PhD Seminar, WS 2023/24

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Introduction

Statement of the conjectures

Let X be a smooth projective variety of dimension d over a finite field $\mathbb{F}_q$. The Weil conjectures are four statements about the zeta function of X, a formal power series with rational coefficients defined as follows:

$$Z_X(t) = \exp\left(\sum_{m\geq 1} N_m \frac{t^m}{m}\right),$$

where $N_m = \#X(\mathbb{F}_{q^m})$.

(1) *Rationality*: there exist polynomials $P_0(t), \dots, P_{2d}(t) \in \mathbb{Z}[t]$ such that

$$Z_X(t) = \frac{P_1(t)P_3(t)\dots P_{2d-1}(t)}{P_0(t)P_2(t)\dots P_{2d}(t)}$$

with $P_0(t) = 1 - t$ and $P_{2d}(t) = 1 - q^d t$.

(2) *Functional equation*: $Z_X(t)$ satisfies the functional equation

$$Z_X\left(\frac{1}{q^d t}\right) = \pm q^{\frac{d\chi}{2}} t^\chi Z_X(t)$$

where $\chi = (X \cdot X)$ is the intersection number of X with itself inside $X \times X$.

(3) *Betti numbers*: Suppose that there exists a number field K and a smooth, proper variety Y defined over $\mathcal{O}_K$ with fiber $Y_\mathfrak{p} \cong X$ at a prime ideal $\mathfrak{p}$ and write $P_r(t) = \prod_{i=1}^{\beta_r}(1 - \alpha_{r,i}t)$. Then the r-th "topological" Betti number of $Y_\mathbb{C}$ coincides with its étale counterpart, i.e. the dimension of $H^r(X, \mathbb{Q}_\ell)$ (which in turn coincides with β_r).

(4) *Riemann hypothesis*: The numbers $\alpha_{r,i}$ are algebraic integers, all of whose conjugates have absolute values $q^{r/2}$.

Connection to algebraic topology

For a variety X over $\mathbb{F}_q$, the fixed points of the m-th power of the Frobenius map Φ^m are precisely the $\mathbb{F}_{q^m}$-valued points of X. Therefore, it makes sense to study the Frobenius endomorphism, and more precisely, to look at its action on the cohomology of X, as the case of complex manifolds suggests. Indeed, in order to study such spaces one usually works with their *homological* properties: if X is a complex manifold of dimension d, then one defines its *cohomology algebra*

$$H^\bullet(X) := \bigoplus_i H^i(X),$$

which is a graded algebra defined over a field K of characteristic 0. This algebra has the following properties:

(1) (a) Each $H^i(X)$ is a finite dimensional K-vector space, and they are 0 unless $0 \leq i \leq 2d$.

(b) We have a natural isomorphism $H^{2d}(X) \cong K$, and the multiplication of $H^\bullet(X)$ should give an identification

$$H^{2d-i}(X) \xrightarrow{\sim} \mathrm{Hom}_K(H^i(X), K).$$

This is known as *Poincaré duality*.

(c) For any two manifolds X, Y, we have an isomorphism of graded algebras (known as the *Künneth formula*):

$$H^\bullet(X) \otimes H^\bullet(Y) \xrightarrow{\sim} H^\bullet(X \times Y)$$

(2) For any morphism $f : X \to X$, we get natural linear maps $f^{(i)} : H^i(X) \to H^i(X)$ which give a homomorphism of graded algebras $f^\bullet : H^\bullet(X) \to H^\bullet(X)$. The set of the fixed points of f is the projection onto X of the intersection of Γ_f (the graph of f) and Δ (the diagonal) in $X \times X$. If Γ_f and Δ intersect transversally at each point, then the number of fixed points of f is given by the following *Lefschetz trace formula*:

$$N = \sum_{i=0}^{2d} (-1)^i \mathrm{Tr}(f^{(i)}),$$

where $\mathrm{Tr}(f^{(i)})$ stands for the trace of the K-endomorphism $f^{(i)}$.

(3) If Y is a submanifold of X of dimension $d-1$, then there exist natural linear mappings $H^i(X) \to H^i(Y)$, which are bijective for $i \leq d-2$ and injective for $i = d-1$.

If we now return to the case where X is an algebraic variety over $\mathbb{F}_q$, and we assume that such a graded algebra $H^\bullet(X)$ can be constructed with the previous properties (and some more), then, as previously mentioned, we can use it to deduce information about the Frobenius maps Φ^m. If $\{\alpha_{ij}\}_j$ is the set of eigenvalues of $\Phi^{(i)}$, the Lefschetz trace formula implies that

$$N_m = \sum_i (-1)^i \sum_j \alpha_{ij}^m.$$

From an elementary computation, it follows that we can write the zeta function of X as follows:

$$Z_X(t) = \frac{P_1(t)P_3(t)\cdots P_{2d-1}(t)}{P_0(t)P_2(t)\cdots P_{2d}(t)},$$

where the $P_i(t)$ are the characteristic polynomials of $\Phi^{(i)}$, which have integer coefficients. By similar arguments one can obtain the functional equation, but the analog of the Riemann hypothesis does *not* follow formally from these cohomological properties: as we will see, one needs a more intricate argument, which was first shown by Deligne.

Why étale cohomology?

Now it is clear that we would like to construct a cohomology theory with the above properties. A first candidate could be to take the cohomology of X as a topological space, but this doesn't give the correct answer. Indeed, the following theorem holds:

Theorem 1. *If X is an irreducible topological space, then $H^r(X, \mathcal{F}) = 0$ for all $r > 0$ and all constant sheaves $\mathcal{F}$.*

The issue is that the Zariski topology is too coarse. The key insight of Grothendieck was to realize that to give a sheaf on X, i.e. a contravariant functor

$$F : X_{\text{Zar}} \to \text{Set}$$

from the category of open subsets of X which possesses some gluing properties, one only needs to know what the open coverings are. Then, instead of refining the topology of X, we can consider functors

$$F : X_{\text{et}} \to \text{Set}$$

on the category of étale maps to X: such morphisms play the role of open subsets. An *étale sheaf* is a functor of this form which satisfies some similar gluing properties, where now a covering of an étale map $Y \to X$ is a family $\{Y_i \to X\}_i$ in X_{et} with morphisms $Y_i \xrightarrow{\varphi_i} Y$ which are jointly surjective and such that

$$\begin{array}{ccc} Y_i & \xrightarrow{\varphi_i} & Y \\ & \searrow & \downarrow \\ & & X \end{array}$$

commutes for all i.

Étale sheaves form an abelian category and we can perform all the operations needed to define the usual sheaf cohomology in this new setting. This produces a cohomology theory, which we write as $H^r(X_{\text{et}}, \mathcal{F})$ and which behaves nicely when working with torsion coefficients (e.g. constant sheaves of the form $\mathbb{Z}/n\mathbb{Z}$). Sadly, it is not so ideal when we try to work with coefficients of the form $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$. Since we want our cohomology algebra to be defined over a field of characteristic zero, we are lead to give some ad hoc definitions:

$$H^r(X_{\mathrm{et}}, \mathbb{Z}_\ell) := \varprojlim_n H^r(X_{\mathrm{et}}, \mathbb{Z}/\ell^n\mathbb{Z}), \quad \text{and} \quad H^r(X, \mathbb{Q}_\ell) := H^r(X, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,$$

and instead work with these groups.

An application: the Ramanujan conjectures

The Weil conjectures have a wide range of deep consequences, and as an example, we are going to present Deligne's proof of the Ramanujan conjecture, concerning the growth of the coefficients of the Fourier expansion of the discriminant modular form

$$\Delta(z) = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n$$

(here $q = e^{2\pi i z}$ is *not* the cardinality of any finite field!). The conjecture affirms that for any prime number p one has $|\tau(p)| < 2p^{11/2}$, or equivalently that the roots of the polynomial

$$H_p(X) = 1 - \tau(p)X + p^{11}X^2$$

have absolute value $p^{-11/2}$. The strategy of the proof is first to find a suitable two dimensional Galois representation W_ℓ of $\mathrm{Gal}(K_\ell/\mathbb{Q})$ such that $H_p(X) = \det(1 - F_p X; W_\ell)$, where K_ℓ is the maximal extension of $\mathbb{Q}$ unramified outside of ℓ and F_p is the inverse of the Frobenius at p, for $p \neq \ell$. Now, W_ℓ is related to the cohomology of the moduli space of elliptic curves with some level structure and the Riemann hypothesis tells us the absolute values of the eigenvalues of the Frobenius action.

Talks

Here is the list of the talks, with an outline and some brief descriptions of the things you should cover. The main reference for the seminar are the notes by Milne [Mil], but they don't have all of the details: you are welcome to also read the book by Milne [EC] or the one by Freitag and Kiehl [FK]. The numbers inside the square brackets next to the titles of the talks indicate which chapters of [Mil] are covered in the talk.

1 Étale cohomology

Talk 1 (11 October): Introduction

In this talk we introduce the subject of the seminar, give a sketch of what we are going to learn and try to find volunteers for the talks without a speaker already.

Talk 2 (18 October): Étale morphisms [2, 3, 4]

The topic of this talk are étale morphisms of schemes: these are some kind of morphisms at the basis of étale cohomology, and they will take the place that open immersions have in the Zariski topology.
As a guideline for this talk, the idea is to first give the important definitions for the case of varieties over fields, and then generalize them to schemes.

- Following the start of [Mil, Chapter 2], discuss briefly the definition of étale morphisms between varieties (you are welcome to not give all of the details). Afterwards, give the definition of flat, unramified and étale morphisms between arbitrary schemes, and maybe mention that these two notions of étale maps agree over algebraically closed fields. State the first basic properties of étale morphisms: [Mil, Prop. 2.11, 2.12, 2.14]. If there is enough time, mention as well [Mil, Cor. 2.16].

- Next, as in the exposition of [Mil, Chapter 3], briefly recall the definition of the topological fundamental group and its classification of covering spaces. Then, define the étale fundamental group, state [Mil, Thm. 3.1] and explain some of the examples in the sections (fields, $\mathbb{A}^1$ and $\mathbb{P}^n$).

- Discuss the idea of local rings for the étale topology in the case of varieties, as in the start of [Mil, Chapter 4]. Prove [Mil, Prop. 4.1]. Define Henselian rings, and state [Mil, Thm. 4.4, Thm. 4.5] without proof. Similarly explain Henselizations, and state [Mil, Cor. 4.14]. Finish the discussion about varieties by defining strict Henselian rings and strict Henselizations as in [Mil, Def. 4.18] and the discussion before it. Finally, finish the talk by giving the definition for the local ring for the étale topology for schemes as in the end of the chapter.

Talk 3 (25 October): Étale sheaves [5, 6, 7, 8]

In this talk we will introduce sites, which give a more general context for sheaf theory than usual topological spaces: the main idea is to replace the category of open subsets of a space X by any category with a notion of coverings for objects. Our particular interest will be in *étale coverings*, consisting of étale morphisms of schemes. Then, we will construct sheaves on this site, and study their properties.

- Define sites and sheaves as in [Mil, Chapter 5] and give the examples of the Zariski, étale (big and small) and flat sites.
- Define and discuss Galois coverings. Prove [Mil, Prop. 6.6], then discuss the following examples of étale sheaves: the structure sheaf on X_{et}, the sheaf induced by a quasi coherent $\mathcal{O}_X$-module, representable sheaves and the sheaves on $\mathrm{Spec}(k)$. Define the stalk of a sheaf and give examples. Also define skyscraper and locally constant sheaves and prove [Mil, Prop. 6.16].
- State that the category of presheaves on X_{et} is abelian, explain how exactness works in $\mathrm{Sh}(X_{\text{et}})$ stating [Mil, Lemma 7.4, 7.5] (proving at least $(b) \Rightarrow (a)$ of 7.4) and state that étale sheaves form an abelian category as well. Give the example of the Kummer sequence. Define and state the existence of sheafification for étale presheaves.
- Define the direct image of sheaves, proving [Mil, Prop. 8.3, Cor. 8.4] and give one (or more) of the examples in [Mil, Ex. 8.5]. Discuss the inverse image of sheaves, mentioning that it is the right adjoint of restriction along étale maps and [Mil, Rem 8.9], and prove [Mil, Prop. 8.12]. Finally define extension by zero and state the results of the section, proving [Mil, Prop. 8.15].

Talk 4 (8 November): Étale cohomology **[9, 10, 11, 12, 13]**

Using the notion of étale sheaves from the previous talk, we can define *étale cohomology*, following the ideas behind sheaf cohomology. Then, we compute the étale cohomology of $\mathbb{G}_m$.

- Define étale cohomology and state its first properties, proving the dimension, exactness and excision axioms (avoid the homotopy axiom if you want).
- Define Čech cohomology and state [Mil, Thm. 10.2, Prop. 10.6, Thm. 10.9]. Try to briefly explain what spectral sequences are and state [Mil, Thm. 10.7]. For a reference for spectral sequences, you may check [Huy, Section 2.3] or [EC, Appendix B].
- State that the first cohomology group of $\mathbb{G}_m$ on a scheme X is canonically isomorphic to the Picard group of X, [Mil, Cor. 11.6]. Decide if and how much you want to prove of it.
- Talk about the higher direct images of sheaves and the Leray spectral sequence, following [Mil, Chapter 12].
- Prove the Weil-Divisor exact sequence for the étale topology and use it to compute the étale cohomology of $\mathbb{G}_m$ on a curve: specifically, prove [Mil, Prop. 13.4, Thm. 13.7] (decide how much of the auxiliary results you want to prove).

Talk 5 (15 November): The cohomology of curves and purity [14, 15, 16]

The two main results of this talk will be Poincaré duality for curves and the existence of the Gysin sequence. Poincaré duality gives a perfect pairing between suitable cohomology groups, and it will be one of the key tools to prove the Weil conjectures.
Somewhat unrelated, cohomological purity is a more technical result that implies the existence of the so called Gysin map, which in turn will be used for the definition of the cycle map.

- Discuss the structure of the Picard group of a curve and use it to compute the cohomology of μ_n. Define cohomology with compact support following [Mil, Def. 18.1], and compute it for μ_n. Next sketch the proof of Poincaré duality for curves: you should state [Mil, Thm. 14.20] (using the general definition of constructible sheaves), explain how the pairing is defined and why $\mathrm{Ext}^r_U(\mathcal{F}, \mu_n) \cong H^r(U_{\mathrm{et}}, \check{\mathcal{F}}(1))$. Then give the proof of [Mil, Thm. 14.7], which is only for locally constant sheaves, and say how to obtain 14.20 from 14.7. You can find a more detailed and complete proof in [EC, V, Chapter 2] if you are not satisfied with the sketch in the notes.
- In the proof of Theorem 14.7 you will need to know that cohomology of curves vanishes in degree higher than 2: this is a special case of [Mil, Thm. 15.1], which you should state (prove it only if you think you have enough time).
- Prove the existence of the Gysin sequence following Milne's exposition and explain [Mil, Ex. 16.3, 16.4].

Talk 6 (22 November): Proper base change and applications [17, 18, 19]

The main result of this section is the base change theorem for proper morphisms, which we will use to define cohomology with compact support and ℓ-adic cohomology.

- Follow [FK, I, Chapter 6], which reduces the proof to two cases: try to give the details of case (2), while discuss case (1) only if you have time. The proof relies on some results that we may not be able to cover, you can decide what to explain and what to blackbox.
- Define cohomology with compact support and prove [Mil, Prop. 18.2, 18.3]. Define the higher direct images with compact support and state [Mil, Prop. 18.4].
- Introduce sheaves of $\mathbb{Z}_\ell$-modules, giving motivation for their construction, and prove [Mil, Thm. 19.2]. Define sheaves of $\mathbb{Q}_\ell$-modules.

Talk 7 (29 November): Smooth base change, Künneth formula, Cycle map [20, 22, 23]

The objective of this talk is to cover three crucial results for the proof of the Lefschetz fixed-point formula. The first is smooth base change, compares the higher direct images of a sheaf in a cartesian diagram (provided one of the two starting maps is smooth). Next, we prove the Künneth formula, which express the cohomology of a product in terms of the cohomology of the factors. Lastly, we define the cycle map, which is a way to associate a cohomology class to an algebraic cycle.

- State the smooth base change theorem and give a sketch of its proof: here is an approach following Chapter VI of [EC]. State [EC, Thm. 4.1], explain that the theorem holds for universally locally acyclic morphisms ([EC, Prop. 4.10]) and that smooth morphisms satisfy this condition ([EC, Thm. 4.15]). Next prove [EC, Cor. 4.2], which gives a comparison of the cohomology of the fibers of a proper and smooth morphism. Finally explain [EC, Rem. 4.21].
- Define the cup product and prove the Künneth formula [Mil, Thm. 22.1].
- Define the cycle map following the first definition in [Mil, Chapter 23], and state properties (a) and (b) before [Mil, Thm. 23.4]. In order to make sense of these properties, you should introduce the *Chow ring*, which is the quotient

  $$CH^*(X) := C^*(X)/(\text{rational equivalences}),$$

  since $C^*(X)$ is *not* a ring under the intersection product. For more details, you can look at [EC, §VI, Chapter 9].

Talk 8 (6 December): Poincaré duality and the Lefschetz fixed-point formula [24, 25]

In this talk we will see the proof of the two main ingredients for the proof of the first three of the Weil conjectures, namely Poincaré duality and the Lefschetz fixed-point formula.

- Prove Poincaré duality, following the exposition of [EC]. When proving [EC, Lemma 11.3] explain in detail how to get the map $\pi : X \to S$. The proof of the theorem is divided into seven steps: feel free to skip the proof of step 4 as it is a bit technical, but try to explain in detail step 5 since it is the key point of the proof, where the induction argument is used.
  Next state [Mil, Rem. 24.2], explaining especially points (b), (c) and (e), which will be used in the proof of the Lefschetz formula.
- Prove the Lefschetz fixed-point formula [Mil, Thm. 25.1].

Note: In the proof of the Lefschetz fixed-point formula, you will work with ℓ-adic cohomology, but our previous results were only proven for torsion coefficients. You should explain, at least in one instance, how to obtain the analogous result for ℓ-adic cohomology from the torsion case.

2 Proof of the Weil conjectures

Talk 9 (20 December): Proof of the Weil conjectures (I) **[26, 27, 28]**

The goal of this talk is to prove the Weil conjectures except for the Riemann hypothesis. As we will see, the proof is an easy consequence of Poincaré duality and of the Lefschetz fixed-point formula.

- Recall the statement of the conjectures as in [Mil, Chapter 26] and then give the proof of the rationality of the zeta function, its functional equation and the comparison between Betti numbers, following [Mil, Chapter 27].
- We now start the proof of the remaining conjecture: prove [Mil, Prop. 28.3], which reduces the proof of the Riemann hypothesis to the case of the middle cohomology of an even dimensional variety.

Talk 10 (10 January): The Main Lemma **[29, 30]**

Before proving the missing conjecture, we need an intermediate result called the main lemma, to which we dedicate this talk.

- State [Mil, Thm. 29.4], which gives a Lefschetz fixed-point type formula for non constant sheaves of $\mathbb{Q}_\ell$-modules. Explain how to deduce it from [Mil, Thm. 29.15], which is an analogous result for sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules, and prove this theorem. Then define the zeta function of a locally constant sheaf of $\mathbb{Q}_\ell$-modules and explain how to obtain [Mil, Thm. 29.6].
- State and prove the main lemma [Mil, Thm. 30.6].

Talk 11 (17 January): Proof of the Weil conjectures (II) **[31, 32, 33]**

Finally, we can prove the Riemann hypothesis for varieties over $\mathbb{F}_p$, hence concluding the proof of the Weil conjectures.

- Introduce Lefschetz pencils, sketching the proof of their existence and of [Mil, Thm. 31.3], which says that up to birational equivalence any variety can be fibered over $\mathbb{P}^1$ via a map $\pi : X \to \mathbb{P}^1$ with nice fibers.
- Next, we want to understand the higher direct images of $\mathbb{Q}_\ell$, as a $\mathbb{Q}_\ell$-module on X, under π: this is done in [Mil, Chapter 32].
- Complete the proof of the Weil conjectures following [Mil, Chapter 33].

3 An application of the Weil conjectures

The reference for the last two talks is the original paper by Deligne [Del]. However, the subdivision of the material is not definitive yet, as we are still trying to understand the argument and how to divide it in two talks. Nevertheless, the rough idea at the moment is to cover most of the background material in the first talk, while the proof and some (eventual) auxiliary results left to discuss will be treated in the second one. Also, for those of you who don't like french papers written in the 70's, there is an English translation which seems decently accurate and which you can find on the webpage of the seminar.
Here is a very vague and not definitive description of the talks.

Talk 12 (24 January): Ramanujan conjecture (I)

Discuss sections 2 and 3 of [Del], where most of the preparatory material for the proof is introduced.

- The Shimura isomorphism [Del, Thm. 2.10]
- The fundamental ℓ-adic representation W_l, [Del, Def. 3.9]
- Hecke operators (??)

Talk 13 (31 January): Ramanujan conjecture (II)

Prove the congruence formula and explain how to obtain Ramanujan's conjecture from the Weil's conjectures.

- The congruence formula [Del, Them 4.9]
- Weil implies Ramanujan

References

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