Department of Mathematics
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Master's Thesis

Algebraic and analytic vector bundles on the Fargues-Fontaine curve

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Chapter 1

Introduction

The goal of this project is to provide a proof for the equivalence between the categories of algebraic and analytic vector bundles on the Fargues-Fontaine curve X where X will be viewed as a scheme. The vector bundles on its analytic counterpart will only show up in the guise of φ-modules over some ring B which figures prominently in the construction of X. Towards that goal we have to introduce a small overview of the necessary auxiliary constructions (i.e. ramified Witt vectors, the algebra B, etc.) with their most important properties that we are going to need for the later proofs. This first chapter is just an overview and we mainly refer to the work of Schneider, Fargues and Fontaine [Sch17], [FF18], [FF14].

In the next chapter we introduce the two functors. The first is algebraization which effectively is passing to a graded submodule and then to the associated $\mathcal{O}_X$ module. The second one is analytization which is composition of the compatible functors pullback, global sections and inverse limit. Our main goal is to prove that they are inverse of each other. This is done in the final chapter where we also show that the functors commute with natural operations such as tensor product and internal hom's.

The main technical challenge is to work out the necessary result from [KL15] (which are used as a blackbox in [FF18]). They imply that both functors are exact and that algebraization takes values in vector bundles (and not just quasi-coherent modules as defined originally). Having those results we can finally proceed to the last part, the proof of the equivalence.

Chapter 2

Reminder on the Fargues-Fontaine curve

In this chapter of the thesis we give a brief summary of the construction of the Fargues-Fontaine curve. For this, the notion of ramified Witt vectors is of central importance.

So let us from now on, unless otherwise specified, fix the following data:

(i) $E|\mathbb{Q}_p$ a finite extension of the p-adic numbers.

(ii) $\mathcal{O} := \mathcal{O}_E$ its ring of integers and k its residue field with $|k| = q$ elements, $q = p^f$ a power of p.

(iii) A uniformizer π of $\mathcal{O}$.

2.1 Ramified Witt Vectors

For this section on the construction of ramified Witt vector, we mostly follow the book [Sch17] as our main source.

For any integer $n \geq 0$ we call

$$\Phi_n(X_0, \ldots, X_n) := X_0^{q^n} + \pi X_1^{q^{n-1}} + \ldots + \pi^n X_n$$

the n-th Witt polynomial.

Let B be an $\mathcal{O}$-algebra and

$$B^{\mathbb{N}_0} := \{(b_0, b_1, \ldots) : b_n \in B\}$$

be the countably infinite direct product of the algebra B with itself (so that addition and multiplication are componentwise). We introduce the following maps:

$$\begin{aligned} f_B : B^{\mathbb{N}_0} &\to B^{\mathbb{N}_0} \\ (b_0, b_1, \dots) &\mapsto (b_1, b_2, \dots) \end{aligned}$$

which is an endomorphism of $\mathcal{O}$-algebras

$$\begin{aligned} v_B : B^{\mathbb{N}_0} &\to B^{\mathbb{N}_0} \\ (b_0, b_1, \dots) &\mapsto (0, \pi b_0, \pi b_1, \dots), \end{aligned}$$

which respects the $\mathcal{O}$-module structure but generally neither multiplication nor the unit element,

$$\begin{aligned} \Phi_n : B^{\mathbb{N}_0} &\to B \\ (b_0, b_1, \dots) &\mapsto \Phi_n(b_0, \dots, b_n), \end{aligned}$$

for $n \geq 0$, and

$$\begin{aligned} \Phi_B : B^{\mathbb{N}_0} &\to B^{\mathbb{N}_0} \\ b &\mapsto (\Phi_0(b), \Phi_1(b), \Phi_2(b), \dots). \end{aligned}$$

Lemma 2.1.1. *(i) If $\pi 1_B$ is not a zero divisor in B then Φ_B is injective.*

(ii) If $\pi 1_B \in B^\times$ then Φ_B is bijective.

Proof. See reference [Sch17] Lemma 1.1.3. □

We will also need that there is a unique the map $\Omega : \mathcal{O} \to \mathcal{O}^{\mathbb{N}_0}$ such that

$$\Phi_{\mathcal{O}}(\Omega(\lambda)) = (\lambda, \lambda, \dots).$$

In fact the uniqueness follows from Lemma 2.1.1(i) because $\Phi_{\mathcal{O}}$ is injective (note that π is not a zero divisor in $\mathcal{O}$). The existence of Ω follows from [Sch17], Proposition 1.1.5.

Then for any $\mathcal{O}$-algebra B we use the canonical homomorphism $\mathcal{O} \to B$ to view $\Omega(\lambda)$ also as an element in $B^{\mathbb{N}_0}$. Next we consider the polynomial $\mathcal{O}$-algebra

$$A := \mathcal{O}[X_0, X_1, \dots, Y_0, Y_1, \dots]$$

in two sets of countably many variables. Then $\pi 1_A$ is not a zero divisor in A.

Let $\mathbf{X} := (X_0, X_1, \dots)$ and $\mathbf{Y} := (Y_0, Y_1, \dots)$ in $A^{\mathbb{N}_0}$. As is shown in [Sch17] page 11, there exist uniquely determined elements $S = (S_n)_n$, $P = (P_n)_n$, $I = (I_n)_n$ and $F = (F_n)_n$ in $A^{\mathbb{N}_0}$ such that:

$$\begin{aligned}
\Phi_A(S) &= \Phi_A(\mathbf{X}) + \Phi_A(\mathbf{Y}),\\
\Phi_A(P) &= \Phi_A(\mathbf{X})\Phi_A(\mathbf{Y}),\\
\Phi_A(I) &= -\Phi_A(\mathbf{X}),\\
\Phi_A(F) &= f_A(\Phi_A(\mathbf{X})),
\end{aligned}$$

or expressed coordinate-wise:

$$\begin{aligned}
\Phi_n(S_0,\dots,S_n) &= \Phi_n(X_0,\dots,X_n) + \Phi_n(Y_0,\dots,Y_n),\\
\Phi_n(P_0,\dots,P_n) &= \Phi_A(X_0,\dots,X_n)\Phi_A(Y_0,\dots,Y_n),\\
\Phi_n(I_0,\dots,I_n) &= -\Phi_n(X_0,\dots,X_n),\\
\Phi_n(F_0,\dots,F_n) &= \Phi_{n+1}(X_0,\dots,X_{n+1}),
\end{aligned}$$

for any $n \geq 0$. And it is proven that S_n, P_n, F_n, I_n are actually polynomials only in the given variables $X_0,\dots,X_n$.

Now returning to the general case let B again be an arbitrary $\mathcal{O}$-algebra. On the one hand we have the $\mathcal{O}$- algebra $(B^{\mathbb{N}_0}, +, \cdot)$ defined as a direct product. On the other hand we define on the set $W_E(B) := B^{\mathbb{N}_0}$ a new ‘addition’

$$(a_n)_n + (b_n)_n := (S_n(a_0,\dots,a_n, b_0,\dots,b_n))_n$$

and a new ‘multiplication’

$$(a_n)_n \cdot (b_n)_n := (P_n(a_0,\dots,a_n, b_0,\dots,b_n))_n.$$

Moreover, we define candidates for the neutral element of the new operations

$$\mathbf{0} := (0, 0, \dots) \quad \textit{and} \quad \mathbf{1} := (1, 0, 0, \dots).$$

Proposition 2.1.2. *(i)* $(W_E(B), +, \cdot)$ *is a (commutative) ring with zero element* $\mathbf{0}$ *and unit element* $\mathbf{1}$*; the additive inverse of* $(b_n)_n$ *is* $(I_n(b_0,\dots,b_n))_n$.

(ii) The map $\Omega : \mathcal{O} \to (W_E(B), +, \cdot)$ *is a ring homomorphism, making* $(W_E(B), +, \cdot)$ *into an* $\mathcal{O}$*-algebra.*

(iii) The map $\Phi_B : W_E(B) \to B^{\mathbb{N}_0}$ *is a homomorphism of* $\mathcal{O}$*-algebras; in particular, for any* $m \geq 0$,

$$\begin{aligned}
\Phi_m : W_E(B) &\to B\\
(b_n)_n &\mapsto \Phi_m(b_0,\dots,b_m)
\end{aligned}$$

is a homomorphism of $\mathcal{O}$-algebras.

(iv) For any $\mathcal{O}$-algebra homomorphism $\rho : B_1 \to B_2$ the map

$$W_E(\rho) : W_E(B_1) \to W_E(B_2)$$

is an $\mathcal{O}$-algebra homomorphism as well.

Proof. See reference [Sch17] Proposition 1.1.8. □

Definition 2.1.3. $(W_E(B), +, \cdot)$ *is called the ring of ramified Witt vectors with coefficients in B.*

Thus we can view $W_E(\cdot)$ as a functor from the category of $\mathcal{O}$-algebras to the category of $\mathcal{O}$-algebras where the construction is $B \mapsto B^{\mathbb{N}_0}$ with "new" operations defined with the help of the polynomials S_n, P_n, F_n, I_n and we are going to see that if we restrict our focus to "good" algebras we get "good" algebras and an explicit description of the elements in $W_E(B)$. Towards that goal we first introduce the necessary language and some important intermediate results.

On $W_E(B)$ we have the maps

$$\begin{aligned} F : W_E(B) &\to W_E(B) \\ (b_n)_n &\mapsto (F_n(b_0, \ldots, b_{n+1}))_n \end{aligned}$$

and

$$\begin{aligned} V : W_E(B) &\to W_E(B) \\ (b_n)_n &\mapsto (0, b_0, b_1, \ldots) \end{aligned}$$

which make the diagrams

$$\begin{array}{ccc} W_E(B) & \xrightarrow{\Phi_B} & B^{\mathbb{N}_0} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f_B} \\ W_E(B) & \xrightarrow{\Phi_B} & B^{\mathbb{N}_0} \end{array} \quad \text{and} \quad \begin{array}{ccc} W_E(B) & \xrightarrow{\Phi_B} & B^{\mathbb{N}_0} \\ {\scriptstyle V}\downarrow & & \downarrow{\scriptstyle v_B} \\ W_E(B) & \xrightarrow{\Phi_B} & B^{\mathbb{N}_0} \end{array}$$

commute

Proposition 2.1.4. *The following hold true:*

(i) F *is an endomorphism of the* $\mathcal{O}$*-algebra* $W_E(B)$.

(ii) V *is an endomorphism of the* $\mathcal{O}$*-module* $W_E(B)$.

(iii) $F(V(b)) = \pi b$ *for any* $b \in W_E(B)$.

(iv) $V(a \cdot F(b)) = V(a) \cdot b$ *for any* $a, b \in W_E(B)$.

(v) $F(b) \equiv b^q \mod \pi W_E(B)$ *for any* $b \in W_E(B)$.

Proof. See reference [Sch17] Proposition 1.1.10. □

Definition 2.1.5. *We call* F *and* V *the Frobenius and the Verschiebung on* $W_E(B)$*, respectively.*

For any $m \geq 0$ define

$$V_m(B) := im(V^m) = \{(b_n)_n \in W_E(B) : b_0 = \ldots = b_{m-1} = 0\}.$$

We then have

$$W_E(B) = V_0(B) \supseteq V_1(B) \supseteq \ldots \quad \textit{and} \qquad \bigcap_{m=0}^{\infty} V_m(B) = 0.$$

By Proposition 2.1.4(ii) and (iv) every $V_m(B)$ is an ideal in $W_E(B)$.

Definition 2.1.6. $W_{E,m}(B) := W_E(B)/V_m(B)$ *is called the ring of ramified Witt vectors of length* m *with coefficients in* B.

Proposition 2.1.7. *(i) the map*

$$\begin{aligned} W_E(B) &\xrightarrow{\cong} \varprojlim_m W_{E,m}(B) \\ b &\mapsto (b + V_m(B))_m \end{aligned}$$

is an isomorphism of $\mathcal{O}$*-algebras.*

(ii) The map $\Phi_0 : W_{E,1}(B) \to B$ *is an isomorphism of* $\mathcal{O}$*-algebras.*

Proof. See reference [Sch17] Lemma 1.1.13 and Exercise 1.1.14. □

Lemma 2.1.8. *The map*

$$\begin{aligned} \tau : B &\to W_E(B) \\ b &\mapsto (b, 0, \ldots) \end{aligned}$$

is multiplicative.

Proof. See reference [Sch17] Lemma 1.1.15. □

Definition 2.1.9. *We call* $\tau(b) \in W_E(B)$ *the Teichmüller representative of* $b \in B$.

If B is a k-algebra then the q-Frobenius

$$\begin{aligned} B &\to B \\ b &\mapsto b^q \end{aligned}$$

is an endomorphism of $\mathcal{O}$-algebras. If this map is bijective, we call B perfect.

Proposition 2.1.10. *For a k-algebra B we have:*

(i) Any $b = (b_n)_n \in W_E(B)$ *satisfies*

$$F(b) = (b_n^q)_n \text{ and } \pi b = F(V(b)) = V(F(b)) = (0, b_0^q, b_1^q, \ldots).$$

(ii) $V_m(B) \cdot V_n(B) \subseteq V_{m+n}(B)$ *for any* $m, n \geq 0$.

(iii) $\pi^m W_E(B) \subseteq V_1(B)^m = \pi^{m-1} V_1(B) \subseteq \pi^{m-1} W_E(B)$ *for any* $m \geq 1$.

(iv) The homomorphisms of $\mathcal{O}$-algebras

$$\begin{aligned} W_E(B) &\xrightarrow{\cong} \varprojlim_m W_E(B)/\pi^m W_E(B) \\ b &\mapsto (b + \pi^m W_E(B))_m \end{aligned}$$

and

$$\begin{aligned} W_E(B) &\xrightarrow{\cong} \varprojlim_m W_E(B)/V_1(B)^m \\ b &\mapsto (b + V_1(B)^m)_m \end{aligned}$$

are bijective.

Proof. See reference [Sch17] Proposition 1.1.18. □

We get an even better picture for the case that B is a perfect k-algebra.

Proposition 2.1.11. *If B is a perfect k-algebra then we have:*

(i) $\pi 1_{W_E(B)} \neq 0$ *is not a zero divisor in* $W_E(B)$.

(ii) For any $b = (b_n)_n \in W_E(B)$ *and* $m \geq 1$,

$$b + V_m(B) = \tau(b_0) + \pi\tau(b_1^{q^{-1}}) + \ \dots + \pi^{m-1}\tau(b_{m-1}^{q^{-(m-1)}}) + V_m(B).$$

(iii) $V_m(B) = \pi^m W_E(B) = V_1(B)^m$. *for any* $m \geq 0$.

Proof. See reference [Sch17] Proposition 1.1.19. □

Proposition 2.1.12. *Let B be a field extension of k; we then have:*

(i) $W_E(B)$ *is an integral domain with a unique maximal ideal, which is equal to* $V_1(B)$ *, and* $W_E(B)/V_1(B) \cong B$.

(ii) If B is perfect then $W_E(B)$ *is a complete discrete valuation ring with maximal ideal* $\pi W_E(B)$ *and residue class field B, and any* $b = (b_n)_n \in W_E(B)$ *has the convergent expansion*

$$b = \sum_{n=0}^{\infty} \pi^n \tau(b_n^{q^{-n}})$$

Proof. See reference [Sch17] Proposition 1.1.21. □

Remark 2.1.13. *If B is a field extension of k then the field of fractions of* $W_E(B)$ *has characteristic zero.*

Proof. See reference [Sch17] Remark 1.1.22. □

2.2 Gauss norms and the construction of the curve

Having an overview of the general construction of ramified Witt vectors we are now going to reduce to the case where the ring of coefficients F is a complete, perfect field containing k. And from there build all the necessary auxiliary constructions to finally define $X = X_{E,F}$ the Fargues-Fontaine curve associated with E, F. For this section we rely mainly on the works of Fargues-Fontaine [FF18], [FF14]. So let's fix the following additional data and change of notation:

(i) Let $(F, |\cdot|)$ be complete, non-trivially valued, non-archimedean, perfect field containing k.

(ii) The Frobenius endomorphims on $W_E(F)$ will be denoted by φ.

(iii) The Teichmüller lift will be denoted by $[x] := \tau(x)$.

Then by Prop.2.1.12 $W_E(F)$ is a complete DVR with maximal ideal $\pi W_E(F)$, and $W_E(F)[\frac{1}{\pi}] = Quot(W_E(F))$ is a complete discretely valued field with uniformizer π, valuation ring $W_E(F)$ and residue class field F. In addition any $x \in W_E(F)[\frac{1}{\pi}]$ can be written as $x = \sum_{n\in\mathbb{Z}} \pi^n[x_n]$, with uniquely determined $x_n \in F$ and $x_n = 0$, for all but finitely many $n < 0$.We indicate such an expansion by $\sum_{n\gg-\infty} \pi^n[x_n]$.

Definition 2.2.1. *Let*

$$B^b := \{x = \sum_{n\gg-\infty} \pi^n[x_n] \in W_E(F)[\frac{1}{\pi}] \mid \sup_n\{|x_n|\} < \infty\}$$

be the subalgebra of "bounded" elements of $W_E(F)[\frac{1}{\pi}]$.

Definition 2.2.2. *For* $0 < \rho \leq 1$ *we define* $|\cdot|_\rho : B^b \to \mathbb{R}_{\geq 0}$ *via*

$$|\sum_{n\gg-\infty} \pi^n[x_n]|_\rho := \sup\{\ |x_n| \cdot \rho^n \ \mid n \in \mathbb{Z}\}$$

Remark 2.2.3. B^b *is easily seen to be a localization of* $W_E(\mathcal{O}_F)$*, hence an* E*-subalgebra of* $W_E(F)[\frac{1}{\pi}]$*. The maps* $|\cdot|_\rho$ *are absolute values on* B^b*, i.e. multiplicative norms. They are usually called Gauss norms (cf. [FF18], Proposition 1.4.3 and Proposition 1.4.9).*

Definition 2.2.4. *For* $x = \sum_{n\gg-\infty} \pi^n[x_n] \in W_E(F)[\frac{1}{\pi}]$ *we set*

$$|x|_0 := \begin{cases} 0 & ,\ for\ \ x = 0 \\ q^{-\min\{n\in\mathbb{Z}|x_n\neq 0\}} & ,\ for\ \ x \neq 0 \end{cases}$$

Definition 2.2.5. *For any non-empty interval* $I \subseteq [0, 1]$ *we define the map* $|\cdot|_I : B^b \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ *by* $|x|_I := \sup\{|x|_\rho \mid \rho \in I\}$.

Lemma 2.2.6. *(i) If* $I = [\sigma, \rho]$ *is compact interval with* $\sigma \neq 0$ *then* $|x|_I = \max\{|x|_\sigma, |x|_\rho\} < \infty$ *for all* $x \in B^b$*.* $|\cdot|_I$ *is a norm on the ring* B^b*, in general only submultiplicative.*

(ii) If $J \subseteq J'$ *then* $|\cdot|_J \leq |\cdot|_{J'}$*.*

Proof. See reference [FF14] page 2. □

Definition 2.2.7. *For any non-empty compact interval* $I \subseteq (0, 1]$ *we denote by* B_I *the completion of* B^b *with respect to* $|\cdot|_I$*.*

Remark 2.2.8. *(i) For any pair of compact intervals $J \subseteq I \subseteq (0,1]$ the map $id_{B^b} : (B^b, |\cdot|_I) \mapsto (B^b, |\cdot|_J)$ is continuous. By the universal property of completions it extends to a continuous homomorphism $\iota_{J,I} : B_I \to B_J$ which makes the following diagram commutative*

$$\begin{tikzcd} & B_J \\ B^b \arrow[ur, "can"] \arrow[r, "can"'] & B_I \arrow[u, "\iota_{J,I}"'] \end{tikzcd}$$

In particular, it is E-linear and uniqueness implies

$$\iota_{K,I} = \iota_{K,J} \circ \iota_{J,I} \quad \textit{for} \quad K \subseteq J \subseteq I.$$

(ii) From Lemma 2.2.6 the map $id_{B^b} : (B^b, |\cdot|_I) \mapsto (B^b, |\cdot|_J)$ is norm decreasing. This continues to be so after completion,
i.e. $\iota_{J,I} : B_I \to B_J$ is also norm decreasing.

Having established that $|\cdot|_I$ is a norm on the ring B^b and the relations between the completions B_I, B_J for compact intervals $J \subseteq I \subseteq (0,1)$. It can be proven that $\iota_{J,I}$ is injective and thus the construction of the projective limit $\varprojlim_{\emptyset \neq I \subseteq (0,1)} B_I$ can be viewed as corresponding to a system ordered by inclusion, satisfying

$$J \subseteq I \Rightarrow B_I \subseteq B_J$$

Definition 2.2.9. *Letting I run through the compact intervals in $(0,1)$, we set*

$$B := \varprojlim_{\emptyset \neq I \subseteq (0,1)} B_I = \{(b_I)_I \in \prod_I B_I \mid \forall J \subseteq I \ : \ b_J = b_I\}.$$

Lemma 2.2.10. *For any J the projection $B \to B_J$, $b \mapsto b_J$ is injective with image $\bigcap_{J \subseteq J'} B_{J'}$ as a subring of B_J. In particular B contains B^b.*

Proof. This is a direct consequence of the injectivity of $\iota_{J,I}$. $\square$

Remark 2.2.11. *We thus can view B as a subring of B_J containing B^b, for any $\emptyset \neq J \subseteq (0,1)$ compact interval. This includes the degenerate case $J = \{\rho\}$ with $\rho \in (0,1)$.*

Returning to the Witt vector construction $W_E(F)[\frac{1}{\pi}]$ we are interested in the behaviour of the Frobenius homomorphism φ. First of all we fix the following notation. For $I \subseteq [0, 1]$

$$\varphi(I) := \{r^q \mid r \in I\}.$$

For example if $I = [\sigma, \rho]$ is a compact interval then $\varphi(I) = [\sigma^q, \rho^q]$. Recall from section 2.1 that Frobenius on $W_E(F)[\frac{1}{\pi}]$ is given explicitly by

$$\varphi(\sum_{n \gg -\infty} \pi^n [x_n]) = \sum_{n \gg -\infty} \pi^n [x_n^q]$$

It restricts to an E-linear automorphism of B^b satisfying

$$\begin{aligned} |\varphi(\sum_{n \gg -\infty} \pi^n [x_n])|_\rho &= |\sum_{n \gg -\infty} \pi^n [x_n^q]|_\rho = \sup_n \{\rho^n \cdot |x_n|^q\} \\ &= \sup_n \{(\rho^{1/q})^n \cdot |x_n|\}^q = |\sum_{n \gg -\infty} \pi^n [x_n]|_{\rho^{1/q}}^q \end{aligned}$$

for any $0 < \rho \leq 1$. Consequently for any non-empty compact interval $I \subseteq (0.1)$

$$\varphi \ : \ (B^b, |\cdot|_I^q) \to (B^b, |\cdot|_{\varphi(I)})$$

is an isometry. By the universal property of completion it extends to an isometry

$$\varphi \ : \ (B_I, |\cdot|_I^q) \to (B_{\varphi(I)}, |\cdot|_{\varphi(I)}).$$

Note that B_I is also the completion with respect to $|\cdot|_I^q$ because it is a power of $|\cdot|_I$ and hence defines the same topology.

These automorphisms fit together to an E-linear automorphism of $B := \varprojlim_I B_I$ given by

$$(b_I)_I \ \mapsto \ (\varphi(b_I))_I$$

Definition 2.2.12. *For $n \geq 0$ the we set*

$$B^{\varphi = \pi^n} \ := \ \{b \in B \mid \varphi(b) = \pi^n b\}.$$

Note that φ is E-linear and hence $B^{\varphi=\pi^n}$ is an E-subspace of B.

Lemma 2.2.13. *The map*

$$\begin{aligned} \bigoplus_{n\geq 0} B^{\varphi=\pi^n} &\to B \\ (b_n)_n &\mapsto \sum_n b_n \end{aligned}$$

is injective and $\bigoplus_{n\geq 0} B^{\varphi=\pi^n}$ *is a graded* E*-subalgebra of* B

Proof. This is basic linear algebra. See also reference [FF18] Définition 6.1.1, Proposition 4.1.3. □

Having introduced all the necessary auxiliary constructions we are ready to give a definition of the Fargues-Fontaine curve.

Definition 2.2.14. *The* E*-scheme*

$$X \ := \ X_{E,F} \ := \ Proj(\bigoplus_{n\geq 0} B^{\varphi=\pi^n})$$

is called the Fargues-Fontaine curve associated with E, F.

We now present the main properties of the curve and the rings B, B_I.

Theorem 2.2.15. *(i) We have*

$$B^\times = (B^b)^\times$$

$$= \{\sum_{n=N}^{\infty} \pi^n [x_n] \mid N \in \mathbb{Z} \ , \ x_N \neq 0 \ , \ \forall n \geq N : |x_n| \leq |x_N|\}.$$

(ii) We have

$$B^{\varphi=1} = E.$$

(iii) For all $n < 0$ *we have*

$$B^{\varphi=\pi^n} = 0.$$

Proof. See reference [FF18] Corollaire 1.9.5 , Proposition 4.1.1 , Proposition 4.1.2. □

Theorem 2.2.16. *Let* $I \subseteq (0, 1)$ *be a compact interval.*

(i) If F *is algebraically closed and* $f \in B_I \setminus \{0\}$ *then there are* $a_1, \ldots, a_d \in F$ *and* $u \in B_I^\times$ *such that* $f = u \cdot \prod_{i=1}^{d} (\pi - [a_i])$.

(ii) B_I *is a PID. If* F *is algebraically closed with* $log_q I \cap |F| = \emptyset$ *then* B_I *is even a field.*

Proof. See reference [FF18] Théorème 2.4.10, 2.5.1 and 3.5.1. □

Theorem 2.2.17. *Assume that* F *is algebraically closed. Then for all* $n \geq 1$ *and* $f \in B^{\varphi=\pi^n} \setminus \{0\}$ *there are*

$$t_1, \dots, t_n \in B^{\varphi=\pi} \setminus \{0\} \text{ such that } f = t_1 \cdot \ldots \cdot t_n.$$

Such a factorization is unique up to multiplication with a unit $u \in E^\times$ *or permutation of the terms* t_i. *Thus,*

$$P := \bigoplus_{n \geq 0} B^{\varphi=\pi^n}$$

a gradded factorial ring.

Proof. See reference [FF18] Théorème 6.2.1 □

Theorem 2.2.18. *For* P *as defined above and* $t \in B^{\varphi=\pi} \setminus \{0\}$. *We have*

(i) *If* F *is algebraically closed then*

$$V_+(t) = Proj(P/tP) \subseteq X = Proj(P)$$

consists of a single point, namely the homogeneous prime ideal $tP \in Proj(P)$.

(ii) *Let us denote the homogeneous localization of* P *at* $\{t^n \mid n \geq 0\}$ *by* $P_{(t)}$. *Then*

$$\mathcal{O}_X(D_+(t)) = P_{(t)}$$

is a Dedekind ring which is not a field. If F *is algebraically closed then it is a PID.*

Proof. See reference [FF18] Théorème 6.5.2(3) and (7), as well as Proposition 7.2.1. □

Theorem 2.2.19. (i) *If* F *is algebraically closed, then* X *is the union* $X = X_1 \cup X_2$ *of two affine open subschemes with* $X_1 \cap X_2 \neq \emptyset$ *and such that for* $i \in \{1, 2\}$ *the ring* $\mathcal{O}_X(X_i)$ *is a PID that is not a field. In fact, we can choose*

$$X_1 = D_+(t) \ , \ X_2 = D_+(s) \text{ for any}$$
$$t \in P_1 \setminus \{0\} \ , \ s \in P_1 \setminus tE$$

(ii) The E-scheme X is :

(a) separated

(b) 1-dimensional

(c) quasi-compact

(d) noetherian

(e) irreducible

(f) regular and hence normal

(iii)

$$H^0(X, \mathcal{O}_X) = E = P_0$$

(iv) If F is algebraically closed and if $|X|$ denotes the set of closed points of X then there is a bijection

$$(B^{\varphi=\pi} \setminus \{0\})/E^\times \to |X|$$
$$tE^\times \mapsto tP \in Proj(P) = X$$

Proof. See reference [FF18] Théorème 6.5.2 and 7.3.3. □

And since we are going to need it later, we now provide a few, somewhat technical, results.

Lemma 2.2.20. *For a finite collection of compact intervals $I, I_1, I_2, \ldots, I_n \subseteq (0,1)$ with $I = \bigcup_{k=1}^n I_k$, the morphism*

$$\coprod_{k=1}^n Spec(B_{I_k}) \to Spec(B_I)$$

is an fpqc covering, i.e. the homomorphism

$$B_I \to \prod_{k=1}^n B_{I_k}$$

of E-algebras is faithfully flat.

Proof. See reference [FF14] lemma 7.15 □

We let $|Y| \subseteq Max(B)$ be the set of ideals generated by a single primitive irreducible element as in [FF14], Def. 2.1 and 2.2. Recall from [FF14] page 11 that there is a norm function $\|\cdot\| : |Y| \to (0,1)$. The space $|Y|$ lies at the heart of the theory of divisors for X. Later on we shall need the following two results where we set

$$|Y|_I = \{y \in |Y| \mid \|y\| \in I\}$$

for any interval $I \subseteq (0,1)$.

Theorem 2.2.21. *For a compact non-empty interval $I \subseteq (0,1)$, if $I = \{\rho\}$ with $\rho \notin |F^\times|$ then B_I is a field, if not then the ring B_I is a principal ideal domain with maximal ideals*

$$\{B_I\mathfrak{m} \mid \mathfrak{m} \in |Y|_I\}.$$

Proof. See reference [FF14] Theorem 3.9 □

Theorem 2.2.22. *For the scheme X there is a bijection*

$$|Y|/\varphi^{\mathbb{Z}} \xrightarrow{\sim} |X|$$

where $|X|$ is the set of closed points of the curve.

Proof. See reference [FF14] Theorem 5.5 □

Chapter 3

φ-Modules and construction of the functors

In this chapter of the thesis we introduce the two relevant categories and state the theorem that they are equivalent. We also construct the two functors that give the equivalence. The actual proof of the theorem is going to be presented in the next chapter.

3.1 φ-Modules

Definition 3.1.1. *A pair (M, φ_M) is called a φ-module over B, if M is a finitely generated projective module over B and $\varphi_M : M \to M$ is a bijective map which is φ-semilinear, meaning that it satisfies:*

(i) for all $m, n \in M$

$$\varphi_M(m+n) = \varphi_M(m) + \varphi_M(n)$$

(ii) for all $m \in M, b \in B$

$$\varphi_M(bm) = \varphi(b) \cdot \varphi_M(m)$$

Definition 3.1.2. *A homomorphism $F : (M, \varphi_M) \to (N, \varphi_N)$ of φ-modules over B is a B-linear map $F : M, \to N$ which commutes with φ_M and φ_N. i.e. the following diagram is commutative*

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ {\scriptstyle\varphi_M}\downarrow & & \downarrow{\scriptstyle\varphi_N} \\ M & \xrightarrow{F} & N \end{array}$$

We denote by φ-Mod_B the category of φ-modules over B and φ-$\mathrm{Hom}_B(M, N)$ the set of φ-module homomorphisms from M to N.

Example 3.1.3. *Let* $\lambda \in \mathbb{Q}$, $\lambda = \frac{d}{h}$ *with* $h, d \in \mathbb{Z},\ h \geq 1$, $gcd(h, d) = 1$ *and set* $M := B^h$. *Let* φ_M *be the unique* φ*-semilinear map which on the standard basis satisfies*

$$\varphi_M(e_i) := \begin{cases} e_{i+1}, & \text{if } 1 \leq i < h \\ \pi^d \cdot e_1, & \text{if } i = h \end{cases}$$

We denote the corresponding object of φ*-*Mod_B *by* $B(\lambda)$.

3.2 Algebraization

We now define the first of the two functors. It is a functor from the category of φ-modules to the category of quasi-coherent $\mathcal{O}_X$-modules given by the composition of two functors:

$$\begin{aligned} \varphi\text{-}Mod_B \ni (M, \varphi_M) \ &\mapsto\ M^{alg} := \bigoplus_{n \geq 0} M^{\varphi_M = \pi^n} \text{ a graded } P\text{-module} \\ &\mapsto \mathcal{F}_M := \widetilde{M^{alg}} \text{ a quasi-coherent } \mathcal{O}_X\text{-module} \end{aligned}$$

Recall that $P = \bigoplus_{n \geq 0} B^{\varphi = \pi^n}$ and $X = Proj(P)$. Note that for $F : M \to N$ a φ-module morphism and for any homogeneous element $e \in M^{\varphi_M = \pi^n}$ we have

$$\varphi_N(F(e)) = F(\varphi_M(f)) = F(\pi^n e) = \pi^n F(e).$$

Thus, for any $n \geq 0$, F restricts to $M^{\varphi_M = \pi^n} \to N^{\varphi_N = \pi^n}$ and induces a morphism of graded P-modules

$$\bigoplus_{n \geq 0} M^{\varphi_M = \pi^n} \to \bigoplus_{n \geq 0} N^{\varphi_N = \pi^n}$$

By $\widetilde{(\cdot)}$ we denote the usual functor from algebraic geometry.

It will later be proven that actually $\mathcal{F}_M$ is a vector bundle (See Cor. 4.2.4).

Remark 3.2.1. *There is an analytic version of the Fargues-Fontaine curve which is an adic space in the sense of [FAR18] chapter 1. It is obtained from an open subspace of* $Spa(W_E(\mathcal{O}_F), W_E(\mathcal{O}_F))$ *by modding out the action of the Frobenius. One can show that the category of of the vector bundles on the adic curve is equivalent to* φ*-*Mod_B *(c.f. [KL15] Theorem 8.2.22). We work with this algebraic description directly in order to avoid the language of adic spaces.*

3.3 Analytization

For the inverse direction we first need some intermediate steps. As usual we set $P_+ := \bigoplus_{n>0} B^{\varphi=\pi^n}$

Lemma 3.3.1. *For any compact interval $I = [\sigma, \rho] \subseteq (0,1)$ with $\sigma \neq \rho$ we have $B^{\varphi=\pi} \cap B_I^\times \neq \emptyset$. In particular P_+ generates the unit ideal in B_I.*

Proof. Otherwise P_+ generates a proper non-empty ideal J of B_I, which in turn must be contained in a maximal ideal $J \subseteq \mathfrak{m}$.

By Thm. 2.2.21 there is an element $f \in B_I$ such that $\mathfrak{m} = (f)$ and hence $div(f)$ is a single point of $|Y|_I$.

Choose an element $g \in B_I$ such that the support of $div(g) \in |Y|_I$ does not meet the φ-orbit of $div(f)$. This is possible because I is not a singleton. Then from Thm.2.2.22 and Thm.2.219 (iv) there is homogeneous element $t \in P_+ \subseteq J \subseteq \mathfrak{m}$ such that $div(t) = div(g)$. But $t = fh$ for some $h \in B_I$. Implies that after restricting to $|Y|_I$ the φ-orbit of $div(g) = div(t) = div(f) + div(h)$ meets $div(f)$, giving us a contradiction. $\square$

Thus taking a homogeneous element of positive degree $f \in P_+$, we have the following chain of inclusions $P_{(f)} \subseteq P_f \subseteq B_f \subseteq B_{I,f}$, which give rise to a morphism of schemes

$$Spec(B_I) \supseteq D(f) \to D_+(f) \subseteq X$$

Using the last lemma we can glue these together to a morphism of E-schemes

$$g_I : \ Spec(B_I) = \bigcup_f D(f) \to \bigcup_f D_+(f) = X.$$

Moreover if $J \subseteq I$ then $B_I \subseteq B_J$ gives us a commutative diagram

$$\begin{tikzcd} Spec(B_J) \arrow[r, "g_J"] \arrow[d, "g_{I,J}"'] & X \\ Spec(B_I) \arrow[ur, "g_I"'] & \end{tikzcd}$$

Now let $\mathcal{F} \in Fib_X$ be a vector bundle on X and denote

$$M_I := \Gamma(Spec(B_I), g_I^* \mathcal{F})$$

the global sections of the pullback of $\mathcal{F}$ to $Spec(B_I)$. Since $Spec(B_I)$ is an affine scheme M_I is a projective finitely generated module such that $\widetilde{M_I} = g_I^*\mathcal{F}$

Remark 3.3.2. *By Theorem 2.2.16 the ring B_I is a PID. Thus M_I is actually free and finitely generated module over B_I*

For $J \subseteq I$ the commutativity of the above diagram gives

$$g_J^*\mathcal{F} = g_{I,J}^* g_I^*\mathcal{F} \Rightarrow M_J \cong B_J \otimes_{B_I} M_I$$

and thus we can define a B_I-linear map

$$M_I \to M_J \cong B_J \otimes_{B_I} M_I \ : \ m \mapsto 1 \otimes m. \tag{$*$}$$

This gives us a projective system of E-vector spaces indexed by the set of non-empty, compact subintervals of $(0,1)$. We denote its projective limit by

$$M_{\mathcal{F}} := \varprojlim_I M_I$$

This construction defines a functor $Fib_X \to Mod_B$. In order to get the desired functor we also need a semilinear bijection $\varphi_{M_{\mathcal{F}}}$. Towards that goal we turn to the Frobenius map $\varphi : B^b \to B^b$ and recall that it extends to an isomorphism $\varphi_I : B_I \to B_{\varphi(I)}$

Lemma 3.3.3. *The following diagram is commutative*

$$\begin{tikzcd} Spec(B_{\varphi(I)}) \arrow[r, "g_{\varphi(I)}"] \arrow[d, "Spec(\varphi_I)"'] & X \\ Spec(B_I) \arrow[ur, "g_I"'] & \end{tikzcd}$$

with $Spec(\varphi_I)$ an isomorphism.

Proof. We are going to prove the commutativity by viewing it locally. So let $n > 0$ and $f \in B^{\varphi=\pi^n}$. Then the isomorphism $\varphi_I : B_I \to B_{\varphi(I)}$ induces an isomomorphism of localizations

$$\varphi_{I,f} \ : \ B_{I,f} \ \to \ B_{\varphi(I),\varphi(f)} = B_{\varphi(I),\pi^n f} = B_{\varphi(I),f}$$

restricting to

$$P_{(f)} \to P_{(\varphi(f))} = P_{(\pi^n f)} = P_{(f)}$$
$$\frac{b}{f^m} \mapsto \frac{\varphi(b)}{\varphi(f^m)}$$

for b a homogeneous element of degree nm, i.e. $f \in B^{\varphi=\pi^n}, b \in B^{\varphi=\pi^{nm}}$. Calculating explicitly we find

$$\frac{\varphi(b)}{\varphi(f^m)} = \frac{\pi^{nm}b}{\pi^{nm}f^m} = \frac{b}{f^m}$$

and thus

$$\varphi_{I,f}\big|_{P_{(f)}} = id_{P_{(f)}}$$

□

Thus for any I we get an isomorphism of quasi-coherent modules

$$g^*_{\varphi(I)}\mathcal{F} \cong Spec(\varphi_I)^* g^*_I \mathcal{F}$$

on $Spec(B_{\varphi(I)})$. On global sections we get a $B_{\varphi(I)}$-linear isomorphism

$$M_{\varphi(I)} \cong B_{\varphi(I)} \otimes_{B_I} M_I$$

Remark 3.3.4. *Here $B_{\varphi(I)}$ is seen as a right B_I-module using the map φ_I, i.e. for all $b' \in B_{\varphi(I)},\ b \in B_I\ :\ b' \cdot b = b'\varphi(b)$*

The above construction is compatible with the inclusion of intervals, meaning that for $J \subseteq I$ the following diagram is commutative

$$\begin{array}{ccc} M_{\varphi(I)} & \xrightarrow{\cong} & B_{\varphi(I)} \otimes_{B_I} M_I \\ \downarrow & & \downarrow \\ M_{\varphi(J)} & \xrightarrow{\cong} & B_{\varphi(J)} \otimes_{B_J} M_J \end{array}$$

Here the vertical maps are the maps constructed before (see $(*)$ after Remark 3.3.2.)

Putting everything together we now have a canonical candidate for a φ_I-semilinear bijection φ_{M_I}

$$\begin{aligned} \varphi_{M_I}\ :\ M_I\ &\to\ B_{\varphi(I)} \otimes_{B_I} M_I \cong M_{\varphi(I)} \\ m\ &\mapsto\ 1 \otimes m \end{aligned}$$

Lemma 3.3.5. *φ_{M_I} is a φ_I-semilinear bijection.*

Proof. • **Bijectivity**
In order to prove that φ_{M_I} is bijective we simply present its inverse

$$\begin{aligned} \varphi^{-1}_{M_I}\ :\ B_{\varphi(I)} \otimes_{B_I} M_I\ &\to\ M_I \\ \sum_i b_i \otimes m_i\ &\mapsto\ \sum_i \varphi_I^{-1}(b_i) m_i \end{aligned}$$

• **φ_I-semilinearity**
Additivity is inhereted from the bi-additivity of the tensor product. Using Remark 3.3.4 we calculate

$$\varphi_{M_I}(bm) = 1 \otimes bm = \varphi_I(b) \otimes m = \varphi_I(b)\varphi_{M_I}(m)$$

for all $b \in B_I,\ m \in M_I$. □

Once again, this construction is compatible with the inclusion of compact subintervals $J \subseteq I$. Thus taking projective limit one has

$$\varphi_{M_\mathcal{F}} := \varprojlim_I \varphi_{M_I} \ :\ M_\mathcal{F} := \varprojlim_I M_I \ \rightarrow\ \varprojlim_I M_{\varphi(I)} = M_\mathcal{F}$$

which is the sought after φ-semilinear bijection on $M_\mathcal{F}$.

It remains to be seen that $(M_\mathcal{F}, \varphi_{M_\mathcal{F}})$ is indeed a φ-module (i.e. M is finitely generated and projective over B), and that the described procedure is functorial (i.e. given a morphism of vector bundles it naturally gives a morphism of φ-modules). For the former we simply quote [FF14] Proposition 7.14. And for the latter we observe that the functor $\mathcal{F} \ \mapsto\ M_\mathcal{F}$ is the composition of the 3 functors, pullback, global section and inverse limit.

Theorem 3.3.6. *The functors $(M \mapsto \mathcal{F}_M)\ :\ \varphi\text{-}Mod_B \to Fib_X$ and $(\mathcal{F} \mapsto M_\mathcal{F})\ :\ Fib_X \to \varphi\text{-}Mod_B$ are well defined inverse equivalences of categories.*

Proof. See the calculations in the end of section 4.2 □

Since we will need it in the next chapter we present a useful property of $M_\mathcal{F}$ which relies on a topological variant of the Mittag-Leffler condition.

Proposition 3.3.7. *The canonical map*

$$M_\mathcal{F} \otimes_B B_I \to M_{\mathcal{F},I}$$

is a bijection.

Proof. See reference [ST03] Cor.3.1 □

Chapter 4

Proof of the equivalence of categories

After presenting all necessary concepts and notations, we are finally ready to begin the proof of the main result of the thesis, the equivalence of categories between φ-modules over B and vector bundles on X.

4.1 Exactness of the functors

One intermediate result that we need is the fact that algebraization actually is a functor to the category of vector bundles. Towards that goal we first need to prove that both functors are actually exact. We start with a technical lemma

Lemma 4.1.1. *Let $[\sigma, \rho] \subseteq (0, 1)$ be a compact interval, $0 < c < 1$ and $x \in B_{[\sigma,\rho]}$. Then there are $y \in B_{[\sigma^q,\rho]}$, $z \in B_{[\sigma,\rho^{1/q}]}$ with*

(i) $x = y + z$

(ii) $|x|_{[\sigma,\rho]} = max\{|y|_{[\sigma,\rho]}, |z|_{[\sigma,\rho]}\}$

(iii) $|\varphi^{-1}(y)|_{[\sigma,\rho]} \leq c^{\frac{1-q}{q}} \cdot |x|_{[\sigma,\rho]}$

(iv) $|\varphi(z)|_{[\sigma,\rho]} \leq c^{q-1} \cdot |x|_{[\sigma,\rho]}$

Here we see $B_{[\sigma^q,\rho]}$, $B_{[\sigma,\rho^{1/q}]} \subseteq B_{[\sigma,\rho]}$ as subrings. Thus the addition makes sense. Furthermore $\varphi^{-1}(B_{[\sigma^q,\rho]}) = B_{[\sigma,\rho^{1/q}]} \subseteq B_{[\sigma,\rho]}$ and $\varphi(B_{[\sigma,\rho^{1/q}]}) = B_{[\sigma^q,\rho]} \subseteq B_{[\sigma,\rho]}$, meaning that

$$\varphi^{-1}(y), \varphi(z) \in B_{[\sigma,\rho]}$$

Proof. From the density of $B^b \subseteq B_{[\sigma,\rho]}$ we can find a sequence $x^{(m)} \in B^b$ with $|x^{(m)}|_{[\sigma,\rho]} \leq |x|_{[\sigma,\rho]}$ and

$$\sum_{m=0}^{\infty} x^{(m)} = x$$

in $B_{[\sigma,\rho]}$. Then writing $x^{(m)}$ in its π-adic expansion

$$x^{(m)} = \sum_{n \gg -\infty} \pi^n [x_n^{(m)}]$$

we can define

$$x_+^{(m)} := \sum_{n \gg -\infty, |x_n^{(m)}| \geq c} \pi^n [x_n^{(m)}] \text{ and } x_-^{(m)} := \sum_{n \gg -\infty, |x_n^{(m)}| < c} \pi^n [x_n^{(m)}]$$

where $x_+^{(m)}, x_-^{(m)} \in B^b$ for any $m \geq 0$. We first show that

$$y := \sum_{m=0}^{\infty} x_+^{(m)}$$

converges in $B_{[\sigma^q,\rho]}$.

So let $0 < \varepsilon \leq 1$. Then there is $M \geq 0$ such that

$$\text{for any } \ m \geq M \ : \ |x^{(m)}|_\rho \leq \varepsilon$$

For $m \geq M$ and $|x_n^{(m)}| \geq c$ we calculate

$$\rho^n \cdot c \leq \rho^n \cdot |x_n^{(m)}| \leq |x^{(m)}|_\rho \leq \varepsilon \leq 1.$$

This gives us a lower bound $n \geq n_0 := \lceil -log(c)/log(\rho) \rceil$ for n. Thus, we can write

$$x_+^{(m)} = \sum_{n \geq n_0, |x_n^{(m)}| \geq c} \pi^n [x_n^{(m)}]$$

and calculate

$$\begin{aligned} |x_+^{(m)}|_{\sigma^q} &\leq \sup_{n \geq n_0} \{\sigma^{qn} |x_n^{(m)}|\} = \sup_{n \geq n_0} \left\{ \left(\frac{\sigma^q}{\rho}\right)^n \rho^n |x_n^{(m)}| \right\} \\ &\leq \left(\frac{\sigma^q}{\rho}\right)^{n_0} \cdot |x^{(m)}|_\rho \ ; \quad \text{since } \sigma^q \leq \rho \\ &\leq \left(\frac{\sigma^q}{\rho}\right)^{n_0} \cdot \varepsilon. \end{aligned}$$

This means

$$|x_+^{(m)}|_{[\sigma^q,\rho]} = \max\{|x_+^{(m)}|_{\sigma^q}, |x_+^{(m)}|_\rho\} \leq \max\left\{\left(\frac{\sigma^q}{\rho}\right)^{n_0}, 1\right\} \cdot \varepsilon$$

This proves the claim that y converges in $B_{[\sigma^q,\rho]}$. Analogously we are going to prove that

$$z := \sum_{m=0}^{\infty} x_-^{(m)}$$

converges in $B_{[\sigma,\rho^{1/q}]}$. For $\varepsilon > 0$ then there is $M \geq 0$ such that

$$\text{for any } \ m \geq M \ : \ |x^{(m)}|_{[\sigma,\rho]} \leq \varepsilon$$

and for $m \geq M$ we calculate

$$\begin{aligned} |x_-^{(m)}|_{\rho^{1/q}} &= \sup_{|x_n^{(m)}|<c} \{\rho^{n/q} \ |x_n^{(m)}|\} = \sup_{|x_n^{(m)}|<c} \{\rho^n \ |x_n^{(m)}|^q\}^{1/q} \\ &< \sup_{|x_n^{(m)}|<c} \{\rho^n \ c^{q-1} \ |x_n^{(m)}|\}^{1/q} \leq c^{(q-1)/q} \cdot |x_-^{(m)}|_\rho^{1/q} \leq c^{(q-1)/q} \cdot \varepsilon^{1/q} \end{aligned}$$

where the strict inequality is derived from

$$|x_n^{(m)}|^q = |x_n^{(m)}|^{q-1} \cdot |x_n^{(m)}| < c^{q-1} \cdot |x_n^{(m)}|.$$

This means

$$|x_-^{(m)}|_{[\sigma,\rho^{1/q}]} = \max\{|x_-^{(m)}|_\sigma, |x_-^{(m)}|_{\rho^{1/q}}\} \leq \max\{\varepsilon, c^{(q-1)/q} \cdot \varepsilon^{1/q}\},$$

proving the claim that z converges in $B_{[\sigma,\rho^{1/q}]}$. From the construction we have $x = y + z$ in $B_{[\sigma,\rho]}$ proving $i)$.

For $ii)$ we explicitly calculate

$$\begin{aligned} &\forall m \geq 0 \ : \ |x_\pm^{(m)}|_{[\sigma,\rho]} \leq |x^{(m)}|_{[\sigma,\rho]} \leq |x|_{[\sigma,\rho]} \\ &\Rightarrow \quad |y|_{[\sigma,\rho]} \leq \sup_{m\geq 0}\{|x_+^{(m)}|_{[\sigma,\rho]}\} \leq |x|_{[\sigma,\rho]} \quad \text{and} \\ &|z|_{[\sigma,\rho]} \leq \sup_{m\geq 0}\{|x_-^{(m)}|_{[\sigma,\rho]}\} \leq |x|_{[\sigma,\rho]} \quad \text{respectively} \\ &\Rightarrow \ \max\{|y|_{[\sigma,\rho]}, |z|_{[\sigma,\rho]}\} \leq |x|_{[\sigma,\rho]} \end{aligned}$$

and using

$$|x|_{[\sigma,\rho]} = |y + z|_{[\sigma,\rho]} \leq \max\{|y|_{[\sigma,\rho]}, |z|_{[\sigma,\rho]}\}$$

we get the reverse inequality proving the equality and thus $ii)$.

For iii) and iv) we directly calculate for $\tau \in [\sigma, \rho]$

$$|\varphi^{-1}(x_+^{(m)})|_\tau = \sup_{|x_n^{(m)}|\geq c} \{\tau^n |x_n^{(m)}|^{1/q}\}$$
$$= \sup_{|x_n^{(m)}|\geq c} \{\tau^n \cdot |x_n^{(m)}| \cdot |x_n^{(m)}|^{\frac{1-q}{q}}\} \leq c^{\frac{1-q}{q}} |x_+^{(m)}|_\tau$$

where the inequality is because $q > 1 \iff 1 - q < 0 \Rightarrow \frac{1-q}{q} < 0$ this implies that after taking supremum over c

$$|\varphi^{-1}(x_+^{(m)})|_{[\sigma,\rho]} \leq c^{\frac{1-q}{q}} |x_+^{(m)}|_{[\sigma,\rho]} \leq c^{\frac{1-q}{q}} |x|_{[\sigma,\rho]}$$

and thus

$$|\varphi^{-1}(y)|_{[\sigma,\rho]} \leq \sup_{m\geq 0} |\varphi^{-1}(x_+^{(m)})|_{[\sigma,\rho]} \leq c^{\frac{1-q}{q}} |x|_{[\sigma,\rho]}$$

Analogously we calculate for $\tau \in [\sigma, \rho]$

$$|\varphi(x_-^{(m)})|_\tau = \sup_{|x_n^{(m)}|<c} \{\tau^n |x_n^{(m)}|^q\}$$
$$\leq \sup_{|x_n^{(m)}|<c} \{\tau^n \cdot |x_n^{(m)}|\} c^{q-1} = c^{q-1} |x_-^{(m)}|_\tau$$

where once again the inequality is derived from

$$|x_n^{(m)}|^q = |x_n^{(m)}|^{q-1} \cdot |x_n^{(m)}| \leq c^{q-1} \cdot |x_n^{(m)}|.$$

After taking supremum over τ we get

$$|\varphi(x_-^{(m)})|_{[\sigma,\rho]} \leq c^{q-1} |x_-^{(m)}|_{[\sigma,\rho]} \leq c^{q-1} |x|_{[\sigma,\rho]}$$

and thus

$$|\varphi(z)|_{[\sigma,\rho]} \leq \sup_{m\geq 0} |\varphi(x_-^{(m)})|_{[\sigma,\rho]} \leq c^{q-1} |x|_{[\sigma,\rho]}$$

$\square$

Now let $M \in \varphi\text{-}Mod_B$. Then for any compact interval $I \subseteq (0, 1)$ we set

$$M_I := M \otimes_B B_I$$

If $J \subseteq I$ is a compact subinterval then $B_I \subseteq B_J$ and we get the B_I-linear map

$$id_M \otimes \iota \ : \ M_I = M \otimes_B B_I \to M \otimes_B B_J = M_J$$

where ι is the inclusion map. Note that $id_M \otimes \iota$ is again injective because M is projective and thus flat. We just write $M_I \subseteq M_J$.

Lemma 4.1.2. *Let $I, J \subseteq (0,1)$ be compact intervals with $I \cap J \neq \emptyset$, then the sequence:*

$$\begin{aligned} 0 \to B_{I\cup J} \to B_I \oplus B_J &\to B_{I\cap J} \to 0 \\ b \mapsto (b,b) & \\ &(b_1, b_2) \mapsto b_1 - b_2 \end{aligned}$$

of B-modules is exact. i.e. when seen as subrings of $B_{I\cap J}$ we have $B_I + B_J = B_{I\cap J}$ and $B_I \cap B_J = B_{I\cup J}$.

Proof. See reference [FF18] Prop.11.2.7 □

Remark 4.1.3. *Since any $M \in \varphi\text{-}Mod_B$ is projective, hence flat over B, Lemma 4.1.2 implies that we get an exact sequence of B-modules:*

$$0 \to M_{I\cup J} \to M_I \oplus M_J \to M_{I\cap J} \to 0$$

i.e. under the natural inclusions into $M_{I\cap J}$ we have

$$M_I + M_J = M_{I\cap J} \quad \textit{and} \quad M_I \cap M_J = M_{I\cup J}$$

Furthermore, we have the bijective map

$$\varphi \ : \ M_I = M \otimes_B B_I \xrightarrow[\cong]{\varphi_M \otimes \varphi} M \otimes_B B_{\varphi(I)} = M_{\varphi(I)}.$$

After composing with the inclusion map $M_{\varphi(I)} \subseteq M_{\varphi(I)\cap I}$, we obtain for any $n \in \mathbb{Z}$ the map

$$\begin{aligned} \pi^{-n} \cdot \varphi - 1 \ : \ M_I &\to M_{\varphi(I)\cap I} \\ x &\mapsto \pi^{-n}\varphi(x) - x \end{aligned}$$

Proposition 4.1.4. *Let $M \in \varphi\text{-}Mod_B$ and $\rho \in (0,1)$. Setting $\sigma := \rho^{q^{\frac{1}{2}}}$ there is $N \in \mathbb{N}$ such that for all $n > N$ the map*

$$\pi^{-n} \cdot \varphi - 1 \ : \ M_{[\sigma,\rho^{1/q}]} \to M_{[\sigma,\rho^{1/q}]\cap\varphi([\sigma,\rho^{1/q}])} = M_{[\sigma,\rho]}$$

is surjective.

Proof. First we fix a set of generators $v_1, \dots, v_m$ of M as a B-module. It is also a set of generators of M_I over B_I for any $I \subseteq (0,1)$ by viewing $B \subseteq B_I$ as a subring. Then we have the following pair of maps:

$$\begin{aligned} \varphi \ &: \ M_{[\sigma,\rho^{1/q}]} \to M_{[\sigma^q,\rho]} \\ \varphi^{-1} \ &: \ M_{[\sigma^q,\rho]} \to M_{[\sigma,\rho^{1/q}]} \end{aligned}$$

We choose $A_{i,j} \in B_{[\sigma,\rho^{1/q}]}$,$B_{i,j} \in B_{[\sigma^q,\rho]}$ for all $1 \leq i,j \leq m$ such that for all $1 \leq j \leq m$

$$\varphi^{-1}(v_j) = \sum_{i=1}^{m} A_{i,j} v_i$$
$$\varphi(v_j) = \sum_{i=1}^{m} B_{i,j} v_i$$

and set

$$c_1 := \max_{i,j} |A_{i,j}|_{[\sigma,\rho^{1/q}]}$$
$$c_2 := \max_{i,j} |B_{i,j}|_{[\sigma^q,\rho]}.$$

Afterwards we pick $N \geq 0$ large enough such that for any $n \geq N$:

$$\rho^n c_1 < 1 \quad \text{and} \quad \rho^{n(q-q^{1/2})-q} \cdot c_2 \cdot c_1^q < 1$$

which is possible because $\rho < 1$, and thus $\rho^n \to 0$. Given $n > N$ we set

$$c := (\rho^{n-1} c_1)^{\frac{q}{q-1}}$$

and we explicitly calculate the following inequalities

(i) $q > 1 \iff q - 1 > 0 \Rightarrow \frac{q}{q-1} > 0$ and
$0 < \rho^{n-1} c_1 \leq \rho^N c_1 < 1 \Rightarrow 0 < c < 1$

(ii) $\rho^n \cdot c_1 \cdot c^{\frac{1-q}{q}} = \rho^n \cdot c_1 \cdot (\rho^{n-1} \cdot c_1)^{-1} = \rho < 1$

(iii) $\sigma^{-n} \cdot c_2 \cdot c^{q-1} = \rho^{-nq^{\frac{1}{2}}} \cdot (\rho^{n-1} c_1)^q \cdot c_2 = \rho^{n(q-q^{\frac{1}{2}})-q} \cdot c_1^q \cdot c_2$
$\leq \rho^{N(q-q^{\frac{1}{2}})-q} \cdot c_1^q \cdot c_2 \quad$ because $n > N$ and $q - q^{\frac{1}{2}} > 0$
$< 1 \quad$ by definition of N

Now let $w \in M_{[\sigma,\rho]}$. We inductively construct $x_1^{(l)}, \ldots, x_m^{(l)} \in B_{[\sigma,\rho]}$ for $l \in \mathbb{N}_0$ as follows:

• <u>$l = 0$:</u> choose $x_1^{(0)}, \ldots, x_m^{(0)} \in M_{[\sigma,\rho]}$ such that

$$w = \sum_{i=1}^{m} x_i^{(0)} v_i$$

• <u>$l \mapsto l + 1$:</u> write $x_i^{(l)} = y_i^{(l)} + z_i^{(l)}$ as in lemma 4.1.1 with the constant c constructed as above and set

$$x_i^{(l+1)} := \pi^n \sum_{j=1}^{m} A_{i,j} \varphi^{-1}(y_j^{(l)}) + \pi^{-n} \sum_{j=1}^{m} B_{i,j} \varphi(z_j^{(l)})$$

Then we set

$$\varepsilon := \max\{\rho^n \cdot c_1 \cdot c^{\frac{1-q}{q}}, \sigma^{-n} \cdot c_2 \cdot c^{q-1}\}$$

and by (ii) and (iii) we have $\varepsilon < 1$. For $l \geq 0$ we explicitly calculate

$$|\pi^n \sum_{j=1}^{m} A_{i,j}\varphi^{-1}(y_j^{(l)})|_{[\sigma,\rho]} \leq \rho^n \cdot c_1 \cdot c^{\frac{1-q}{q}} \cdot \max_j |x_j^{(l)}|_{[\sigma,\rho]} \leq \varepsilon \cdot \max_j |x_j^{(l)}|_{[\sigma,\rho]}$$

where the first inequality is derived from lemma 4.1.1. Likewise we calculate

$$|\pi^{-n} \sum_{j=1}^{m} B_{i,j}\varphi(z_j^{(l)})|_{[\sigma,\rho]} \leq \sigma^{-n} \cdot c_2 \cdot c^{q-1} \cdot \max_j |x_j^{(l)}|_{[\sigma,\rho]} \leq \varepsilon \cdot \max_j |x_j^{(l)}|_{[\sigma,\rho]}$$

Putting both inequalities together we obtain

$$\begin{aligned}\max\{|y_i^{(l+1)}|_{[\sigma,\rho]}, |z_i^{(l+1)}|_{[\sigma,\rho]}\} &= |x_i^{(l+1)}|_{[\sigma,\rho]} \\ \leq \max\{|\pi^n \sum_{j=1}^{m} A_{i,j}\varphi^{-1}(y_j^{(l)})|_{[\sigma,\rho]}, &|\pi^{-n} \sum_{j=1}^{m} B_{i,j}\varphi(z_j^{(l)})|_{[\sigma,\rho]}\} \\ &\leq \varepsilon \cdot \max_j |x_j^{(l)}|_{[\sigma,\rho]}\end{aligned}$$

Thus we inductively have

$$\max\{|y_i^{(l+1)}|_{[\sigma,\rho]}, |z_i^{(l+1)}|_{[\sigma,\rho]}\} \leq \varepsilon^{l+1} \cdot \max_j |x_j^{(0)}|_{[\sigma,\rho]}$$

for any $l \geq 0$ and $1 \leq i \leq m$. This implies that for $1 \leq i \leq m$

$$y_i := \sum_{l=0}^{\infty} y_i^{(l)} \quad \text{and} \quad z_i := \sum_{l=0}^{\infty} z_i^{(l)}$$

converge in $B_{[\sigma,\rho]}$. Whence meaning $\varphi^{-1}(y_i) = \sum_{l=0}^{\infty} \varphi^{-1}(y_i^{(l)})$ converges in $B_{[\sigma^{1/q},\rho^{1/q}]}$. On the other hand we can use lemma 4.1.1 once again to construct the following bounds:

$$\begin{aligned}|\varphi^{-1}(y_i^{(l)})|_{[\sigma,\rho]} \leq c^{\frac{1-q}{q}} \cdot |x_i^{(l)}|_{[\sigma,\rho]} &\leq c^{\frac{1-q}{q}} \cdot \max_j\{|x_j^{(l)}|_{[\sigma,\rho]}\} \\ &\leq c^{\frac{1-q}{q}} \cdot \varepsilon^l \cdot \max_j\{|x_j^{(0)}|_{[\sigma,\rho]}\}\end{aligned}$$

showing that $\sum_{l=0}^{\infty} \varphi^{-1}(y_i^{(l)})$ converges in $B_{[\sigma,\rho]}$. Since $|\cdot|_\sigma \leq |\cdot|_{[\sigma,\rho]}$ and likewise $|\cdot|_{\rho^{1/q}} \leq |\cdot|_{[\sigma^{1/q},\rho^{1/q}]}$ we have that $\sum_{l=0}^{\infty} \varphi^{-1}(y_i^{(l)})$ converges with respect to the norms $|\cdot|_\sigma$ and $|\cdot|_{\rho^{1/q}}$. But using Lemma 2.2.6 we have

$$|\cdot|_{[\sigma,\rho^{1/q}]} = \max\{|\cdot|_{\rho^{1/q}}, |\cdot|_\sigma\}$$

implying that $\sum\limits_{l=0}^{\infty} \varphi^{-1}(y_i^{(l)})$ converges in $B_{[\sigma,\rho^{1/q}]}$

Analogously one calculates for $\varphi(z_i) = \sum\limits_{l=0}^{\infty} \varphi(z_i^{(l)})$ that it doesn't only converge in $B_{[\sigma^q,\rho^q]}$, but from 4.1.1 once again

$$|\varphi(z_i^{(l)})|_{[\sigma,\rho]} \leq c^{q-1} \cdot |x_i^{(l)}|_{[\sigma,\rho]} \leq c^{q-1} \cdot \max_j\{|x_j^{(l)}|_{[\sigma,\rho]}\}$$
$$\leq c^{q-1} \cdot \varepsilon^l \cdot \max_j\{|x_j^{(0)}|_{[\sigma,\rho]}\}$$

so that it also converges in $B_{[\sigma,\rho]}$. Since $|\cdot|_{\sigma^q} \leq |\cdot|_{[\sigma^q,\rho^q]}$ and $|\cdot|_{\rho} \leq |\cdot|_{[\sigma,\rho]}$ we see that $\varphi(z_i) = \sum\limits_{l=0}^{\infty} \varphi(z_i^{(l)})$ converges with respect to the norms $|\cdot|_{\sigma^q}$ and $|\cdot|_{\rho}$. But using Lemma 2.2.6 we have

$$|\cdot|_{[\sigma^q,\rho]} = \max\{|\cdot|_{\rho}, |\cdot|_{\sigma^q}\}$$

implying that $\varphi(z_i) = \sum\limits_{l=0}^{\infty} \varphi(z_i^{(l)})$ converges in $B_{[\sigma^q,\rho]}$.

Thus

$$z_i = \varphi^{-1}(\varphi(z_i)) \in B_{[\sigma,\rho^{1/q}]}$$

Set

$$v := -\pi^n \sum_{j=1}^{m} \varphi^{-1}(y_j)\varphi^{-1}(v_j) + \sum_{j=1}^{m} z_j v_j \in M_{[\sigma,\rho^{1/q}]}$$

then

$$v = -\pi^n \sum_{j=1}^{m}\sum_{l=0}^{\infty}\sum_{i=1}^{m} \varphi^{-1}(y_j^{(l)}) \cdot A_{i,j} \cdot v_i + \sum_{j=1}^{m} z_j v_j$$
$$= -\sum_{i=1}^{m}\sum_{l=0}^{\infty}(x_i^{(l+1)} - \pi^{-n}\sum_{j=1}^{m} B_{i,j}\varphi(z_j^{(l)})) \cdot v_i + \sum_{j=1}^{m}\sum_{l=0}^{\infty} z_j^{(l)} v_j =$$
$$-\sum_{i=1}^{m}\sum_{l=1}^{\infty} y_i^{(l)} v_i - \sum_{i=1}^{m}\sum_{l=1}^{\infty} z_i^{(l)} v_i + \pi^{-n}\sum_{i=1}^{m}\varphi(z_i)\cdot\varphi(v_i) + \sum_{i=1}^{m}\sum_{l=0}^{\infty} z_i^{(l)} v_i$$
$$= -\sum_{i=1}^{m}\sum_{l=1}^{\infty} y_i^{(l)} v_i + \sum_{i=1}^{m} z_i^{(0)} v_i + \pi^{-n}\sum_{i=1}^{m}\varphi(z_i)\cdot\varphi(v_i)$$
$$= -\sum_{i=1}^{m}\sum_{l=0}^{\infty} y_i^{(l)} v_i + \sum_{i=1}^{m} x_i^{(0)} v_i + \pi^{-n}\sum_{i=1}^{m}\varphi(z_i)\cdot\varphi(v_i)$$
$$= \pi^{-n}\varphi(v) + w$$
$$\Rightarrow w = v - \pi^{-n}\varphi(v)$$

where the first equality is from the definition of y_i (as a sum on the l variable) and $A_{i,j}$, the second equality is from definition of $x_i^{(l+1)}$ and z_i (as a sum on the l variable), the third is from using the $x = y + z$ equation from lemma 4.1.1, using the definition of $B_{i,j}$, contracting the l-summation and renaming the last two counters j to i, the fifth equation is derived by adding and subtracting $\sum_{i=1}^{m} y_i^{(0)} v_i$ $\square$

Theorem 4.1.5. *Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of φ-modules over B. Then there is $N \geq 0$ such that for any $n > N$ the sequence

$$0 \to M_1^{\varphi=\pi^n} \to M_2^{\varphi=\pi^n} \to M_3^{\varphi=\pi^n} \to 0$$

of E-vector spaces is exact.

Proof. First we note that since the original sequence is a sequence of φ-modules over B (i.e. morphisms commute with φ), the restriction to any homogeneous component produces a sequence

$$M_1^{\varphi=\pi^n} \to M_2^{\varphi=\pi^n} \to M_3^{\varphi=\pi^n}$$

of E-subspaces. Since the maps are the restrictions of the original maps we get left exactness for any $n \geq 0$. Hence we only need to prove the surjectivity statement, i.e. for any surjective map $f : M_2 \to M_3$ of φ-modules over B we can find $N \geq 0$ such that the restricted map

$$f \ : \ M_2^{\varphi=\pi^n} \to M_3^{\varphi=\pi^n}$$

is surjective for any $n \geq N$.

Choose an arbitrary $\rho \in (0, 1)$ and set $\sigma := \rho^{q^{\frac{1}{2}}}$. By Proposition 4.1.4 applied to M_1 we get $N \geq 0$ such that the map

$$\varphi - \pi^n \ : \ M_{1,[\sigma,\rho^{1/q}]} \to M_{1,[\sigma,\rho]}$$

is surjective for any $n \geq N$. Now let $m \in M_3^{\varphi=\pi^n}$ and consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & M_{1,[\sigma,\rho^{1/q}]} \arrow[r] \arrow[d, "\varphi-\pi^n"'] & M_{2,[\sigma,\rho^{1/q}]} \arrow[r] \arrow[d, "\varphi-\pi^n"'] & M_{3,[\sigma,\rho^{1/q}]} \arrow[r] \arrow[d, "\varphi-\pi^n"'] & 0 \\
0 \arrow[r] & M_{1,[\sigma,\rho]} \arrow[r] & M_{2,[\sigma,\rho]} \arrow[r] & M_{3,[\sigma,\rho]} \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Using the snake lemma we get the surjectivity of

$$M_{2,[\sigma,\rho^{1/q}]}^{\varphi=\pi^n} \to M_{3,[\sigma,\rho^{1/q}]}^{\varphi=\pi^n}$$

In particular there is $m' \in M_{2,[\sigma,\rho^{1/q}]}^{\varphi=\pi^n}$ with

$$(f \otimes id_{B_{[\sigma,\rho^{1/q}]}})(m') = m \ \in M_3^{\varphi=\pi^n} \subseteq M_{3,[\sigma,\rho^{1/q}]}^{\varphi=\pi^n}$$

and we more precisely calculate that since $m' \in M_{2,[\sigma,\rho^{1/q}]}^{\varphi=\pi^n}$

$$\varphi(m') = \pi^n m' \ \in M_{2,[\sigma,\rho^{1/q}]} \cap M_{2,[\sigma^q,\rho]} = M_{2,[\sigma,\rho^{1/q}]\cup[\sigma^q,\rho]} = M_{2,[\sigma^q,\rho^{1/q}]}$$

and thus inductively

$$m' \ \in M_{2,[\sigma^{q^n},\rho^{1/q^n}]}$$

for any $n \in \mathbb{N}$. And using that

$$\bigcap_{n\in\mathbb{N}} M_{2,[\sigma^{q^n},\rho^{1/q^n}]} = \bigcap_{I\subseteq(0,1) comp.} M_{2,I} = M_2$$

thus

$$m' \ \in M_2^{\varphi=\pi^n} \quad \text{with} \quad f(m') = m$$

□

Corollary 4.1.6. *The functor* $(M \mapsto \mathcal{F}_M) \ : \ \varphi\text{-}Mod_B \to QCoh_X$ *is exact.*

Proof. We work locally. Let $f \in B^{\varphi=\pi} \setminus \{0\}$. Then $\mathcal{F}_M\big|_{D_+(f)} = \widetilde{M_{(f)}^{alg}}$. Since the principle open sets $D_+(f)$ cover X and $\tilde{(\cdot)}$ is exact on affine schemes, we need to prove that the functor

$$M \ \mapsto \ M_{(f)}^{alg}$$

is exact. As explained earlier (see proof of thm.4.1.5) the functor $(\cdot)^{alg}$ is left exact and homogeneous localization is even exact. Therefore it is only left to prove that $(\cdot)^{alg}$ preserves surjections. So let $g \ : M \to N$ be a surjective morphism of φ-modules, and $x \in N_{(f)}^{alg}$. Write x as

$$x = \frac{y}{f^m} \quad \text{for suitable } m \in \mathbb{N}_0, \ y \in N^{\varphi=\pi^m}.$$

Then for any $r \in \mathbb{N}_0$

$$f^r \cdot y \ \in N^{\varphi=\pi^{m+r}}$$

and choosing r large enough, the restriction of g to the $m+r$ homogeneous component is surjective by Theorem 4.1.5. In particular there is $z \in M^{\varphi=\pi^{m+r}}$ such that $g(z) = f^r \cdot y$. Computing in the localization we get

$$g_{(f)}\left(\frac{z}{f^{r+m}}\right) = \frac{f^r \cdot y}{f^{r+m}} = \frac{y}{f^m} = x$$

□

Proposition 4.1.7. *The functor* $(\mathcal{F} \to M_{\mathcal{F}}) \ : \ Fib_X \to \varphi\text{-}Mod_B$ *is exact and the canonical* B_I*-linear map* $B_I \otimes_B M_{\mathcal{F}} \to M_I$ *is bijective for any compact interval* $I \subseteq (0,1)$.

Proof. By thm.2.2.18 for any $f \in B^{\varphi=\pi} \setminus \{0\}$, the localization $P_{(f)}$ is a Dedekind domain. Using once again the inclusions of integral domains introduced in the analytization we have:

$$P_{(f)} \subseteq P_f \subseteq B_f \subseteq B_{I,f}$$

In particular we can see $B_{I,f}$ as a torsion-free $P_{(f)}$-module.

$\Rightarrow$ $B_{I,f}$ is flat because $P_{(f)}$ is Dedekind
$\Rightarrow$ the functor $(\cdot) \otimes_{P_{(f)}} B_{I,f}$ is exact
$\Rightarrow$ the functor $g_I^* \ : \ QCoh_X \to QCoh_{Spec(B_I)}$ is exact

Global sections are an exact functor for quasi-coherent modules over affine schemes. Thus it only remains to show that the projective limit is exact as a functor which is true because our projective system satisfies the generalized Mittag-Leffler condition of [ST03] chap.3 thm.A. □

4.2 Proof of the equivalence of categories

Having proven that the two functors defined in chapter 2 (algebraization, analytization) are exact, we proceed to prove that algebraization is actually a functor to the category of vector bundles.

In particular both compositions are valid. In the last part of the section we can explicitly calculate that the functors are inverse of each other.

Proposition 4.2.1. *If* $M \in \varphi\text{-}Mod_B$ *is a* φ*-module, then there is* $N \in \mathbb{N}$ *such that* $\forall n \geq N$ *there are finitely many elements of* $M^{\varphi=\pi^n}$ *which generate* M *as a* B*-module.*

Proof. Let $\rho \in (0,1)$. Using the notation of Proposition 4.1.4 we set $\sigma = \rho^{q^{\frac{1}{2}}}$ and let $N \in \mathbb{N}$ be the natural number such that:

$$\text{for all } n > N : \pi^{-n}\varphi - 1 \ : \ M_{[\sigma,\rho^{1/q}]} \to M_{[\sigma,\rho]} \text{ is surjective}$$

Recall that the quantities c_1, c_2, c, ε were defined by

$$c_1 := \max_{i,j} |A_{i,j}|_{[\sigma,\rho^{1/q}]}$$
$$c_2 := \max_{i,j} |B_{i,j}|_{[\sigma^q,\rho]}$$
$$c := (\rho^{n-1}c_1)^{\frac{q}{q-1}}$$
$$0 < \varepsilon := \max\{\rho^n \cdot c_1 \cdot c^{\frac{q-1}{q}}, \sigma^{-n} \cdot c_2 \cdot c^{q-1}\} < 1$$

where $n > N$ is fixed. Choosing $z \in F$ with $0 < |z| < 1$ sufficiently close to c we have

$$0 < \varepsilon' := \max\{\rho^n c_1 |z|^{\frac{1-q}{q}}, \sigma^{-n} c_2 |z|^{q-1}\} < 1.$$

Recall that we have used a set of generators $v_1, v_2, \ldots, v_m$ of M over B which also generate $M_{[\sigma,\rho]}$ over $B_{[\sigma,\rho]}$. Then we fix a $k \in \{1, 2, \ldots, m\}$ and set

$$x_1^{(0)} = x_2^{(0)} = \ldots = x_m^{(0)} = 0$$

whence

$$\sum_{i=1}^m x_i^{(0)} v_i = 0.$$

We define the following decompositions

$$x_i^{(0)} = y_i^{(0)} + z_i^{(0)} \text{ by } \ y_i^{(0)} = -\delta_{i,k} \cdot [z], \ z_i^{(0)} = \delta_{i,k} \cdot [z]$$

as a base case. As in the proof of Proposition 4.1.4 we inductively define

$$x_i^{(l+1)} := \pi^n \sum_{j=1}^m A_{i,j} \varphi^{-1}(y_j^{(l)}) + \pi^{-n} \sum_{j=1}^m B_{i,j} \varphi(z_j^{(l)}) \text{ for all } l \geq 0.$$

As in the proof of Proposition 4.1.4 the element

$$u_k = -\pi^n \sum_{j=1}^m \sum_{l=0}^\infty \sum_{i=1}^m \varphi^{-1}(y_j^{(l)}) \cdot A_{i,j} \cdot v_i + \sum_{j=1}^m \sum_{l=0}^\infty z_j^{(l)} v_j$$

converges in $M_{[\sigma,\rho^{1/q}]}$ and satisfies

$$(\varphi - \pi^n)(u_k) = \sum_{i=1}^m x_i^{(0)} v_i = 0$$

meaning that $\varphi(u_k) = \pi^n \cdot u_k$. Mimicking the proof of thm 4.1.5 we get

$$u_k \in \bigcap_{n\in\mathbb{N}} M_{[\sigma^{q^n}, \rho^{1/q^n}]} = \bigcap_{I\subseteq(0,1) comp.} M_I = M$$

and thus $u_k \in M^{\varphi=\pi^n}$. We claim that $u_1, \dots, u_m$ is a generating system for $M_{[\sigma,\rho]}$ over $B_{[\sigma,\rho]}$. To see this we compute the following upper bounds:
For any $1 \le i \le m$ we have

(i) $|x_i^{(1)}|_{[\sigma,\rho]} = |-\pi^n A_{ik}\varphi^{-1}([z]) + \pi^{-n}B_{ik}\varphi([z])|_{[\sigma,\rho]}$
$\le \max\{\rho^n c_1|z|^{1/q}, \sigma^{-n}c_2|z|^q\} = \varepsilon' \cdot |z| < |z|$.

Using the inequality (i) as well as the upper bounds in the proof of prop 4.1.4 we have:

(ii) $|z_i^{(l)}|_{[\sigma,\rho]} \le \varepsilon^{l-1} \cdot \max_j\{|x_j^{(1)}|\} \le \varepsilon^{l-1} \cdot \varepsilon' \cdot |z| < |z|$, for all $l \ge 1$.

(iii) $|-\pi^n \sum\limits_{j=1}^{m}\sum\limits_{i=1}^{m} \varphi^{-1}(y_j^{(0)})A_{i,j}|_{[\sigma,\rho]} = |-\pi^n \sum\limits_{i=1}^{m} \varphi^{-1}([z])A_{i,k}|_{[\sigma,\rho]} < |z|$.

(iv) $|-\pi^n \sum_{j=1}^m \sum_{i=1}^m \varphi^{-1}(y_j^{(l)}) \cdot A_{i,j}|_{[\sigma,\rho]} \le \varepsilon^{l-1} \cdot \max_j\{|x_j^{(1)}|\}$
$\le \varepsilon^{l-1} \cdot \varepsilon' \cdot |z| < |z|$, for all $l \ge 1$.

By definition of u_k we can thus write

$$u_k = [z] \cdot (y_k \cdot v_k + \sum_{i\neq k} C_{ik} \cdot v_i)$$

with $C_{ik} \in B_{[\sigma,\rho]}$ with $|C_{ik}|_{[\sigma,\rho]} < 1$ and $|y_k|_{[\sigma,\rho]} = 1$. In fact, y_k is a unit in $B_{[\sigma,\rho]}$ because we can write it as

$$y_k = 1 + f$$

with

$$f = \sum_{l=1}^{\infty} z_k^{(l)} \cdot [z^{-1}] - \pi^n \sum_{j=1}^{m}\sum_{l=0}^{\infty} \varphi^{-1}(y_j^{(l)}) \cdot A_{k,j} \cdot [z^{-1}]$$

and thus $|f| < 1$. Then from the geometric series we find the inverse

$$y_k^{-1} = \sum_{i=0}^{\infty}(-1)^i f^i$$

which converges in $B_{[\sigma,\rho]}$ and proves that y_k is a unit. Now consider the matrix $C = (C_{ik})$ with diagonal elements $C_{kk} = y_k$. From the definition of determinant we have

$$\det(C) = y + x, \quad \text{where } y := \prod_{k=1}^{m} y_k,\ x \in B_{[\sigma,\rho]} \text{ with } |x|_{[\sigma,\rho]} < 1.$$

Note that y is still of the form $1+e$ with $|e|_{[\sigma,\rho]} < 1$ whence y is a unit with $|y| = |y^{-1}| = 1$. This implies $\det(C) = y \cdot (1 + \frac{x}{y})$ with $|\frac{x}{y}| < 1$ so that also $\det(C)$ is a unit and the matrix C is invertible. Therefore $u_1, u_2, \ldots, u_m$ are a generating set for $M_{[\sigma,\rho]}$ over $B_{[\sigma,\rho]}$.

Duplicating the previous process for $\rho_1 = \sigma$, we get a second family $u'_1, u'_2, \ldots, u'_m \in M^{\varphi=\pi^n}$ that generates $M_{[\rho^q,\sigma]}$ over $B_{[\rho^q,\sigma]}$. Then consider the B-linear map

$$\begin{aligned} F:\ & B^{2m} \to M \\ (b_1, b_2, \ldots, b_m, b'_1, b'_2, \ldots, b'_m) &\mapsto \sum_{i=1}^{m} b_i u_i + \sum_{i=1}^{m} b'_i u'_i. \end{aligned}$$

It remains to show that F is surjective. Since the inverse limit is exact in our situation (see [ST03] chap.3 thm.A) it suffices to show that the induced B_I-linear map

$$F_I \ := \ F \otimes id_{B_I} : \quad B_I^{2m} \to M_I$$

is surjective for any compact interval $I \subseteq (0,1)$.

Choose $l \in \mathbb{N}$ large enough so that $I \subseteq [\rho^{q^l}, \rho^{q^{-l}}] =: J$. If F_J is surjective, then so is $F_I = F_J \otimes id_{B_I}$. Thus we may assume

$$I = J = [\rho^{q^l}, \rho^{q^{-l}}].$$

Now let

$$I_1 := [\rho^q, \sigma], \quad I_2 := [\sigma, \rho]$$
$$\text{so that } \ I = \bigcup_{i=-l}^{l-1} \varphi^i([\rho^q, \rho]) = \bigcup_{i=-l}^{l-1} \varphi^i(I_1) \cup \bigcup_{i=-l}^{l-1} \varphi^i(I_2).$$

By Lemma 2.2.20 the ring homomorphism

$$B_I \to \prod_{i=-l}^{l-1} B_{\varphi^i(I_1)} \times \prod_{i=-l}^{l-1} B_{\varphi^i(I_2)}$$

is faithfully flat. Therefore it suffices to prove surjectivity after base change to the product of rings. Looking at the components, it suffices to prove surjectivity after base change to $B_{\varphi^i(I_j)}$ for all possible i, j. Then the claim is that $\{u_1, u_2, \dots, u_m, u'_1, u'_2, \dots, u'_m\}$ generate $M_{\varphi^i(I_j)}$ over $B_{\varphi^i(I_j)}$.

For $j = 1$ and i arbitrary we already have

$$M_{\varphi^i(I_1)} = \varphi^i(M_{I_1}) = \varphi^i(\sum_{t=1}^{m} B_{I_1} u'_t) = \sum_{t=1}^{m} \varphi^i(B_{I_1}) u'_t = \sum_{t=1}^{m} B_{\varphi^i(I_1)} u'_t$$

using $\varphi^i(u'_t) = \pi^{n \cdot i} u'_t$. Analogously for $j = 2$ and i arbitrary we have

$$M_{\varphi^i(I_2)} = \varphi^i(M_{I_2}) = \varphi^i(\sum_{t=1}^{m} B_{I_2} u_t) = \sum_{t=1}^{m} \varphi^i(B_{I_2}) u_t = \sum_{t=1}^{m} B_{\varphi^i(I_2)} u_t$$

using $\varphi^i(u_t) = \pi^{n \cdot i} u_t$ □

Lemma 4.2.2. *Let $f \in P \setminus \{0\}$ be homogeneous of degree 1 and let $I = [\sigma, \rho] \subseteq (0, 1)$ be a compact interval with $\sigma \leq \rho^q$. Then the ring homomorphism*

$$P_{(f)} \hookrightarrow P_f \hookrightarrow B_{I,f}$$

is faithfully flat.

Proof. Firstly we show that the map is flat. For that we use the injectivity of the homomorphism, which implies

$$\begin{aligned} & B_{I,f} \quad \text{is torsion free over} \quad P_{(f)} \\ \Rightarrow & B_{I,f} \text{ is flat over } P_{(f)}, \text{ because } P_{(f)} \text{ is Dedekind} \end{aligned}$$

Let $C := \hat{\bar{F}}$ be the completion of an algebraic closure, then by Krasner's lemma C is algebraically closed.

We indicate by a subscript F (resp. C) all rings and schemes defined using F (resp. C) as an input.

Then the inclusion

$$F \hookrightarrow C$$

of complete valued fields gives rise to ring homomorphims

$$\begin{aligned} B_{F,J} &\to B_{C,J} \\ B_F &\to B_C \\ P_F &\to P_C \end{aligned}$$

and to a homomorphism of schemes

$$X_C \to X_F$$

fitting into a commutative diagram

$$\begin{tikzcd} Spec(B_{C,I}) \arrow[r] \arrow[d] & X_C \arrow[d] \\ Spec(B_{F,I}) \arrow[r] & X_F \end{tikzcd}$$

For the complete construction we refer [FF18] §7.6-7.7.

Furthermore we need some more results from [FF18], namely that the morphisms

$$X_C \to X_F$$
$$Spec(B_{C,I}) \to X_C$$

are surjective (see [FF18], page 271 and Prop.6.7.1(2)). The commutativity of the above diagram implies that

$$Spec(B_{F,I}) \to X_F$$

is surjective. Thus we can return to the case $X = X_F$ (i.e. drop the subscripts). By construction we have

$$g_I(D(f)) \subseteq D_+(f)$$

and claim that it is actually an equality. So let

$$\mathfrak{p} \in Spec(B_I) \text{ with } g_I(\mathfrak{p}) \in D_+(f)$$

and choose $g \in P \setminus \{0\}$ homogeneous of degree 1 with $\mathfrak{p} \in D(g)$. Then by construction we have

$$g_I(\mathfrak{p}) = \mathfrak{p}B_{I,g} \cap P_{(g)} \in D_+(g) = Spec(P_{(g)}).$$

By means of contradiction assume $\mathfrak{p} \notin D(f)$ meaning that $f \in \mathfrak{p}$ and hence

$$\frac{f}{g} \in \mathfrak{p}B_{I,g} \cap P_{(g)} = g_I(\mathfrak{p})$$

contradicting $g_I(\mathfrak{p}) \in D_+(f)$. Thus $\mathfrak{p} \in D(f)$ and $g_I(D(f)) = D_+(f)$. Altogether the surjection $g_I : Spec(B_I) \to X$ restricts to a surjection

$$Spec(B_{I,f}) = D(f) = g_I^{-1}(D_+(f)) \to D_+(f) = Spec(P_{(f)})$$

implying that the flat ring homomorphism is $P_{(f)} \to B_{I,f}$ is in fact faithfully flat. □

Now let $M \in \varphi\text{-}Mod_B$ be a φ-Module, $I \subseteq (0,1)$ a compact interval and

$$M_I := M \otimes_B B_I.$$

For any of $f \in B^{\varphi=\pi}$ we define the $B_{I,f}$-linear map

$$\begin{aligned} M_{(f)}^{(alg)} \otimes_{P_{(f)}} B_{I,f} &\to M_I \otimes_{B_I} B_{I,f} \cong M \otimes_B B_{I,f} \\ m \otimes b &\mapsto m \otimes b \end{aligned}$$

Theorem 4.2.3. *Let $M \in \varphi\text{-}Mod_B$, $I = [\sigma, \rho] \subseteq (0,1)$ a compact interval with $\rho^q \leq \sigma$ and $f \in B^{\varphi=\pi} \setminus \{0\}$. The canonical map constructed above is a bijection and $M_{(f)}^{(alg)}$ is a finitely generated projective $P_{(f)}$-module.*

Proof. We first prove that the map is a bijection. For this we use Proposition 4.2.1 and find a finite set of generators $\{w_1, w_2, \ldots, w_m\}$ of M over B with $w_i \in M^{\varphi=\pi^n}$ for n chosen suitably large. Then we can just see $f^{-n}w_i$ as an element of $M_{(f)}^{alg}$, thus finding preimages, and thus proving the surjectivity.

Using Example 3.1.3 we consider the φ-module $B(n)^m$ and the natural surjection of φ-modules

$$\begin{aligned} B(n)^m &\twoheadrightarrow M \\ (b_1, b_2, \ldots, b_m) &\mapsto \sum_{i=1}^m b_i \cdot w_i \end{aligned}$$

Denoting by M' the kernel of this map we get a short exact sequence

$$0 \to M' \to B(n)^m \to M \to 0$$

of φ-modules over B. By Theorem 4.1.5 and the exactness of homogeneous localization we get the exact sequence

$$0 \to (M')_{(f)}^{(alg)} \to (B(n)^m)_{(f)}^{(alg)} \to M_{(f)}^{(alg)} \to 0$$

of $P_{(f)}$-modules. Thus $M_{(f)}^{(alg)}$ is a finitely generated $P_{(f)}$-module because the underlying $P_{(f)}$-module in the middle is simply $P_{(f)}^m$. Furthermore we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & (M')_{(f)}^{(alg)} \otimes_{P_{(f)}} B_{I,f} & \longrightarrow & (B(n)^m)_{(f)}^{(alg)} \otimes_{P_{(f)}} B_{I,f} & \longrightarrow & M_{(f)}^{(alg)} \otimes_{P_{(f)}} B_{I,f} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M'_I \otimes_{B_I} B_{I,f} & \longrightarrow & (B(n)^m)_I \otimes_{B_I} B_{I,f} & \longrightarrow & M_I \otimes_{B_I} B_{I,f} & \longrightarrow & 0 \end{array}$$

with the vertical arrows being the canonical homomorphisms defined above. For the second row we use the formula

$$N_I \otimes_{B_I} B_{I,f} = N \otimes_B B_{I,f}$$

and the fact that M is a flat B-module. We point out that the left vertical arrow is surjective by the first part of the proof applied ti M', and the middle vertical arrow is bijective, too. In fact, the underlying $P_{(f)}$-module (resp. B_I-module) of $(B(n)^m)^{(alg)}_{(f)}$ (resp. $(B(n)^m)_I$) is just $P^m_{(f)}$ (resp. B^m_I), and the vertical map is the canonical isomorphism

$$P^m_{(f)} \otimes_{P_{(f)}} B_{I,f} \cong B^m_{I,f} \cong B^m_I \otimes_{B_I} B_{I,f}.$$

It now follows from snake lemma that the right vertical map is injective, completing the proof that the canonical homomorphism defined above is a bijection.

Thus it only remains to prove that $M^{(alg)}_{(f)}$ is a projective module. Since projectivity can be checked after a faithfully flat base change this follows from Lemma 4.2.2 because

$$M^{(alg)}_{(f)} \otimes_{P_{(f)}} B_{I,f} \cong M \otimes_B B_{I,f}$$

is a finitely generated projective $B_{I,f}$-module (because M is a finitely generated projective B-module). □

Corollary 4.2.4. $\mathcal{F}_M \in Fib_X$

Proof. Letting $f \in B^{\varphi=\pi} \setminus \{0\}$ vary we have the open cover

$$X = \bigcup_f D_+(f)$$

of X. For the restrictions to $D_+(f)$ we see from Theorem 4.2.3 that

$$\mathcal{F}_M\big|_{D_+(f)} \cong \widetilde{M^{(alg)}_{(f)}}$$

is associated to a finitely generated projective module over $\mathcal{O}_X(D_+(f)) = P_{(f)}$. Thus $\mathcal{F}_M\big|_{D_+(f)}$ is a vector bundle over $D_+(f)$ and $\mathcal{F}_M$ is a vector bundle over X. □

For the next part we are going to need the following lemma from commutative algebra

Lemma 4.2.5. *If M is a finitely generated projective B-module, then $M \otimes_B (-)$ commutes with limits. In particular if M is a φ-module,and $(N_I)_I$ is a compatible family of B_I-modules where $I \subseteq (0,1)$ runs through all compact intervals then*

$$M \otimes_B (\varprojlim_I N_I) \cong \varprojlim_I (M \otimes_B N_I).$$

Proof. Consider the comparison map

$$M \otimes_B \varprojlim_I N_I \to \varprojlim_I (M \otimes_B N_I)$$

Any finitely generated projective module is a direct summand of a finitely generated free module. Since the comparison map commutes with finite direct sums we can reduce the problem to the case $M = B$. In this case the statement is trivially true, because

$$\varprojlim_I N_I \cong B \otimes_B \varprojlim_I N_I \to \varprojlim_I (B \otimes_B N_I) \cong \varprojlim_I N_I$$

is the identity map. $\square$

We now proceed to explicitly calculate the compositions of the two functors

- $\underline{M \mapsto \mathcal{F}_M \mapsto M_{\mathcal{F}_M}}$:

Let $M \in \varphi\text{-}Mod_B$, $I \subseteq (0,1)$ be a compact interval and $f \in B^{\varphi=\pi} \setminus \{0\}$. Then Theorem 4.2.3 gives an isomorphism

$$\mathcal{F}_M(D_+(f)) \otimes_{P_{(f)}} B_{I,f} \cong M \otimes_B B_{I,f}$$

of $P_{(f)}$-modules. Letting f vary (using $Spec(B_I) = \bigcup_f D(f)$ by Lemma 3.3.1) gives us

$$(M_{\mathcal{F}_M})_I \cong M \otimes_B B_I$$

and therefore after taking inverse limits

$$M_{\mathcal{F}_M} := \varprojlim_I (M_{\mathcal{F}_M})_I \cong \varprojlim_I (M \otimes_B B_I) \cong M.$$

Here we used Lemma 4.2.5 in the last step.

- $\underline{\mathcal{F} \mapsto M_{\mathcal{F}} \mapsto \mathcal{F}_{M_{\mathcal{F}}}}$: We fix $f \in B^{\varphi=\pi} \setminus \{0\}$ and $\mathcal{F} \in Fib_X$ a vector bundle. Consider the $P_{(f)}$-linear injection

$$M^{alg}_{\mathcal{F},(f)} \hookrightarrow M^{alg}_{\mathcal{F},f} \hookrightarrow M_{\mathcal{F},f}.$$

Since $\varphi(f) = \pi f$ and π is a unit in B, the automorphism $\varphi : B \to B$ extends to an automorphism $\varphi_f : B_f \to B_f$. Likewise $\varphi_{M_\mathcal{F}} : M_\mathcal{F} \to M_\mathcal{F}$ extends to a φ-semilinear automorphism

$$\varphi_{M_{\mathcal{F},f}} = \varphi_{M_\mathcal{F}} \otimes \varphi_f : M_{\mathcal{F},f} = M_\mathcal{F} \otimes_B B_f \to M_\mathcal{F} \otimes_B B_f = M_{\mathcal{F},f}.$$

We claim that

$$M^{alg}_{\mathcal{F},(f)} = M^{\varphi_{M_{\mathcal{F},f}}=1}_{\mathcal{F},f}$$

for the first inclusion let $x \in M^{alg}_{\mathcal{F},(f)}$. There exist $n \geq 0$ and $m \in M_\mathcal{F}^{\varphi_{M_\mathcal{F}}=\pi^n}$ such that $x = \frac{m}{f^n}$. This gives

$$\varphi_{M_{\mathcal{F},f}}(x) = \frac{\varphi_{M_\mathcal{F}}(m)}{\varphi(f)^n} = \frac{\pi^n m}{\pi^n f^n} = \frac{m}{f^n} = x,$$

i.e. $x \in M^{\varphi_{M_{\mathcal{F},f}}=1}_{\mathcal{F},f}$. For the reverse inclusion let $x \in M^{\varphi_{M_{\mathcal{F},f}}=1}_{\mathcal{F},f} \subseteq M_{\mathcal{F},f}$. There exist $n \geq 0$ and $m \in M_\mathcal{F}$ such that $x = \frac{m}{f^n}$. This gives

$$\frac{m}{f^n} = x = \varphi_{M_{\mathcal{F},f}}(x) = \frac{\varphi_{M_\mathcal{F}}(m)}{\varphi(f)^n} = \frac{\varphi_{M_\mathcal{F}}(m)}{\pi^n f^n},$$

i.e. $\varphi_{M_\mathcal{F}}(m) = \pi^n m$ because $M_\mathcal{F}$ is a torsion-free B-module. Thus

$$x = \frac{m}{f^n} \in M^{alg}_{\mathcal{F},(f)},$$

proving the claim.

Using Theorem 4.2.3 we define the canonical $P_{(f)}$-linear map

$$\begin{gathered}\mathcal{F}(D_+(f)) \to \mathcal{F}(D_+(f)) \otimes_{P_{(f)}} B_{I,f} \cong M_\mathcal{F} \otimes_B B_{I,f} \cong M_{\mathcal{F},f} \\ x \mapsto x \otimes 1\end{gathered}$$

By construction of $\varphi_{M_\mathcal{F}}$ the following diagram commutes

$$\begin{tikzcd}
\mathcal{F}(D_+(f)) \otimes_{P_{(f)}} B_{I,f} \arrow[r, "\cong"] \arrow[d, "id\otimes\varphi"'] & M_{\mathcal{F},f} \arrow[d, "\varphi_{M_{\mathcal{F},f}}"] \\
\mathcal{F}(D_+(f)) \otimes_{P_{(f)}} B_{I,f} \arrow[r, "\cong"] & M_{\mathcal{F},f}
\end{tikzcd}$$

Therefore the canonical map defined above takes values in $M^{\varphi_{M_{\mathcal{F},f}}=1}_{\mathcal{F},f} = M^{alg}_{\mathcal{F},(f)}$ meaning that we have a $P_{(f)}$-linear map

$$\mathcal{F}(D_+(f)) \to M^{alg}_{\mathcal{F},(f)} = \mathcal{F}_{M_\mathcal{F}}(D_+(f)).$$

The associated homomorphisms of module sheaves

$$\mathcal{F}\big|_{D_+(f)} = \widetilde{\mathcal{F}(D_+(f))} \to \widetilde{M^{alg}_{\mathcal{F},(f)}} = \mathcal{F}_{M_{\mathcal{F}}}\big|_{D_+(f)}$$

glue to a homomorphism

$$\mathcal{F} \to \mathcal{F}_{M_{\mathcal{F}}}$$

of $\mathcal{O}_X$-modules. In order to see that it is an isomorphism, it suffices to see that the $P_{(f)}$-linear map $\mathcal{F}(D_+(f)) \to M^{alg}_{\mathcal{F},(f)}$ is an isomorphism. Using lemma 4.2.2 (for suitably chosen I) it is enough to see that the map

$$\mathcal{F}(D_+(f)) \otimes_{P_{(f)}} B_{I,f} \to M^{alg}_{\mathcal{F},(f)} \otimes_{P_{(f)}} B_{I,f}$$

obtained after base change is bijective. However, composing this map with the isomorphism

$$M^{alg}_{\mathcal{F},(f)} \otimes_{P_{(f)}} B_{I,f} \to M_{\mathcal{F},I} \otimes_{B_I} B_{I,f} = \mathcal{F}(D_+(f)) \otimes_{P_{(f)}} B_{I,f}$$

of Theorem 4.2.3 gives the identity.

4.3 Commutativity of the functors with tensor products and internal hom's.

In this final section we will show that the two functors commute with tensor products and internal hom's.

Firstly we turn our attention to tensor products. And define the tensor product of two φ-modules as follows.

Definition 4.3.1. *Given two φ-modules $(M, \varphi_M), (N, \varphi_N)$ over B we define their tensor product in the category φ-Mod_B as $(M \otimes_B N, \varphi_M \otimes \varphi_N)$.*

Here $M \otimes_B N$ is finitely generated projective as a tensor product of finitely generated projective modules. On simple tensors the $\varphi_M \otimes \varphi_N$ is given by

$$(\varphi_M \otimes \varphi_N)(m \otimes_B n) = \varphi_M(m) \otimes_B \varphi_N(n)$$

and has inverse $(\varphi_M^{-1} \otimes \varphi_N^{-1})$. Finally we calculate

$$\begin{aligned}(\varphi_M \otimes \varphi_N)(b(m \otimes n)) &= (\varphi_M \otimes \varphi_N)(b \cdot m \otimes n) = \varphi_M(b \cdot m) \otimes \varphi_N(n) \\ &= (\varphi(b) \cdot \varphi_M(m)) \otimes \varphi_N(n) = \varphi(b) \cdot (\varphi_M(m) \otimes \varphi_N(n)) \\ &= \varphi(b) \cdot (\varphi_M \otimes \varphi_N)(m \otimes n),\end{aligned}$$

showing that $\varphi_M \otimes \varphi_N$ is semilinear.

Proposition 4.3.2. *Algebraization commutes with tensor products.*

Proof. Let $M, N \in \varphi\text{-}Mod_B$. Then there is a natural homomorphism of P-modules $M^{alg} \otimes_P N^{alg} \to (M \otimes_B N)^{alg}$ satisfying $x \otimes y \mapsto x \otimes y$. Applying $\tilde{(\cdot)}$ we get a homomorphism of $\mathcal{O}_X$-modules

$$\widetilde{M^{alg} \otimes_P N^{alg}} \cong \widetilde{M^{alg}} \otimes_{\mathcal{O}_X} \widetilde{N^{alg}} = \mathcal{F}_M \otimes_{\mathcal{O}_X} \mathcal{F}_N \to \mathcal{F}_{M \otimes_B N} = \widetilde{(M \otimes_B N)^{alg}}.$$

In order to see that it is an isomorphism it suffices to see that for any $f \in B^{\varphi=\pi} \setminus \{0\}$ the homomorphism of $P_{(f)}$-modules

$$(\mathrm{F}_M \otimes_{\mathcal{O}_X} \mathcal{F}_N)(D_+(f)) = M^{alg}_{(f)} \otimes_{P_{(f)}} N^{alg}_{(f)} \to (M \otimes_B N)^{alg}_{(f)} = \mathcal{F}_{M \otimes_B N}(D_+(f))$$

is bijective. Using lemma 4.2.2 for a suitably chosen compact interval I, it is enough to see this bijectivity after base change to $B_{I,f}$. But the map one obtains is the isomorphism

$$\begin{aligned} M^{alg}_{(f)} \otimes_{P_{(f)}} N^{alg}_{(f)} \otimes_{P_{(f)}} B_{I,f} &\cong (M^{alg}_{(f)} \otimes_{P_{(f)}} B_{I,f}) \otimes_{B_{I,f}} (N^{alg}_{(f)} \otimes_{P_{(f)}} B_{I,f}) \\ &\cong (M \otimes_B B_{I,f}) \otimes_{B_{I,f}} (N \otimes_B B_{I,f}) \cong M \otimes_B N \otimes_B B_{I,f} \\ &\cong (M \otimes_B N)^{alg}_{(f)} \otimes_{P_{(f)}} B_{I,f} \end{aligned}$$

from Theorem 4.2.3 applied to M, N and $M \otimes_B N$. □

Proposition 4.3.3. *Analytization commutes with tensor products.*

Proof. Since pullback and the $\tilde{(\cdot)}$ construction on affine schemes commute with tensor products, we have:

$$M_{\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, I} = M_{\mathcal{F}, I} \otimes_{B_I} M_{\mathcal{G}, I}.$$

Therefore, Proposition 3.3.7 gives

$$M_{\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, I} = M_{\mathcal{F}, I} \otimes_{B_I} M_{\mathcal{G}, I} \cong M_{\mathcal{F}} \otimes_B B_I \otimes_{B_I} M_{\mathcal{G}, I} \cong M_{\mathcal{F}} \otimes_B M_{\mathcal{G}, I}.$$

Because of Lemma 4.2.5 we obtain

$$\begin{aligned} M_{\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G}} = \varprojlim_I (M_{\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G},I}) \cong \varprojlim_I (M_{\mathcal{F}} \otimes_B M_{\mathcal{G},I}) \\ \cong M_{\mathcal{F}} \otimes_B \varprojlim_I M_{\mathcal{G},I} = M_{\mathcal{F}} \otimes_B M_{\mathcal{G}} \end{aligned}$$

□

Next we turn our attention to internal hom's.

Definition 4.3.4. *Given two φ-modules $(M, \varphi_M), (N, \varphi_N)$ we define their internal hom in the category of φ-Mod_B as*

$$\underline{\mathrm{Hom}}(M, N) := \mathrm{Hom}_B(M, N)$$

as a B-module with the φ-linear automorphism

$$\varphi_{\underline{\mathrm{Hom}}(M,N)} := \quad (f \mapsto \varphi_N \circ f \circ (\varphi_M)^{-1}).$$

Here $\mathrm{Hom}_B(M, N)$ is again a finitely generated, projective B-module. To see that one uses the characterization of finitely generated projective modules as direct summands of finitely generated free modules and commutativity with finite direct sums of both arguments of internal hom's. Moreover, $\mathrm{Hom}_B(B, B) \cong B$. For the map $\varphi_{\underline{\mathrm{Hom}}(M,N)}$, the additivity of $\varphi_{\underline{\mathrm{Hom}}(M,N)}(f)$ is a direct consequence of the fact that it is the composition of three additive maps. And for the scalars one calculates

$$\begin{aligned} \varphi_{\underline{\mathrm{Hom}}(M,N)}(f)(bm) &= \varphi_N \circ f \circ \varphi_M^{-1}(bm) = \varphi_N(f(\varphi_M^{-1}(bm))) \\ &= \varphi_N(f(\varphi^{-1}(b) \cdot \varphi_M^{-1}(m))) = \varphi_N(\varphi^{-1}(b) \cdot f(\varphi_M^{-1}(m))) \\ &= \varphi(\varphi^{-1}(b)) \cdot \varphi_N(f(\varphi_M^{-1}(m))) = b \cdot \varphi_{\underline{\mathrm{Hom}}(M,N)}(f)(m). \end{aligned}$$

This shows that the map $\varphi_{\underline{\mathrm{Hom}}(M,N)}(f)$ is B-linear and therefore $\varphi_{\underline{\mathrm{Hom}}(M,N)}$ is well defined. Furthermore it is additive since:

$$\begin{aligned} \varphi_{\underline{\mathrm{Hom}}(M,N)}(f+g)(m) &= \varphi_N \circ (f+g) \circ (\varphi_M)^{-1}(m) = \varphi_N(f + g(\varphi_M^{-1}(m))) \\ &= \varphi_N(f(\varphi_M^{-1}(m)) + g(\varphi_M^{-1}(m))) = \varphi_N(f(\varphi_M^{-1}(m))) + \varphi_N(g(\varphi_M^{-1}(m))) \\ &= \varphi_{\underline{\mathrm{Hom}}(M,N)}(f)(m) + \varphi_{\underline{\mathrm{Hom}}(M,N)}(g)(m). \end{aligned}$$

For the semilinearity one calculates

$$\begin{aligned} \varphi_{\underline{\mathrm{Hom}}(M,N)}(bf)(m) &= \varphi_N \circ (bf) \circ (\varphi_M)^{-1}(m) = \varphi_N(bf(\varphi_M^{-1}(m))) \\ &= \varphi(b) \cdot \varphi_N(f(\varphi_M^{-1}(m))) = (\varphi(b) \cdot \varphi_{\underline{\mathrm{Hom}}(M,N)}(f))(m). \end{aligned}$$

As for the surjectivity of $\varphi_{\underline{\mathrm{Hom}}(M,N)}$, given $g \in \mathrm{Hom}_B(M,N)$ one defines $f := (\varphi_N)^{-1} \circ g \circ \varphi_M$ and checks as above that f is B-linear. Now

$$\varphi_{\underline{\mathrm{Hom}}(M,N)}((\varphi_N)^{-1} \circ g \circ \varphi_M) = \varphi_N \circ (\varphi_N)^{-1} \circ g \circ \varphi_M \circ (\varphi_M)^{-1} = g.$$

As for the injectivity, given $f \in ker(\varphi_{\underline{\mathrm{Hom}}(M,N)})$ we have

$$\varphi_N \circ f \circ \varphi_M^{-1} = \varphi_{\underline{\mathrm{Hom}}(M,N)}(f) = 0 \iff f = (\varphi_N)^{-1} \circ 0 \circ \varphi_M = 0.$$

A special case of internal hom which warrants mentioning is the dual.

Definition 4.3.5. *Given a φ-module (M, φ_M) we define its dual φ-module as*

$$\begin{aligned} M^\vee &:= \underline{\mathrm{Hom}}(M,B) := \mathrm{Hom}_B(M,B) \\ \varphi_{M^\vee} : &\quad f \mapsto \varphi \circ f \circ (\varphi_M)^{-1} \end{aligned}$$

We now prove the commutativity of the two functors with the internal hom's. Towards that goal we use a formula relating internal hom's, duals and tensor products. In our particular case they have the form

$$\begin{aligned} \underline{\mathrm{Hom}}(M,N) &= \mathrm{Hom}_B(M,N) \cong M^\vee \otimes_B N \\ \underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G}) &= \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \cong \mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathcal{G}. \end{aligned}$$

Since commutativity with the tensor product is already proven, commutativity with internal hom's can be reduced to commutativity with the dual. For algebraization we do not need this trick.

Proposition 4.3.6. *Algebraization commutes with internal hom's.*

Proof. We need to show

$$\widetilde{(\underline{\mathrm{Hom}}_B(M,N))^{alg}} \cong \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}_M, \mathcal{F}_N).$$

Let $f \in B^{\varphi=\pi} \setminus \{0\}$ be arbitary, $n \geq 0$ and $g \in \underline{\mathrm{Hom}}_B(M,N)^{\varphi=\pi^n}$, i.e. $\varphi_N \circ g \circ \varphi_M^{-1} = \pi^n g$.Then

$$\begin{aligned} \frac{g}{f^n} : &\quad M^{alg}_{(f)} \to N^{alg}_{(f)} \\ &\frac{x}{f^m} \mapsto \frac{g(x)}{f^{n+m}} \end{aligned}$$

is a well-defined $P_{(f)}$-linear map. Thus we get a $P_{(f)}$-linear map

$$T: \quad \underline{\mathrm{Hom}}_B(M,N)^{alg}_{(f)} \to \underline{\mathrm{Hom}}_{P_{(f)}}(M^{alg}_{(f)}, N^{alg}_{(f)})$$

where the left hand side is

$$\underline{\mathrm{Hom}}_B(M,N)^{alg}_{(f)} = \widetilde{\underline{\mathrm{Hom}}_B(M,N)^{alg}}(D_+(f))$$

and the right hand side is

$$\underline{\mathrm{Hom}}_{P_{(f)}}(M^{alg}_{(f)}, N^{alg}_{(f)}) = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}_M, \mathcal{F}_N)(D_+(f)).$$

Applying $\tilde{(\cdot)}$ to T and letting f vary we obtain a homomorphism of $\mathcal{O}_X$-modules

$$\widetilde{\underline{\mathrm{Hom}}_B(M,N)^{alg}} \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}_M, \mathcal{F}_N).$$

In order to see that it is bijective it suffices to see that T is bijective. By Lemma 4.2.2 it is enough to show the bijectivity of T after base change to $B_{I,f}$ for a suitably chosen compact interval I. But after the base change T is the isomorphism

$$\begin{gathered}\underline{\mathrm{Hom}}_B(M,N)^{alg}_{(f)} \otimes_{P_{(f)}} B_{I,f} \cong \underline{\mathrm{Hom}}_B(M,N) \otimes_B B_{I,f} \cong \\ \underline{\mathrm{Hom}}_{B_{I,f}}(M \otimes_B B_{I,f}, N \otimes_B B_{I,f}) \cong \\ \underline{\mathrm{Hom}}_{B_{I,f}}(M^{alg}_{(f)} \otimes_{P_{(f)}} B_{I,f}, N^{alg}_{(f)} \otimes_{P_{(f)}} B_{I,f}) \cong \\ \underline{\mathrm{Hom}}_{P_{(f)}}(M^{alg}_{(f)}, N^{alg}_{(f)}) \otimes_{P_{(f)}} B_{I,f}\end{gathered}$$

obtained from Theorem 4.2.3 applied to M, N and $\underline{\mathrm{Hom}}_B(M,N)$ $\square$

Proposition 4.3.7. *Analytization commutes with internal hom's.*

Proof. Using the above formula relating internal hom's, duals and tensor products Proposition 4.3.3 gives

$$M_{\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G})} \cong M_{\mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathcal{G}} \cong M_{\mathcal{F}^\vee} \otimes_B M_{\mathcal{G}}.$$

We recall that analytization is the composition of inverse limit, global sections and pullback. Pullback and the $\tilde{(\cdot)}$-construction on affine schemes commute with duals. i.e.

$$\begin{gathered}M_{\mathcal{F}^\vee, I} \cong (M_{\mathcal{F},I})^\vee = \mathrm{Hom}_{B_I}(M_{\mathcal{F},I}, B_I) \overset{(\star)}{\cong} \mathrm{Hom}_{B_I}(M_{\mathcal{F}} \otimes_B B_I, B_I) \\ \cong \mathrm{Hom}_B(M_{\mathcal{F}}, B_I)\end{gathered}$$

where the ($\star$) isomorphism is the one obtained from Proposition 3.3.7. Now we calculate

$$\begin{aligned} M_{\mathcal{F}^\vee} &= \varprojlim_I (M_{\mathcal{F}^\vee,I}) \cong \varprojlim_I ((M_{\mathcal{F},I})^\vee) \cong \varprojlim_I \mathrm{Hom}_B(M_{\mathcal{F}}, B_I) \\ &\cong \mathrm{Hom}_B(M_{\mathcal{F}}, \varprojlim_I B_I) = \mathrm{Hom}_B(M_{\mathcal{F}}, B) = (M_{\mathcal{F}})^\vee \end{aligned}$$

and thus

$$M_{\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G})} \cong (M_{\mathcal{F}})^\vee \otimes_B M_{\mathcal{G}} \cong \underline{\mathrm{Hom}}(M_{\mathcal{F}}, M_{\mathcal{G}})$$

□

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