangle := \eta[a] + 1 \in K_0^{\mathrm{MW}}(F)$.
- The ring (resp. group) morphism $K_0^{\mathrm{MW}}(F) \to \mathrm{GW}(F)$ (resp. $K_n^{\mathrm{MW}}(F) \to \mathrm{W}(F)$ with $n < 0$) which for each $a \in F^*$ maps $\langle a \rangle$ (resp. $\eta^{-n}\langle a \rangle$) to the class in $\mathrm{GW}(F)$ (resp. in $\mathrm{W}(F)$) of the symmetric bilinear form $\begin{cases} F \times F & \to & F \\ (x, y) & \mapsto & axy \end{cases}$ is an isomorphism.

Contents



The unknot



The trefoil knot

Knot theory in a nutshell

Topological objects of interest are knots and links.

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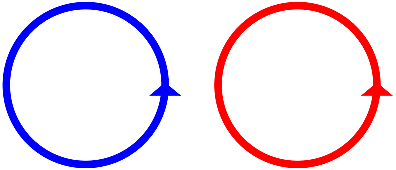
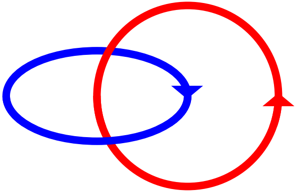
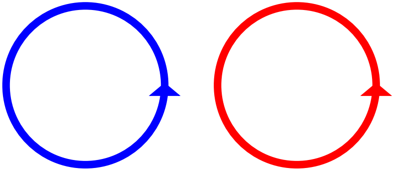
- An **oriented knot** is a knot with a "continuous" local trivialization of its tangent bundle, or equivalently of its normal bundle (the ambient space being oriented). There are two orientation classes.
- A **link** is a finite union of disjoint knots. A link is **oriented** if all its components (i.e. its knots) are oriented.



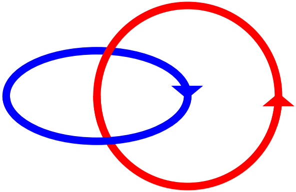
The unlink with two components (linking number $= 0$)



The Hopf link (linking number $= 1$)

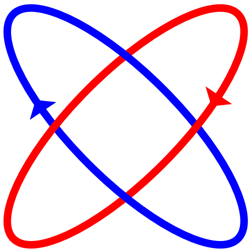


The unlink with two components (linking number $= 0$)

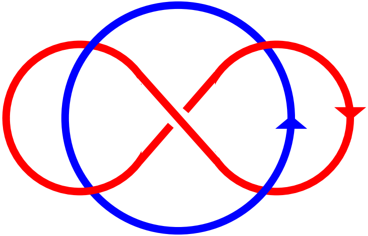


The Hopf link (linking number $= 1$)

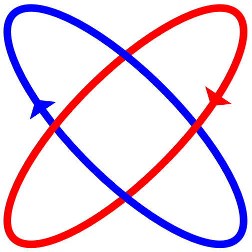
The **linking number** of an oriented link with two components is the number of times one of the components turns around the other component (the sign indicating the direction).



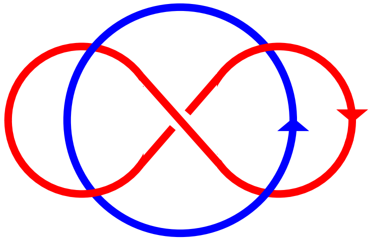
The Solomon link (linking number $= 2$)



The Whitehead link (linking number $= 0$)



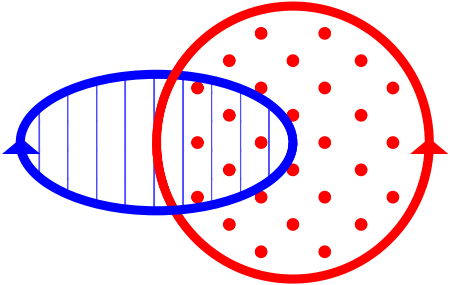
The Solomon link (linking number $= 2$)



The Whitehead link (linking number $= 0$)

The linking number is a complete invariant of oriented links with two components for link homotopy (i.e. $L = K_1 \sqcup K_2$ and $L' = K_1' \sqcup K_2'$ are link homotopic if and only if they have the same linking number).

Defining the linking number: Seifert surfaces



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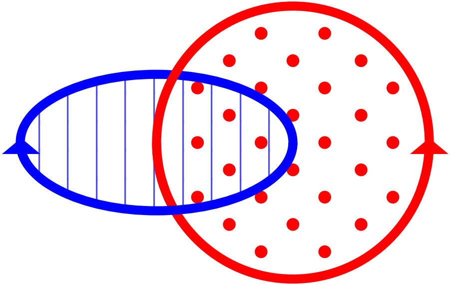
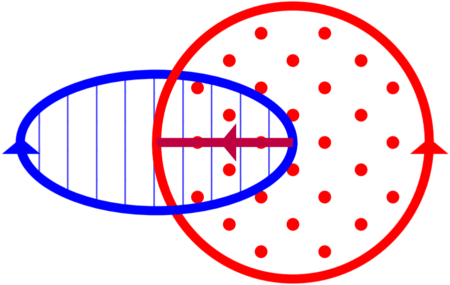


The class S_1 in $H^1(\mathbb{S}^3 \setminus L) \simeq H_2^{\mathrm{BM}}(\mathbb{S}^3, L)$ of Seifert surfaces of the oriented knot K_1 is the unique class that is sent by the boundary map to the (oriented) fundamental class of K_1 in $H^0(K_1) \subset H^0(L)$.

Defining the linking number: intersection of S. surfaces



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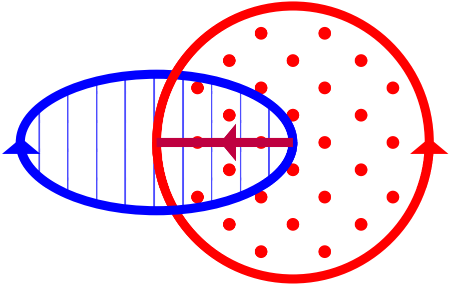
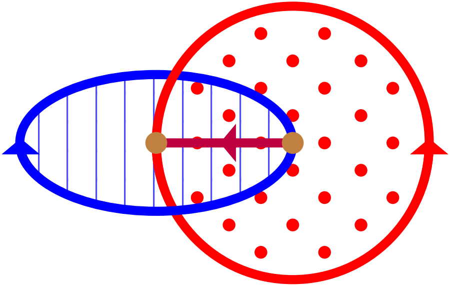
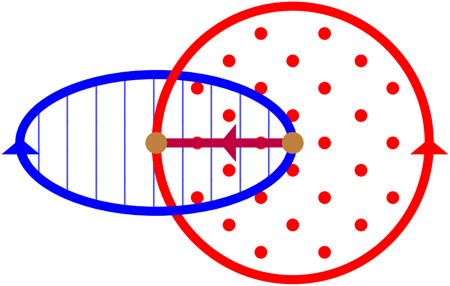


This corresponds to the cup-product $S_1 \cup S_2 \in H^2(\mathbb{S}^3 \setminus L)$.

Defining the linking number: boundary of int. of S. surf.



Defining the linking number: boundary of int. of S. surf.



This corresponds to $\partial(S_1 \cup S_2) \in H^1(L) \simeq H^1(K_1) \oplus H^1(K_2)$, which we call the **linking class**.

The linking number

The linking number

The linking number of L is the image of the part of the linking class which is in $H^1(K_1)$ by the composite of the morphism $(i_1)_* : H^1(K_1) \to H^3(\mathbb{S}^3)$ induced by the inclusion $i_1 : K_1 \to \mathbb{S}^3$ and of the "right-hand rule" isomorphism $r : H^3(\mathbb{S}^3) \to \mathbb{Z}$.

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What about the other number?

The image of the part of the linking class which is in $H^1(K_2)$ by the composite of the morphism $(i_2)_* : H^1(K_2) \to H^3(\mathbb{S}^3)$ induced by the inclusion $i_2 : K_2 \to \mathbb{S}^3$ and of the isomorphism $r : H^3(\mathbb{S}^3) \to \mathbb{Z}$ is equal to the opposite of the linking number.

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The absolute value of the linking number does not depend on the orientation of the link.

The linking couple

The linking couple

The linking couple is the image of the linking class by the isomorphism $h_1 \oplus h_2 : H^1(K_1) \oplus H^1(K_2) \to \mathbb{Z} \oplus \mathbb{Z}$ which is induced by the volume forms ω_{K_1} of K_1 and ω_{K_2} of K_2.

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Important fact

The linking couple is equal to $(\pm n, \pm n)$ with n the linking number.

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Important fact

The linking couple is equal to $(\pm n, \pm n)$ with n the linking number.

The absolute value of either component of the linking couple is equal to the absolute value of the linking number.

Linking numbers in general

The linking number can actually be defined in a much more general case:

- if M^n is an oriented n-dimensional manifold (as defined in [Seifert and Threlfall, *Lehrbuch der Topologie / A textbook of top.* Chapter X], e.g. $\mathbb{S}^n$, $\mathbb{RP}^n$ (if n is odd, for orientability) or $\mathbb{CP}^{\frac{n}{2}}$ (if n is even))

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- and if A^{k-1} and B^{n-k} are disjoint oriented homologically trivial submanifolds of M^n of respective dimensions $k-1$ and $n-k$
- then the linking number of A^{k-1} and B^{n-k} is the intersection number of C^k with B^{n-k}, where C^k is a k-dimensional singular chain of boundary A^{k-1} (e.g. C^k is a k-dimensional oriented submanifold of M^n whose oriented boundary is A^{k-1}).

Linking of spheres

- If $M^n = \mathbb{S}^n$ (with a fixed orientation) and A^{k-1} (resp. B^{n-k}) is an oriented (closed) topological subspace of M^n which is homeomorphic to $\mathbb{S}^{k-1}$ (resp. $\mathbb{S}^{n-k}$), with A^{k-1} and B^{n-k} disjoint, then there is a linking number of the "higher-dimensional knots" A^{k-1} and B^{n-k}.

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- If in addition $k-1 = n-k$, i.e. n is odd and $k = \frac{n+1}{2}$, then one can define the linking class, the linking number and the linking couple (of $\mathbb{S}^m \sqcup \mathbb{S}^m$ in $\mathbb{S}^{2m+1}$) in a similar manner to what was done before, and:
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Contents

The singular complex and the Rost-Schmid complex

Classical algebraic topology

Each topological space X has a singular cochain complex:

$$\dots \longrightarrow \mathcal{C}^i(X) \longrightarrow \mathcal{C}^{i+1}(X) \longrightarrow \dots$$

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Each smooth F-scheme X has a Rost-Schmid complex *for each integer $j \in \mathbb{Z}$ and invertible $\mathcal{O}_X$-module $\mathcal{L}$*:

$$\dots \longrightarrow \bigoplus_{p \in X^{(i)}} K^{\mathrm{MW}}_{j-i}(\kappa(p)) \otimes_{\mathbb{Z}[\kappa(p)^*]} \mathbb{Z}[(\nu_p \otimes \mathcal{L}_{|p}) \setminus \{0\}]$$

$$\downarrow$$

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The singular cohomology ring and the Rost-Schmid ring

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The i-th cohomology group $H^i(X)$ of X is the i-th cohomology group of the singular cochain complex of X.

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Motivic algebraic topology

The i-th Rost-Schmid group $H^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}\})$ of X with respect to j and $\mathcal{L}$ is the i-th cohomology group of the Rost-Schmid complex of X w.r.t. j and $\mathcal{L}$. We denote $H^i(X, \underline{K}_j^{\mathrm{MW}}) := H^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{O}_X\})$.

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Classical algebraic topology

Let (Z, i, X, j, U) be a boundary triple. We have the following long exact sequence (where ∂ is the boundary map):

$$\ldots \longrightarrow H^n(Z) \xrightarrow{i_*} H^{n+d_X-d_Z}(X) \xrightarrow{j^*} H^{n+d_X-d_Z}(U) \xrightarrow{\partial} H^{n+1}(Z) \longrightarrow$$

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Let (Z, i, X, j, U) be a boundary triple. We have the localization long exact sequence (where ∂ is the boundary map):

$$\ldots \longrightarrow H^n(Z, \underline{K}_m^{\mathrm{MW}}\{\nu_Z\}) \xrightarrow{i_*} H^{n+d_X-d_Z}(X, \underline{K}_{m+d_X-d_Z}^{\mathrm{MW}}) \xrightarrow{j^*}$$

$$\xrightarrow{j^*} H^{n+d_X-d_Z}(U, \underline{K}_{m+d_X-d_Z}^{\mathrm{MW}}) \xrightarrow{\partial} H^{n+1}(Z, \underline{K}_m^{\mathrm{MW}}\{\nu_Z\}) \longrightarrow \ldots$$

Links in algebraic geometry

Let F be a perfect field and X be a smooth finite-type irred. F-scheme.

Link with two components

A link with two components is a couple of disjoint smooth finite-type irreducible closed F-subschemes Z_1 and Z_2 of X such that:

- Z_1 and Z_2 have the same codimension c in X;
- $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$ and $H^{c}(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$ for some $j_1 \leq 0$;
- $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$ and $H^{c}(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$ for some $j_2 \leq 0$.

Oriented links in algebraic geometry

An orientation o_i of Z_i is an isomorphism from the determinant (i.e. the maximal exterior power) of the normal sheaf $\mathcal{N}_{Z_i/X}$ of Z_i in X to the tensor product of an invertible $\mathcal{O}_{Z_i}$-module $\mathcal{L}_i$ with itself:

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Orientation classes

Two orientations $o_i : \nu_{Z_i} \to \mathcal{L}_i \otimes \mathcal{L}_i$ and $o_i' : \nu_{Z_i} \to \mathcal{L}_i' \otimes \mathcal{L}_i'$ of Z_i represent the same orientation class of Z_i if there exists an isomorphism $\psi : \mathcal{L}_i \simeq \mathcal{L}_i'$ such that $(\psi \otimes \psi) \circ o_i = o_i'$.

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Two orientations $o_i : \nu_{Z_i} \to \mathcal{L}_i \otimes \mathcal{L}_i$ and $o'_i : \nu_{Z_i} \to \mathcal{L}'_i \otimes \mathcal{L}'_i$ of Z_i represent the same orientation class of Z_i if there exists an isomorphism $\psi : \mathcal{L}_i \simeq \mathcal{L}'_i$ such that $(\psi \otimes \psi) \circ o_i = o'_i$.

The link (Z_1, Z_2) together with an orientation class $\overline{o_1}$ of Z_1 and an orientation class $\overline{o_2}$ of Z_2 is an oriented link with two components.

Oriented fundamental classes and Seifert classes

Let $i \in \{1, 2\}$.

Definition

- We define the oriented fundamental class $[o_i]_{j_i}$ with respect to $j_i \leq 0$ as the unique class in $H^0(Z_i, \underline{K}_{j_i}^{\mathrm{MW}}\{\nu_{Z_i}\})$ that is sent by $\widetilde{o}_i$ to the class of η^{-j_i} in $H^0(Z_i, \underline{K}_{j_i}^{\mathrm{MW}})$.

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- We define the Seifert class $\mathcal{S}_{o_i, j_i}$ with respect to j_i as the unique class in $H^{c-1}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_i+c})$ that is sent by the boundary map ∂ to the oriented fundamental class $[o_i]_{j_i} \in H^0(Z, \underline{K}^{\mathrm{MW}}_{j_i}\{\nu_Z\})$.

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The assumptions $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_i+c}) = 0$ and $H^c(X, \underline{K}^{\mathrm{MW}}_{j_i+c}) = 0$ made earlier are there to ensure the unicity and the existence resp. of the Seifert class.

The (ambient) quadratic linking class / degree

The quadratic linking class (/degree after an iso. fixed once and for all)

We define the quadratic linking class with respect to (j_1, j_2) as the image of the intersection product $\mathcal{S}_{o_1,j_1} \cdot \mathcal{S}_{o_2,j_2}$ by the boundary map $\partial : H^{2c-2}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2c}) \to H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\})$.

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For this to be interesting, it is important that $H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\}) \neq 0$.

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The ambient quadratic linking class (/degree after a fixed iso.)

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Which closed immersions of smooth models of motivic spheres have a (potentially nontrivial) quadratic linking class?

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- $Q_{2n+1} := \mathrm{Spec}(F[x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}]/(\sum_{i=1}^{n+1} x_i y_i - 1))$
- Q_{2n+1} is a smooth model of $S^n \wedge \mathbb{G}_m^{\wedge(n+1)}$

Which closed immersions of smooth models of motivic spheres have a (potentially nontrivial) quadratic linking class?

- $\mathbb{A}^{n+1}_F \setminus \{0\}$ is a smooth model of $S^n \wedge \mathbb{G}_m^{\wedge(n+1)}$
- $Q_{2n+1} := \mathrm{Spec}(F[x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}]/(\sum_{i=1}^{n+1} x_i y_i - 1))$
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- $Q_{2n} := \mathrm{Spec}(F[x_1, \ldots, x_n, y_1, \ldots, y_n, z]/(\sum_{i=1}^{n} x_i y_i - z(1+z)))$
- Q_{2n} is a smooth model of $S^n \wedge \mathbb{G}_m^{\wedge n}$

- $\mathbb{A}^n_F \setminus \{0\} \sqcup \mathbb{A}^n_F \setminus \{0\} \to \mathbb{A}^{2n}_F \setminus \{0\}$ with $n \geq 2$ (and ambient qlc ✓);

- $\mathbb{A}^n_F \setminus \{0\} \sqcup \mathbb{A}^n_F \setminus \{0\} \to \mathbb{A}^{2n}_F \setminus \{0\}$ with $n \geq 2$ (and ambient qlc $\checkmark$);
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In the cases $Q_n \sqcup Q_n \to Q_{n+\lfloor \frac{n}{2} \rfloor+1} = X$ with $n \in \{2,3,4\}$, the only conditions which are not verified are the ones which are there to ensure the existence of Seifert classes ($H^c(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$ and $H^c(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$).

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In these settings, the ambient quadratic linking degree is in $\mathrm{W}(F)$ or in $\mathrm{GW}(F)$ and each component of the quadratic linking degree is either in the zero group, in $\mathrm{W}(F)$, in $\mathrm{GW}(F)$ or in $K^{\mathrm{MW}}_1(F)$.

The Hopf link in algebraic geometry

We fix coordinates x, y, z, t for $\mathbb{A}^4_F$ and u, v for $\mathbb{A}^2_F$ once and for all.

- The image of the Hopf link:

$$\{x = 0, y = 0\} \sqcup \{z = 0, t = 0\} \subset \mathbb{A}^4_F \setminus \{0\}$$

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$$o_1 : \overline{x}^* \wedge \overline{y}^* \mapsto 1 \otimes 1, o_2 : \overline{z}^* \wedge \overline{t}^* \mapsto 1 \otimes 1$$

The (amb.) quadratic linking degree (cpl.) of the Hopf link

Or. fund. cl.	$\eta \otimes (\overline{x}^* \wedge \overline{y}^*)$	|	$\eta \otimes (\overline{z}^* \wedge \overline{t}^*)$
Seifert cl.	$\langle x \rangle \otimes \overline{y}^*$	|	$\langle z \rangle \otimes \overline{t}^*$
Apply int. prod.	$\langle xz \rangle \otimes (\overline{t}^* \wedge \overline{y}^*)$		
Quad. lk. class	$-\langle z \rangle \eta \otimes (\overline{t}^* \wedge \overline{x}^* \wedge \overline{y}^*)$	$\oplus$	$\langle x \rangle \eta \otimes (\overline{y}^* \wedge \overline{z}^* \wedge \overline{t}^*)$
Apply $(i_1)_*$	$-\langle z \rangle \eta \otimes (\overline{t}^* \wedge \overline{x}^* \wedge \overline{y}^*)$		
Apply ∂	$-\eta^2 \otimes (\overline{x}^* \wedge \overline{y}^* \wedge \overline{z}^* \wedge \overline{t}^*)$		
Amb. qld.	-1		
Quad. lk. class	$-\langle z \rangle \eta \otimes (\overline{t}^* \wedge \overline{x}^* \wedge \overline{y}^*)$	$\oplus$	$\langle x \rangle \eta \otimes (\overline{y}^* \wedge \overline{z}^* \wedge \overline{t}^*)$
Apply $\widetilde{o_1} \oplus \widetilde{o_2}$	$-\langle z \rangle \eta \otimes \overline{t}^*$	$\oplus$	$\langle x \rangle \eta \otimes \overline{y}^*$
Apply $\varphi_1^* \oplus \varphi_2^*$	$-\langle u \rangle \eta \otimes \overline{v}^*$	$\oplus$	$\langle u \rangle \eta \otimes \overline{v}^*$
Apply $\partial \oplus \partial$	$-\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$	$\oplus$	$\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$
Qld. couple	-1	$\oplus$	1

Another Hopf link in algebraic geometry

From now on, F is a perfect field of characteristic different from 2. Recall that we fixed coordinates x, y, z, t for $\mathbb{A}^4_F$ and u, v for $\mathbb{A}^2_F$.

- The image is different from the Hopf link we saw before:

$$\{z = x, t = y\} \sqcup \{z = -x, t = -y\} \subset \mathbb{A}^4_F \setminus \{0\}$$

  But the change of coordinates $x' = z - x$, $y' = t - y$, $z' = z + x$, $t' = t + y$ would give $\{x' = 0, y' = 0\} \sqcup \{z' = 0, t' = 0\} \subset \mathbb{A}^4_F \setminus \{0\}$.

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$$o_1 : \overline{z - x}^* \wedge \overline{t - y}^* \mapsto 1, o_2 : \overline{z + x}^* \wedge \overline{t + y}^* \mapsto 1$$

- This Hopf link is an analogue of the Hopf link in knot theory! In knot theory, the Hopf link is given by $\{z = x, t = y\} \sqcup \{z = -x, t = -y\}$ in $\mathbb{S}^3_\varepsilon = \{(x, y, z, t) \in \mathbb{R}^4, x^2 + y^2 + z^2 + t^2 = \varepsilon^2\}$ for ε small enough and has linking number 1.

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The Solomon link in algebraic geometry

- In knot theory, the Solomon link is given by $\{z = x^2 - y^2, t = 2xy\} \sqcup \{z = -x^2 + y^2, t = -2xy\}$ in $\mathbb{S}^3_\varepsilon$ for ε small enough and has linking number 2.

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- More generally, we have analogues over $\mathbb{R}$ of the torus links $T(2, 2n)$ (of linking number n); the ambient quadratic linking degree of $T(2, 2n)$ is $-n \in \mathrm{W}(\mathbb{R}) \simeq \mathbb{Z}$ and the quadratic linking degree of $T(2, 2n)$ is $(n, -n) \in \mathrm{W}(\mathbb{R}) \oplus \mathrm{W}(\mathbb{R}) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Examples of $Q_2 \sqcup Q_2 \to \mathbb{A}^4_F \setminus \{0\}$ ($j_1 = -1 = j_2$)

Assume $F \neq \mathbb{Z}/2\mathbb{Z}$. Let $a \neq b \in F^*$. $Z_1 = \{xy = z(z+1), t = a\}$ and $Z_2 = \{xy = z(z+1), t = b\}$ are of ambient quadratic linking degree 0 and of quadratic linking degree $(0, 0)$.

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Assume the characteristic of F to be different from 2 and 3. $Z_1 = \{xy = z(z+1), t = 1\}$ and $Z_2 = \{xy = t(t+1), z = 2\}$ (with the orientation classes and parametrisations which you can guess) are of ambient quadratic linking degree 0 and of quadratic linking degree $(-1, -1) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$.

Examples of $Q_2 \sqcup Q_2 \to Q_4$ ($j_1 = -1 = j_2$)

For both examples, assume F of characteristic different from 2. Recall that $Q_4 = \operatorname{Spec}(F[x_1, y_1, x_2, y_2, z]/(x_1y_1 + x_2y_2 - z(z+1)))$.

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For both examples, assume F of characteristic different from 2. Recall that $Q_4 = \mathrm{Spec}(F[x_1, y_1, x_2, y_2, z]/(x_1 y_1 + x_2 y_2 - z(z+1)))$.

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For both examples, assume F of characteristic different from 2. Recall that $Q_4 = \mathrm{Spec}(F[x_1, y_1, x_2, y_2, z]/(x_1y_1 + x_2y_2 - z(z+1)))$.

$Z_1 = \{x_1y_1 = z(z+1), x_2 = 1\}$ and $Z_2 = \{x_1y_1 = z(z+1), x_2 = -1\}$ are of quadratic linking degree $(0, 0)$.

$Z_1 = \{x_1y_1 = (z-1)z, y_2 = 1\}$ and $Z_2 = \{x_1y_1 = (z+1)(z+2), x_2 = 1\}$ (with the orientation classes and parametrisations which you can guess) are of quadratic linking degree $(\langle 2\rangle, \langle 2\rangle) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$.

Thanks for your attention!