3. COHOMOLOGY: Definition and the basic properties

We showed that the category of sheaves of abelian groups on X_{et}: $Sh(X_{et})$ is an abelian category with enough injectives. The functor

$$\begin{aligned} Sh(X_{et}) &\to Ab \\ \mathcal{F} &\mapsto \Gamma(X, \mathcal{F}) \end{aligned} \quad \text{is left exact.}$$

We define $H^r(X_{et}, -)$ the rth right derived functor.
Explicitely: given a sheaf $\mathcal{F}$, choose an injective resolution

$$0 \to \mathcal{F} \to I^0 \to I^1 \to I^2 \to \cdots$$

$$\Downarrow \text{ apply the functor } \Gamma(X,-)$$

$$0 \to \Gamma(X, I^0) \xrightarrow{d^0} \Gamma(X, I^1) \xrightarrow{d^1} \Gamma(X, I^2) \to \cdots$$

This complex is no longer exact (in general), and $H^r(X, \mathcal{F})$ is defined to be its rth cohomology group, i.e.

$$H^r(X, \mathcal{F}) := \frac{\ker d^r}{\operatorname{im} d^{r-1}}$$

We have the following properties:

- $\forall \mathcal{F}$ sheaf : $H^0(X_{et}, \mathcal{F}) = \Gamma(X, \mathcal{F})$

  $\left(H^0(X, \mathcal{F}) = \ker d^0 = \Gamma(X, \mathcal{F}) \text{ since } H^0(X,-) \text{ is left exact} \right.$
  $\left. 0 \to \mathcal{F} \to I^0 \to I^1 \Rightarrow \underbrace{0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, I^0) \to \Gamma(X, I^1)}_{\text{(exact)}} \right)$

- If I is injective $\Rightarrow H^r(X_{et}, I) = 0 \quad \forall r > 0$

  (choose the injective resolution $0 \to I \to I \to 0$)

- a ses of sheaves $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ gives rise to a long exact sequence

$$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \to H^1(X, \mathcal{F}') \to \cdots$$

  and this association is functorial

Remark: one should check that the above definition is a good one: namely it doesn't depend on the choice of the injective resolution $\rightsquigarrow$ general theory: different choices of resolutions yeld naturally isomorphic functors

Remark. $\varphi: U \to X$ étale morphism

$\varphi^*: Sh(X_{et}) \to Sh(U_{et})$ is exact and preserves injectives

$$\begin{tikzcd} Sh(X_{et}) \arrow[r, "\varphi^*"] \arrow[rr, bend right, "\Gamma(U,-)"'] & Sh(U_{et}) \arrow[r, "\Gamma(U,-)"] & Ab \end{tikzcd}$$

(φ^* is just restriction)

$\Rightarrow$ the right derived functors of $\mathcal{F} \mapsto \mathcal{F}(U): Sh(X_{et}) \to Ab$ are $\mathcal{F} \mapsto H^r(U_{et}, \mathcal{F}|_U) \overset{\text{not}}{=} H^r(U_{et}, \mathcal{F})$

The DIMENSION AXIOM

let $x = \operatorname{Spec} k$, and $\bar{x} = \operatorname{Spec} k^{sep}$ for some separable closure k^{sep} of k.

we saw that the functor

$$\mathcal{F} \longmapsto M_{\mathcal{F}} \overset{\text{def}}{=} \mathcal{F}_{\bar{x}} = \varinjlim_{\substack{k \subseteq k' \subseteq k^{sep} \\ k'|k \text{ finite and Galois}}} \mathcal{F}(k')$$

defines an equivalence from the category of sheaves on x_{et} to the category of discrete G-modules where $G = \operatorname{Gal}(k^{sep}|k)$.

we have $(M_{\mathcal{F}})^G = \Gamma(x, \mathcal{F})$ $\Rightarrow$ the derived functors of $M \mapsto M^G$ and $\mathcal{F} \mapsto \Gamma(x, \mathcal{F})$ correspond, namely

$$H^r(x, \mathcal{F}) \simeq H^r(G, M_{\mathcal{F}}).$$

To have

$$H^r(x, \mathcal{F}) = 0 \quad \forall r > 0 \quad \forall \mathcal{F} \text{ sheaf}$$

we showed take x to be a geometric point. (?)

[GEOMETRIC POINTS in étale nte $\longleftrightarrow$ POINTS of a topological space]

$(U, \mathcal{F}) \to H^{r+1},$

<u>Remark</u>. $\varphi : U \to X$ étale morphism ②

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$\Rightarrow$ the right derived functors of $\mathcal{F} \mapsto \mathcal{F}(U) : Sh(X_{et}) \to Ab$ are $\mathcal{F} \mapsto H^r(U_{et}, \mathcal{F}|U) \overset{not}{=} H^r(U_{et}, \mathcal{F})$

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[GEOMETRIC POINTS in étale nte $\longleftrightarrow$ POINTS of a topological space]

EXACTNESS AXIOM ③

Z closed subvariety of X, $U := X \setminus Z$.

$\mathcal{F}$ sheaf on X_{et} $\Rightarrow$ define $\Gamma_Z(X, \mathcal{F}) := \ker\left(\Gamma(X,\mathcal{F}) \to \Gamma(U,\mathcal{F})\right)$

"group of sections of $\mathcal{F}$ with support on Z"

Denote with $H^r_Z(X, -)$ the "cohomology of $\mathcal{F}$ with support on Z"

<u>Theorem</u>: For any sheaf $\mathcal{F}$ on X_{et} and closed $Z \subseteq X$, there is a long exact sequence

$$\cdots \to H^r_Z(X,\mathcal{F}) \to H^r(X,\mathcal{F}) \to H^r(U,\mathcal{F}) \to H^{r+1}_Z(X,\mathcal{F}) \to \cdots$$

Ext - Groups

fix a sheaf $\mathcal{F}_0$ ~~and a ses~~

$$\mathcal{F} \longmapsto Hom_X(\mathcal{F}_0, \mathcal{F}) : Sh(X_{et}) \to Ab \quad \text{is left exact}$$

$\Rightarrow$ $Ext^r(\mathcal{F}_0, -) \overset{def}{:=} r$th right derived functor. Then:

- $Ext^0(\mathcal{F}_0, \mathcal{F}) = Hom(\mathcal{F}_0, \mathcal{F})$, for any sheaf $\mathcal{F}$
- I injective $\Rightarrow$ $Ext^r(\mathcal{F}_0, I) = 0 \quad \forall r > 0$
- $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ induces

$$\to Ext^r_X(\mathcal{F}_0, \mathcal{F}') \to Ext^r_X(\mathcal{F}_0, \mathcal{F}) \to Ext^r_X(\mathcal{F}_0, \mathcal{F}'') \to \cdots$$

<u>EXAMPLE</u> $\mathbb{Z}$: costant sheaf on X. $\mathcal{F}$ any sheaf on X

Consider the map ~~$Hom_{\mathbb{Z}}$~~ $Hom_X(\mathbb{Z}, \mathcal{F}) \xrightarrow{\simeq} \mathcal{F}(X)$

$$\alpha \longmapsto \alpha(1)$$

$\Rightarrow$ $Hom_X(\mathbb{Z}, -) = \Gamma(X, -)$

$\Rightarrow$ $Ext^r_X(\mathbb{Z}, -) \simeq H^r(X_{et}, -)$

<u>Prop</u>: ses $0 \to \mathcal{F}_0' \to \mathcal{F}_0 \to \mathcal{F}_0'' \to 0$

$\Longrightarrow$ $\cdots \to Ext^r_X(\mathcal{F}_0'', \mathcal{F}) \to Ext^r_X(\mathcal{F}_0, \mathcal{F}) \to Ext^r_X(\mathcal{F}_0', \mathcal{F}) \to \cdots$

($\forall$ any sheaf $\mathcal{F}$.)

$Ext^{r+1}_X(\mathcal{F}_0'', \mathcal{F})$ et

Proof of the theorem :

(4)

$$U \xrightarrow{j} X \xleftarrow{i} Z$$

$\mathbb{Z}$ constant sheaf on X

Consider the ses

$$0 \to \underset{\mathcal{F}_0'}{j_! j^* \mathbb{Z}} \to \underset{\mathcal{F}_0}{\mathbb{Z}} \to \underset{\mathcal{F}_0''}{i_* i^* \mathbb{Z}} \to 0 \qquad (*)$$

$\mathcal{F}$ sheaf on X_{et}

$$\mathrm{Hom}_X(j_! j^* \mathbb{Z}, \mathcal{F}) = \mathrm{Hom}_U(j^* \mathbb{Z}, j^* \mathcal{F}) = \mathcal{F}(U)$$

$$\Rightarrow \quad \mathrm{Ext}^r_X(j_! j^* \mathbb{Z}, \mathcal{F}) = H^r(U_{et}, \mathcal{F}) \quad \cdot 3$$

applying the functor $\mathrm{Hom}(-, \mathcal{F})$

From $(*)$, we get

$$0 \to \mathrm{Hom}(i_* i^* \mathbb{Z}, \mathcal{F}) \to \mathrm{Hom}(\mathbb{Z}, \mathcal{F}) \to \mathrm{Hom}(j_! j^* \mathbb{Z}, \mathcal{F}).$$

($\mathrm{Hom}(-,\mathcal{F})$ is left exact)

$$\Rightarrow \quad \mathrm{Hom}(i_* i^* \mathbb{Z}, \mathcal{F}) = \mathrm{Ker}\left(\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F})\right)$$

|| by def

$\Gamma_Z(X, \mathcal{F})$

$$\Rightarrow \quad \mathrm{Ext}^r_X(i_* i^* \mathbb{Z}, \mathcal{F}) = H^r_Z(X, \mathcal{F}) \quad \cdot 1$$

$\leadsto$ (0) becomes $\cdots \to H^r_Z(X, \mathcal{F}) \to H^r_X(X, \mathcal{F}) \to H^r(U_{et}, \mathcal{F}) \to \cdots$

$\cdot 2$

$\to$

<u>Corollary</u>: x a closed point of X. For any sheaf $\mathcal{F}$ on X, there is an isomorphism $H^r_x(X, \mathcal{F}) \to H^r_x(\operatorname{Spec} \mathcal{O}^h_{X,x}, \mathcal{F})$ where $\mathcal{O}^h_{X,x}$ is the Henselization of $\mathcal{O}_{X,x}$.

To prove the corollary one needs:

- $H^r_x(X, \mathcal{F}) = H^r_u(\mathcal{U}, \mathcal{F})$ for any étale nbh $(\mathcal{U}, u)$ of x s.t. u is the only point of $\mathcal{U}$ mapped to x
- The following <u>proposition</u>:

Let I be a directed set, $(X_i)_{i \in I}$ an inverse system of X-schemes. Assume that all X_i are quasicompact and that the maps $X_i \leftarrow X_j$ are all affine. Let $X_\infty := \varprojlim X_i$ and for any sheaf $\mathcal{F}$ on X, let $\mathcal{F}_i$ be its inverse image on X_i, $i \in I \cup \{\infty\}$. Then

$$\varinjlim H^r(X_{i,et}, \mathcal{F}_i) \simeq H^r(X_\infty, \mathcal{F}_\infty)$$

$= H^0(X, \mathcal{F})$

$\boxed{\text{ČECH COHOMOLOGY}}$

Let $\mathcal{U} = (U_i \to X)_{i \in I}$ be an étale covering of X and P be a presheaf of abelian groups on X_{et}. Define $\in PreSh(X_{et})$

$$C^r(\mathcal{U}, P) = \prod_{(i_0, \dots, i_r) \in I^{r+1}} P(U_{i_0 \dots i_r}), \quad \text{where} \quad U_{i_0, \dots, i_r} = U_{i_0} \times_X \dots \times_X U_{i_r}.$$

For $s = (s_{i_0, \dots, i_r}) \in C^r(\mathcal{U}, P)$, define $d^r s \in C^{r+1}(\mathcal{U}, P)$ by the rule

$$(d^r s)_{i_0, \dots, i_{r+1}} = \sum_{j=0}^{r+1} (-1)^j \operatorname{res}_j (s_{i_0 \dots i_{j-1} i_{j+1} \dots i_{r+1}}), \quad \text{where}$$

res_j is the restruction map corresponding to the projection map $U_{i_0 \dots i_{r+1}} \longrightarrow U_{i_0 \dots i_{j-1} i_{j+1} \dots i_{r+1}}$.

One can verify directly that

$$\operatorname{res}_j : P(U_{i_0, \dots i_{j-1} i_{j+1} \dots i_{r+1}}) \to P(U_{i_0 \dots i_{r+1}})$$

$$C^\bullet(\mathcal{U}, P) = C^0(\mathcal{U}, P) \xrightarrow{d^0} C^1(\mathcal{U}, P) \xrightarrow{d^1} \dots$$ is a complex

Define $\check{H}^r(\mathcal{U}, P) = H^r(C^\bullet(\mathcal{U}, P))$ "Čech cohomology group of P relative to the covering $\mathcal{U}$"

Rmk. $\check{H}^0(\mathcal{U}, P) = \text{eq}\left(\prod P(U_i) \rightrightarrows \prod P(U_{ij})\right)$

If $\mathcal{F}$ is a sheaf $\Rightarrow \check{H}^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F})$

Def "refinement of coverings"
A second covering $\mathcal{V} = (V_j \to X)_{j \in J}$ of X is called a refinement of $\mathcal{U}$ if there is a map $\tau : J \to I$ such that $V_j \to X$ factors through $U_{\tau j}$ $\forall j \in J$

$$\begin{tikzcd} V_j \arrow[r] \arrow[dr] & X \\ & U_{\tau j} \arrow[u] \end{tikzcd}$$

The choice of τ and X-morphisms $V_j \to U_{\tau j}$ for each j deter=

mines a map of complexes:

$$\tau^\bullet : \check{C}^\bullet(\mathcal{U}, P) \to \check{C}^\bullet(\mathcal{V}, P)$$

$$(\tau^r s)_{j_0 \dots j_r} = s_{\tau j_0 \dots \tau j_r}\Big|_{V_{j_0 \dots j_r}}$$

One can verify that the map on cohomology groups

$$\rho(\mathcal{V}, \mathcal{U}) : \check{H}^r(\mathcal{U}, P) \to \check{H}^r(\mathcal{V}, P)$$ is independent

of all choices.

We pass to the limit over all coverings and then obtain the Čech cohomology groups

$$\check{H}^r(X, P) \overset{\text{def}}{=} \varinjlim \check{H}^r(\mathcal{U}, P)$$

They have the properties

- $\check{H}^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F}) \quad \forall\, \mathcal{F}$ sheaf on X
- $\check{H}^r(X, I) = 0, \quad r > 0 \quad \forall\, I$ injective sheaf
- Prop: $\check{H}^r(X_{et}, -)$ is the r-th right derived functor of $P \to \check{H}^0(X_{et}, P) : \mathrm{PreSh}(X_{et}) \to \mathrm{Ab}$

<u>Theorem</u> Assume that every finite subset of X is contained in an open affine and that X is quasi-compact (e.g. X a quasi-projective variety). Then

$$\check{H}^r(X, \mathcal{F}) \simeq H^r(X, \mathcal{F}) \quad \forall\, r, \ \forall\, \mathcal{F} \text{ sheaf.}$$

<u>Proposition</u>: there's an isomorphism $H^1(X, \mathcal{F}) \to \check{H}^1(X, \mathcal{F})$

$\boxed{\text{SPECTRAL SEQUENCES}}$ It's the collection of

- A family $(E_r^{p,q})$ of objects of an abelian category
  $p, q \geq 0 \quad r \geq 1 \ (\geq 2) \quad p, q, r \in \mathbb{Z}$
- Morphisms $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,\, q-r+1}$ s.t.

$$d_r^{p+r,\, q-r+1} \circ d_r^{p,q} = 0$$

$$E_{r+1}^{p,q} = \frac{\ker(d_r^{p,q})}{\operatorname{im}(d_r^{p-r,q+r-1})}$$

For each (p,q) $\exists\, r_0$: $\forall\, r \geq r_0$ $d_r^{p,q} = d_r^{p-r,q+r-1} = 0$

$$\Rightarrow E_{r_0}^{p,q} = E_{r_0+1}^{p,q} = \cdots \overset{\text{def}}{=} E_{\infty}^{p,q}$$

- A family of objects (E^m), $m \geq 0$ and for each E^m a filtration

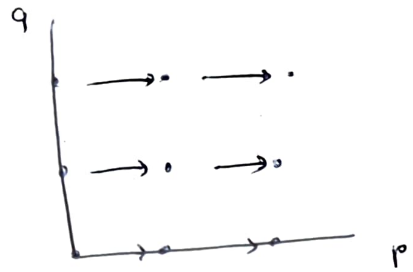
$$E^m = E_0^m \supseteq E_1^m \supseteq E_2^m \supseteq \cdots \supseteq E_m^m \supseteq 0 .$$

such that

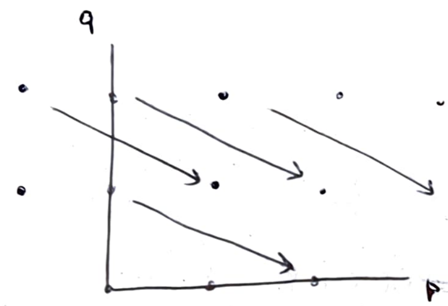
$$E_p^m / E_{p+1}^m = E_\infty^{p,m-p}$$

Such a spectral sequence is written $E_{1\,(2)}^{p,q} \Rightarrow E^m$

page 1



page 2



<u>Theorem</u>: A, B, C abelian categories and assume that A, B have enough injectives, $F: A \to B$, $G: B \to C$ left exact functors and assume that $(R^rG)(FI) = 0$ for $r > 0$ if I is injective, then there is a spectral sequence

$$E_2^{r,s} := (R^rG)(R^sF)(A) \Rightarrow R^{r+s}(FG)(A) =: E^{r+s}$$

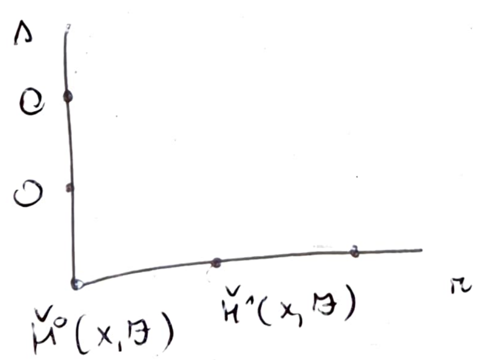
In our case we have

$$Sh(X_{et}) \xrightarrow{i} Pre\,Sh(X_{et}) \xrightarrow{\check{H}^0(X_{et},-)} Ab$$

NOTATION: $\underline{H}^r(-)$ is the r^{th} right derived functor of i

FACT $\check{H}^0(X_{et}, \underline{H}^s(\mathcal{F})) = 0$ for $s > 0$

page 2 of our spectr. sequence



Theorem $H^r(X_{et}, \mathbb{G}_m) = \begin{cases} \Gamma(X, \mathcal{O}_X^*)^\times & r = 0 \\ \mathrm{Pic}(X) & r = 1 \\ 0 & r > 1 \end{cases}$ (12)

for a connected nonsingular curve X over an algebraically closed field $k = \bar{k}$.

Proof $r = 0$ ✓

$r = 1$ follows from a more general statement:

$$\underbrace{L_m(X_{zar})}_{} \overset{1:1}{\longleftrightarrow} \check{H}^1(X_{et}, GL_m)$$
$$\cong \quad H^1(X_{et}, GL_m)$$

// def

" isomorphism classes of locally free sheaves of $\mathcal{O}_X$-modules of rank m on X for the Zariski topology"

$m = 1$ $\mathrm{Pic}(X) = L_1(X_{zar})$

$r > 1$? The Weil divisor exact sequence

A integrally closed integral domain $\Rightarrow$

$$A = \bigcap_{\substack{p \text{ prime} \\ ht(p) = 1}} A_p$$

$\longrightarrow$ the integral closure in its field of fractions is A itself.

We have an exact sequence

$$0 \to A^\times \to K^\times \xrightarrow{f} \bigoplus_{ht(p)=1} \mathbb{Z} \qquad (\times)$$

$$(K = \mathrm{Frac}(A))$$

f is in general not surjective (ex: A Dedekind domain $\Rightarrow$ its cokernel is the ideal class group of A)

If A integral domain

f in $(\times)$ is also surjective $\Leftrightarrow$ A is a UFD.

The exact sequence for the Zariski topology

Recall: A variety is said to be normal if $\Gamma(U, \mathcal{O}_X)$ is an integrally closed integral domain for every connected open affine $U \subseteq X$, equivalently if $\mathcal{O}_{X,x}$ is an integrally closed integral domain for all x in X

Assume X to be connected and normal

$\Rightarrow$ $\exists$ a field K of rational functions on X that is the field of fractions of $\Gamma(U, \mathcal{O}_X)$ for any open affine $U \subseteq X$

A prime (weil) divisor on X is a closed irreducible subvariety Z of codim 1, a weil divisor is an element $D = \sum m_Z Z$ of the free abelian group generated by the prime divisors.

we have a bijection for any $\emptyset \neq U \overset{open}{\subseteq} X$

$$\left\{\begin{matrix}\text{prime divisor of } X \\ \text{meeting } U\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \{\text{prime divisor of } U\}$$

$$Z \longmapsto Z \cap U$$

$$\left(\underset{\text{closure in } X}{\bar{p}} \longleftarrow p \right)$$

If U is an open affine with $\Gamma(U, \mathcal{O}_X) = A$, then there's a bijection:

$$\left\{\begin{matrix}\text{prime ideals of } A \\ \text{of height } 1\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \{\text{prime divisors of } U\}$$

$$p \longmapsto V(p) \text{ zero set of } p$$

$$I(Z) \longleftarrow Z$$

$I(Z)$ = functions zero on Z

Every prime ~~ideal~~ divisor Z on X defines a discrete valuation ord_Z on K

Intuitively, for $f \in K$, $\text{ord}_Z(f)$ is the order of the zero (or the pole) of f along Z.

Prop : There's a sequence of sheaves on X_{Zar}

$$0 \to \mathcal{O}_X^\times \to k^\times \to \mathrm{Div} \to 0 \qquad (*)$$

where $\Gamma(U, \cancel{k^\times}) = k^\times$ for all non empty open U and $\mathrm{Div}(U)$ is the group of divisors on U

The sequence is always left exact, it's exact when X is regular (e.g. a nonsingular variety)

Proof: For any open affine U in X, with $\Gamma(U, \mathcal{O}_X) = A$

$$0 \to A^\times \to k^\times \to \bigoplus_{ht(p)=1} \mathbb{Z} \to 0$$

(sequence of sections over U)

For any $x \in X$, the sequence of stalks at x is as above but with A replaced by $\mathcal{O}_{X,x}$

($\mathcal{O}_{X,x}$ is an integrally closed integral domain $\Rightarrow$ left exact
if $\mathcal{O}_{X,x}$ regular $\Rightarrow$ exact)

X irreducible variety, $g : \eta \to X$ generic point (it belongs to all nonempty subsets of X opens)

Prop : the sequence $(*)$ can be rewritten as

and write z as the generic point of a prime divisor Z

$i_z : z \to X$ the inclusion

$$0 \to \mathcal{O}_X^\times \to g_* k^\times \to \bigoplus_{\mathrm{codim}(Z)=1} i_{z*}\mathbb{Z} \to 0$$

The exact sequence for étale topology

Prop : For any connected normal variety (or scheme) X, there is a sequence of sheaves on X_{et}

$$0 \to \mathbb{G}_m \to g_* \mathbb{G}_{m,K} \to \bigoplus_{\mathrm{codim}\, Z=1} i_{z*}\mathbb{Z} \to 0$$

always left exact, and exact if X is regular (i.e. a nonsingular variety)

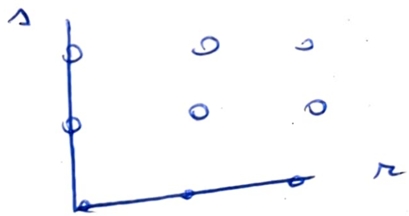
We want to show $H^r(X_{et}, g_* \mathbb{G}_{m,\eta}) = 0$ $\quad \forall r > 0$

$$H^r(X_{et}, Div_X) = 0$$

x closed point $\Rightarrow$ $H^r(X_{et}, i_{x*}\mathcal{F}) \underset{i_{x*} \text{ exact}}{=} H^r(x_{et}, \mathcal{F}) = 0$

$\Rightarrow$ $H^r(X_{et}, Div_X) = 0$ for $r > 0$

<u>FACT</u> $R^s g_* \mathbb{G}_{m,\eta} = 0$ $\forall s > 0$, from the leray spectral sequence, looking at page 2 we have



$\Rightarrow$ $s = 0$ $\quad H^r(X_{et}, g_* \mathbb{G}_m) \simeq H^r(\eta_{et}, \mathbb{G}_m)$

$\simeq H^r(Gal(k^{sep}/k), (k^{sep})^\times)$

$\eta = Spec\, k$

$= 0, \quad r \geq 1$