Babyseminar WS 2023-24 — Talk 2

Étale morphisms

§ 1 Definitions & first properties

Fix $k = k^{al}$ alg. closed field and let $\varphi : W \to V$ be a morphism of nonsingular algebraic varieties $/k$.

Def: (i) φ is **étale** at $Q \in W$ if the map $d\varphi_Q : Tg_Q(W) \to Tg_{\varphi(Q)}(V)$ is an isomorphism (of k vect. spaces)

(ii) φ is **étale** if it is étale at every point of W

Ex: Assume that locally φ corresponds to a map of reduced k-alg. of finite type $F : A \to B = \dfrac{A[X]}{(g(X))}$ $g(X) \in A[X]$ monic polynomial

with $A = \dfrac{k[Y_1, \ldots Y_m]}{I}$ $I = \sqrt{I} = (f_1, \ldots, f_r)$

$$\Rightarrow B = \frac{k[Y_1, \ldots Y_m, X]}{(g(X)) + I}$$

If $Q \in \mathrm{Spm}(B)$ is of the form $Q = (y_1, \ldots y_m ; x)$ then $\varphi(Q) = (y_1, \ldots y_m) \in \mathrm{Spm}(A)$

An explicit description of $d\varphi_Q$ is as follows:

$$Tg_Q(W) \cong \left\{ (a_1, \ldots, a_m; b) \in k^{m+1} \;\middle|\; \begin{array}{l} \sum_{i=1}^m \frac{\partial f_j}{\partial Y_i}(\varphi(Q)) \cdot a_i = 0 \quad j = 1 \ldots r \\ \sum_{i=1}^m \frac{\partial g}{\partial Y_i}(Q) \cdot a_i + \frac{\partial g}{\partial X}(Q) \cdot b = 0 \end{array} \right\}$$

$d\varphi_Q \downarrow$ projection onto the first m coordinates

$$Tg_{\varphi(Q)}(V) \cong \left\{ (a_1, \ldots, a_m) \in k^m \;\middle|\; \sum_{i=1}^m \frac{\partial f_j}{\partial Y_i}(\varphi(Q)) \cdot a_i = 0 \quad j = 1 \ldots r \right\}$$

Hence φ étale at $Q \iff \dfrac{\partial g}{\partial X}(Q) \neq 0$

Now we try to generalize the notion of étale morphism to schemes

<u>Recall</u>: a morphism of schemes $\varphi : Y \to X$ is flat if $\varphi_y^{\#} : \mathcal{O}_{X,\varphi(y)} \to \mathcal{O}_{Y,y}$ is flat $\forall y \in Y$

<u>Rmk</u>: (i) Since a ring hom $A \xrightarrow{f} B$ is flat iff $A_{f^{-1}(m)} \to B_m$ flat $\forall m \in \operatorname{Spec}(B)$ maximal ideal, flatness of φ can be checked at closed points

(ii) morally, to give a flat morphism $\varphi : Y \to X$ of varieties over k corresponds to giving a "continuous family" of varieties $\{ Y_x = \varphi^{-1}(x) \}_{x \in X}$

⚠ From now on <u>all</u> rings are <u>noetherian</u> and <u>all schemes</u> are (locally) <u>noetherian</u>

<u>Def</u> (i) A local homomorphism of local rings $f : A \to B$ is unramified if

(a) $f(m_A) \cdot B = m_B$ (b) $A/m_A \hookrightarrow B/m_B$ finite separable field ext.

(c) B is essentially of finite type over A (i.e. B localization of a finite type A-alg. at a prime)

(ii) A morphism $\varphi : Y \to X$ of schemes which is (locally) of finite type is unramified if $\forall y \in Y$ $\mathcal{O}_{X,\varphi(y)} \to \mathcal{O}_{X,x}$ is unramified

<u>Def</u> A morphism of scheme $\varphi : Y \to X$ (locally) of finite type is étale if it is flat & unramified.

<u>Proposition 1</u>: If $\varphi : W \to V$ is a morphism of nonsingular varieties over $k = k^{al}$, then the two notions of étaleness for φ agree.

<u>Proof</u> (idea) $d\varphi_Q : (m_Q/m_Q^2)^\vee \to (m_{\varphi(Q)}/m_{\varphi(Q)}^2)^\vee$ isomorphism $\iff$ $\hat{\mathcal{O}}_{V,\varphi(Q)} \xrightarrow{\simeq} \hat{\mathcal{O}}_{W,Q}$ (*)

Then one is left to prove that:

$\hat{\varphi}_Q : \hat{\mathcal{O}}_{V,\varphi(Q)} \xrightarrow{\simeq} \hat{\mathcal{O}}_{W,Q}$ $\iff$ $\varphi_Q : \mathcal{O}_{V,\varphi(Q)} \to \mathcal{O}_{W,Q}$ is flat and unramified

Once proven that φ_Q flat and unramified $\Rightarrow$ $\hat{\varphi}_Q$ injective, unramified and $\hat{\mathcal{O}}_{W,Q}$ finite as $\hat{\mathcal{O}}_{V,\varphi(Q)}$-alg., the conclusion follows from Nakayama's lemma ∎

<u>Rmk</u> (*) defines étaleness at Q for any morphism $\varphi : W \to V$ of (possibly singular) varieties

Proposition 2: (i) open immersions are étale

(ii) being étale is stable under composition and base change

(iii) If $Z \xrightarrow{\psi} Y \xrightarrow{\varphi} X$ is étale and φ is étale $\Rightarrow \psi$ is étale

Proposition 3: Let $\varphi : Y \to X$ be an étale morphism (of finite type).

(i) $\forall y \in Y$ $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,\varphi(y)}$ have the same Krull dim.

(ii) φ is quasi-finite (i.e. $\# \varphi^{-1}(x)$ finite $\forall x \in X$)

(iii) φ is open

(iv) X reduced | normal | regular $\Rightarrow$ Y is reduced | normal | regular

Proposition 4: $\varphi : Y \to X$ morphism of finite type, then

$\{ y \in Y \mid \varphi$ is étale (i.e. flat and unram.) at $y \} \subseteq Y$ is open

Proposition 5 Consider a diagram

$$\begin{tikzcd} Y \arrow[r, "q"] \arrow[d, shift right, "\varphi"'] \arrow[d, shift left, "\varphi'"] & S \\ X \arrow[ur, "p"'] & \end{tikzcd}$$

with p étale and separated, Y connected

If $\exists y \in Y$ st $\varphi(y) = \varphi'(y) = x$ and the maps $k(x) \rightrightarrows k(y)$ induced by φ and φ' coincide $\Rightarrow \varphi = \varphi'$

Proof (idea)

$$\begin{tikzcd} Y \arrow[r, shift left, "{(1,\varphi)}"] \arrow[r, shift right, "{(1,\varphi')}"'] \arrow[rr, bend right, "1_Y"'] & Y \times_S X \arrow[r, "p_Y"] & Y \end{tikzcd}$$

- p_Y étale & separated
- $(1,\varphi)$, $(1,\varphi')$ sections of p_Y

Y connected $\Rightarrow (1,\varphi) = (1,\varphi') \Rightarrow \varphi = \varphi'$

$\blacksquare$

Rmk: Locally any étale morphism of schemes is always "standard étale", i.e. of the form

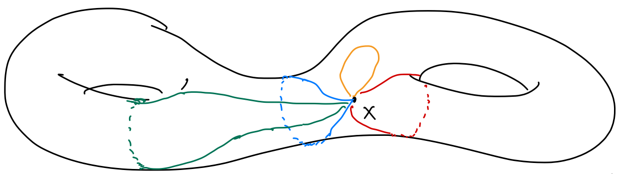
$$A \longrightarrow \frac{A[X]}{(g(X))}[b^{-1}] \qquad g(X) \in A[X] \text{ monic}$$

$b \in \dfrac{A[X]}{(g(X))}$ s.t. $g'(X) \in \left(\dfrac{A[X]}{(g(X))}[b^{-1}] \right)^{\times}$

This can be used to give one line proofs of many of the above statements

§ 2. The étale fundamental group

Recall: if X is a (decent) connected topological space and $x \in X$, one can construct the fundamental group $\pi_1(X, x)$ as the group of homotopy classes of loops in X based at x.



There is a natural isomorphism $\mathrm{Aut}_X(\hat{X}) \xrightarrow{\sim} \pi_1(X, x)$ where $(\hat{X}, \hat{x}) \to (X, x)$ is the so-called universal cover of (X, x) and a way of rephrasing this is as follows

the functor

$$\begin{array}{ccc} \mathrm{Cov}(X) & \longrightarrow & \mathrm{Aut}_X(\hat{X})\text{-fsets} \\ Y & \longmapsto & \mathrm{Hom}_X(\hat{X}, Y) \end{array}$$

category of coverings of X with finitely many connected components

$\rightsquigarrow$ only finitely many orbits

$\alpha * f := f \circ \alpha \quad \alpha \in \mathrm{Aut}_X(\hat{X}) \quad f : \hat{X} \to Y$ over X

is an equivalence of categories

Grothendieck's idea: find an analogue for alg. varieties / schemes !!

Rmk: If $\varphi : Y \to X$ ($Y \neq \emptyset$) with X connected is étale and finite $\Rightarrow$ φ open and closed $\Rightarrow \varphi$ surjective $\rightsquigarrow$ analogue of finite covering space

One can mimic the topological situation.

Set $\mathrm{Fét}/X$ category
- objects: finite étale maps $Y \to X$
- morphisms: X-morphisms

Fix a geometric point $\bar{x} : \mathrm{Spec}(\Omega) \to X$, then we get a functor

$\mathrm{Fét}/X \xrightarrow{F} \mathrm{sets} \qquad F(Y \xrightarrow{\varphi} X) = \{ \bar{y} : \mathrm{Spec}(\Omega) \to Y \text{ over } X \text{ with } \bar{x} \} = \text{"}\varphi^{-1}(\bar{x})\text{"}$

Unlike in the topological case, this functor is only pro-representable i.e.

$\exists$ projective system $(X_i \xrightarrow{\varphi_i} X)_{i\in I}$ I directed set such that

$\forall$ $Y \xrightarrow{\varphi} X$ $\quad F(Y \xrightarrow{\varphi} X) = \varinjlim_{i\in I} \mathrm{Hom}_X(X_i, Y)$

WLOG $X_i \xrightarrow{\varphi_i} X$ Galois $\forall i \in I$ (i.e. $\deg \varphi_i = \# \mathrm{Aut}_X(X_i)$)

Def: $\pi_1(X, \bar{x}) := \varprojlim_i \mathrm{Aut}_X(X_i)$ étale fundamental group of X (it is a profinite group)

Theorem 5: $(Y \xrightarrow{\varphi} X) \rightsquigarrow F(Y \xrightarrow{\varphi} X)$ defines an equivalence of categories

$$F\acute{e}t/_X \xrightarrow{\simeq} \text{finite discrete } \pi_1(X, \bar{x})\text{-sets}$$

Examples: (i) $X = \mathrm{Spec}(K)$ K field $\Rightarrow$ $\pi_1(K, \bar{x}) \simeq \mathrm{Gal}(K^{sep}/K)$

where $K \subset K^{al}$ fixed alg. closure of K

and K^{sep} separable closure of K inside K^{al}

[In the next examples we will need Riemann-Hurwitz formula: given a finite separable morphism $\varphi: Y \to X$ between smooth proj curves over an alg. closed field k which is tamely ramified, it holds $2g(Y) - 2 = (\deg \varphi)(2g(X) - 2) + \sum_{p\in X}(e_p - 1)$ $g(-)$ genus, e_p ram. index at p]

(ii) $X = \mathbb{P}^1_k$ $k = k^{al}$ then every finite étale covering $Y \xrightarrow{\varphi} X$ of degree n is given by a smooth proj curve Y/k. Riemann-Hurwitz formula gives

$2g(Y) - 2 = -2 \cdot n$ $\quad \Rightarrow g(Y) = 0$ and $n = 1$ $\Rightarrow \varphi$ isomorphism

$\Rightarrow$ $\pi_1(X, \bar{x})$ trivial

(iii) $X = \mathbb{A}^1_k$ $k = k^{al}$, char $k = 0$ then every finite étale covering $Y \xrightarrow{\varphi} X$ extends to a map $\bar{Y} \xrightarrow{\bar{\varphi}} \mathbb{P}^1_k$ where $\bar{Y}$ is a smooth proj curve and $\bar{\varphi}$ can be branched over ∞,

Riemann-Hurwitz $\Rightarrow 2g(\bar{Y}) - 2 = -2 \cdot \deg(\varphi) + e_\infty - 1$ $\quad$ e_∞ = ramification index at ∞

$\Rightarrow$ $2g(\bar{Y}) - 2 \leq -\deg\varphi - 1$ $\quad$ $e_\infty \leq \deg(\varphi)$

$\Rightarrow$ $g(\bar{Y}) = 0$, $\deg(\varphi) = 1$ $\quad e_\infty = 1$ $\Rightarrow$ φ iso $\Rightarrow$ $\pi_1(\mathbb{A}^1_k, \bar{x})$ trivial

(iv) $X = \mathbb{A}^1_k \setminus \{0\}$ $\quad k = k^{al}$ $\quad$ char $k = 0$

As before any finite étale covering $Y \xrightarrow{\varphi} X$ extends to $\bar{Y} \xrightarrow{\bar\varphi} \mathbb{P}^1_k$

$\bar{Y}$ smooth proj curve, $\bar\varphi$ can be branched over 0 and ∞.

Riemann–Hurwitz gives: $2g(\bar Y) - 2 = -2\deg(\varphi) + e_0 + e_\infty - 2$

$\Rightarrow 2g(\bar Y) - 2 \leq -2 \Rightarrow g(\bar Y) = 0$ and $2\deg(\varphi) = e_0 + e_\infty$

$\Rightarrow$ $Y \cong X = \mathbb{A}^1_k \setminus \{0\}$ and $\deg \varphi = e_0 = e_\infty$

so $\varphi : \mathbb{A}^1_k \setminus \{0\} \to \mathbb{A}^1_k \setminus \{0\}$ of $\deg \varphi = n$ corresponds to $t \mapsto c \cdot t^{\pm n}$ $\quad c \in k^*$

$\mathrm{Aut}_X(X \xrightarrow{\varphi} X) \cong \mu_n(k) \quad \Rightarrow \quad \pi_1(X, \bar x) \cong \hat{\mathbb{Z}}$

<u>NB</u>: (iii) & (iv) change in char > 0; KEY WORD: Artin-Schreier

<u>Theorem 6</u> If X is a smooth alg. variety over $\mathbb{C}$ then $\exists$ natural iso

$$\pi_1^{ét}(X, \bar x) \cong \pi_1^{top}(\underbrace{X^{an}}_{\text{complex topology}}, x)^{\wedge} \leftarrow \text{profinite completion}$$

§ 3. <u>Henselian rings</u>

<u>Def</u>: A local ring (A, m_A) is **henselian** if, for every $f(t) \in A[t]$ monic polynomial, if the image $\bar f(t) \in A/m_A[t]$ admits a factorization $\bar f = g_0 \cdot h_0$ with g_0, h_0 monic and s.t. $(g_0, h_0) = 1$, then $\exists\, g, h \in A[t]$ monic s.t. $f = g \cdot h$ and $\bar g = g_0$, $\bar h = h_0$.

<u>Rmk</u>: g, h in the above definition are unique and coprime in $A[t]$

<u>Ex.</u> (A, m_A) complete local ring $\Rightarrow (A, m_A)$ henselian (Hensel's lemma)

<u>Proposition 7</u>: Let (A, m_A) be a local ring with residue field $k_A = A/m_A$; TFAE

(i) (A, m_A) henselian

(ii) if any $f(t) \in A[t]$ not nec. monic is such that $\bar f = g_0 \cdot h_0$ with g_0 monic and $(g_0, h_0) = 1 \Rightarrow f = gh$ with g monic, $\bar g = g_0$, $\bar h = h_0$

(iii) given $f_1, \dots, f_n \in A[T_1, \dots, T_n]$, every common zero $\underline{x}_0 \in (k_A)^n$ of the $\bar{f}_i$'s such that $\mathrm{Jac}(\bar{f}_1, \dots, \bar{f}_n)(\underline{x}_0) \in GL_n(k_A)$ lifts to a common zero of the f_i's in A^n

(iv) If B is an étale A-algebra and $B/m_A \cdot B \cong k_A \times \bar{B}'$ for some k_A-algebra $\bar{B}'$ then $\exists$ a decomposition $B \cong A \times B'$ B' A-alg lifting the decomposition $B/m_A B \cong k_A \times \bar{B}'$

Proof: omitted (see Milne, Prop. 4.11)

Def: Let (A, m_A) be a local ring. A morphism of local rings $A \to A^h$ with A^h henselian is called henselization if it satisfies the univ. property:

$$\begin{tikzcd} A \arrow[r, "\forall"] \arrow[dr] & B \\ & A^h \arrow[u, dashed, "\exists!"'] \end{tikzcd} \quad B \text{ henselian}$$

Proposition 8 (i) $A^h \cong \varinjlim_{(B,q)} B$ where the limit is over pairs (B, q) where B is an étale A-algebra and $q \in \mathrm{Spec}(B)$ st $q \cap A = m_A$ and $A/m_A \to B/q$ is o

(ii) $A^h \cong \bigcap_{\substack{A \subseteq B \subseteq \hat{A} \\ m_A \subseteq m_B \subseteq \hat{m}_A}} B$ B local henselian

Def (i) A local ring (A, m_A) is strictly henselian if it is henselian and it's residue field is separably closed.

(ii) A strict henselization of (A, m_A) is a morphism of local rings $A \to A^{sh}$ st. $(A^{sh}, m_{A^{sh}})$ is strictly henselian such that every local morphism $A \to B$ with with (B, m_B) strictly henselian factors through A^{sh}, with factorization uniquely determined by the extension of residue fields $A^{sh}/m_{A^{sh}} \to B/m_B$.

Ex: $A = \mathbb{Z}_p \rightsquigarrow$ can choose $A^{sh} = W(\bar{\mathbb{F}}_p)$ (Witt vectors)

Rmk Unlike henselization, strict henselization is <u>NOT</u> unique up to unique isom.

§ 4. The local ring for the étale topology

X scheme, $\bar{x} : \mathrm{Spec}(\Omega) \to X$ geometric point (Ω separably closed field)

<u>Def</u> (i) An étale neighbourhood of $\bar{x}$ is a pair $(U \to X, \bar{u})$ where $U \to X$ étale and $\bar{u} : \mathrm{Spec}(\Omega) \to U$ geom. point s.t.

$$\begin{tikzcd} \mathrm{Spec}(\Omega) \arrow[r, "\bar{u}"] \arrow[dr, "\bar{x}"'] & U \arrow[d] \arrow[dl, phantom, "\circlearrowright"] \\ & X \end{tikzcd}$$

(ii) the local ring at $\bar{x}$ for the étale topology is

$$\mathcal{O}_{X,\bar{x}} := \varinjlim_{(U,\bar{u})} \mathcal{O}_U(U)$$

where in the limit we only consider U connected and affine

<u>Proposition 9</u> If $x \in X$ is the image of $\bar{x}$, then $\mathcal{O}_{X,\bar{x}} \cong \mathcal{O}_{X,x}^{sh}$

<u>Proof</u>: omitted