Matrix factorizations and the singularity category of hypersurfaces in principal ideal domains

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Matrikel Nummer: 3154922

September 10, 2025

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This thesis is dedicated to my Father, who, due to poverty, never got to finish his studies as a mathematician, and to my mother, who now owes the equivalent of tens of thousands of euros to predatory banks in order to pay for my education.

I would like to thank Professor Dr. Jan Kohlhaase for his patience and guidence and consistent support. I would also like to thank Professor Dr. Görtz for his valuable comments and for teaching me algebraic geometry, as well as Florian leptien for wasting several hours of his time trying to work out the singularity category of $k[x, y]/(xy)$ with me to no avail. Finally, I would like to thank my dear friend Sharban Mattiety for his C_∞ emotional support throughout my mathematical journey.

Contents

1 Introduction

[1]

The study of singularities has long been a central theme in commutative algebra and algebraic geometry, with profound implications for the classification of schemes and the structure of their derived categories. A particularly rich field of research is the *singularity category* $\mathrm{D_{sg}}(X)$ of a scheme X, which measures the failure of X to be non-singular by capturing the homological complexity of its singularities. For affine schemes $\mathrm{Spec}(R)$, this category is defined as the Verdier quotient

$$\mathrm{D_{sg}}(R) = \mathrm{D^b}(\text{mod-}R) \,/\, \mathrm{Perf}(R),$$

where $\mathrm{D^b}(\text{mod-}R)$ is the bounded derived category of finitely generated R-modules and $\mathrm{Perf}(R)$ is the full subcategory of perfect complexes. This quotient isolates the complexes that arise solely from the singular nature of R.

A groundbreaking advance in this field was the discovery by Eisenbud, Buchweitz, and others that for *hypersurface rings*-rings of the form $R/(f)$ where R is regular and f is a non-zero divisor-the singularity category admits a completely explicit and computable model via *matrix factorizations*, which were defined in [Eis80]. A matrix factorization of f is a pair of maps between free R-modules whose composition is multiplication by f. Despite their elementary definition, these objects encode deep information about the singularity of $R/(f)$. The seminal result is an equivalence of categories:

$$\mathrm{HMF}(R, f) \cong \mathrm{D_{sg}}(R/(f)),$$

where $\mathrm{HMF}(R, f)$ denotes the homotopy category of matrix factorizations of f. This equivalence was established in two parts:

1. An equivalence of categories between $D_{sg}(R/f)$ and $\underline{\mathrm{MCM}(R/f)}$, the stable category of maximal Cohen-Macaulay modules. This was done in [Buc86, Theorem 4.4.1].

2. An equivalence of categories between $\underline{\mathrm{MCM}(R/f)}$ and $\mathrm{HFM}(R/f)$. This was done in [Eis80, Corollary 6.3].

The historical development of this subject intertwines with the evolution of homological algebra. The concept of matrix factorizations first emerged in the work of Eisenbud [Eis80], who showed that over a regular local ring R, the minimal $R/(x)$-free resolution of a finitely generated R-module M is periodic after $\dim R + 1$ of period two exactly when the periodicity of M is equivalent to being maximal Cohen-Macaulay with no free summands [Eis80, Theorem 6.1]. This periodicity is captured by matrix factorizations. Later, Buchweitz [Buc86] and Orlov [Orl04] independently developed the theory of singularity categories, with Orlov extending the framework to global geometric settings. The equivalence between matrix factorizations and singularity categories for hypersurfaces was firmly established in this period, providing a powerful tool for studying singularities.

Meanwhile, the theory of Cohen-Macaulay modules and maximal Cohen–Macaulay (MCM) modules played a crucial role. For a hypersurface ring $R/(f)$, the stable category of MCM modules is equivalent to the singularity category [Buc86]. Matrix factorizations naturally yield MCM modules via the cokernel of their second map, and this construction

[1]The introduction was written using an LLM, and has been proof read by the author of this thesis. None of the statements, though, were discovered nor proven by using an LLM.

induces the equivalence. This connection has been exploited to study the representation theory of singularities, with applications in algebraic geometry, invariant theory, and even string theory through Landau-Ginzburg models.

In this thesis, we focus, for the first time to our knowledge, on the basic case where the base ring R is a *principal ideal domain* (PID). This choice is motivated by the fact that PIDs provide the simplest yet non-trivial setting where the theory of matrix factorizations becomes exceptionally tractable. Existance of the Smith normal form ensures that every matrix over a PID is diagonalizable, allowing us to reduce the problem to the relatively simple rank one case.

Our main contribution is a comprehensive classification of the homotopy category of matrix factorizations over a PID R for a non-zero element $f \in R$. We show that every matrix factorization is isomorphic to a direct sum of rank-one factorizations (R, R, α, β) where $\alpha\beta = f$. We then describe morphisms, homotopies, and homotopy equivalences between these factorizations in explicit number-theoretic terms, involving greatest common divisors, least common multiples, and valuations. Key results include:

- A characterization of homotopy equivalences between rank-one factorizations via divisibility conditions on the parameters α and β.
- A computation of the endomorphism rings in the homotopy category, showing that

  $$\mathrm{End}_{\mathrm{HMF}}(M) \;\cong\; R/(\alpha, \beta)$$

  for a rank-one factorization $M = (R, R, \alpha, \beta)$.
- A classification of indecomposable objects up to homotopy equivalence, particularly when $f = p^n$ is a prime power.
- A formula for the number of homotopy classes of rank-one factorizations in terms of the prime factorization of f, confirming that the singularity category vanishes if and only if f is square-free (i.e. $R/(f)$ is regular) [Buc86].

This classification not only elucidates the structure of $\mathrm{D}_{\mathrm{sg}}(R/(f))$ for PIDs but also serves as an example of how matrix factorizations can render abstract homological algebra concrete and computable. It illustrates the profound interplay between algebra and arithmetic, where concepts like prime factorization and valuation directly govern homological properties.

Through this work, we aim to demonstrate the beauty and power of matrix factorizations as a tool for understanding singularities, and to provide a clear, elementary exposition that makes this advanced topic accessible to students and researchers alike. We assume the reader is familiar with the definition of the singularity category and its prerequisites, namely: abelian and triangulated categories, as well as graduate level commutative algebra.

2 Review of Commutative Algebra

The path towards singularity categories is built upon a solid foundation of commutative algebra. This chapter serves to establish this essential groundwork. We recall the fundamental concepts of *regular sequences* and the *depth* of a module, which measure the homological complexity within an ideal. We then review the notions of *Krull dimension*, *regular* and *Cohen-Macaulay rings*, which provide the geometric and algebraic context for measuring regularity and singularity.

A central tool constructed here is the *Koszul complex*, which is sensitive to regular sequences. Its homology provides a powerful method for computing depth and understanding the homological properties of a ring. We also discuss *projective dimension* and *global dimension*, culminating in the celebrated theorem that a local ring is regular if and only if its global dimension is finite (and equal to its Krull dimension). This equivalence is crucial, as it directly links the geometric notion of smoothness (regularity) to the homological property of having finite projective resolutions for all modules. Understanding this link is the primary motivation for the entire thesis: to study what happens homologically when a ring is *not* regular. For the rest of this section, let R be a noetherian ring with 1 and M and R-module. We closely follow [Eis95, Section 16-19] and [Kap74, Chapter 3].

Definition 1. An **M-regular sequence** of length n in R is an n-tuple $(x_1, \ldots, x_n)$ of elements in R such that:

1. $(x_1, \ldots, x_n)M \neq M$, and
2. x_i is not a zero divisor on $M/(x_1, \ldots, x_{i-1})$ for all i.

Example 2. For $R = k[X_1, \ldots, X_n]$, the sequence given by the variables $(X_1, \ldots, X_n)$ is clearly regular over $M = R$. In fact, given a ring R containing a field k, any regular R-sequence consists of independent indeterminates over the field k.

Proposition 3. Let I, J be ideals in R. Then $(M/IM)/J(M/IM) \cong M/(I + J)M$

Proof. It is straight forward to see that the map $M \to M/IM \to (M/IM)/J(M/IM)$ is surjective with kernel $I + J$. □

Corollary 4. A sequence $(x_1, \ldots, x_n)$ is M-regular if and only if the sequences $(x_1, \ldots, x_i)$ and $(x_{i+1}, \ldots, x_n)$ are M-regular and $M/(x_1, \ldots, x_i)M$-regular respectively.

Proposition 5. If x_1, x_2 form an M-regular sequence, then x_1 is not a zero divisor in M/x_2M.

Proof. Suppose there exists $\bar{t} \in M/x_2M$ with $x_1\bar{t} \equiv 0 \pmod{x_2}$ and let t be a preimage of $\bar{t}$ in M. Then

$$\begin{aligned} & x_1 t \in x_2 M \\ \longrightarrow\, & x_1 t = x_2 u \quad \text{for some } u \in M \\ \longrightarrow\, & u \in x_1 M \quad \text{since } x_2 \text{ is not a zero divisor modulo } x_1 \\ \longrightarrow\, & x_1 t = x_1 x_2 m \quad \text{for some } m \in M \\ \longrightarrow\, & t = x_2 m \\ \longrightarrow\, & \bar{t} \equiv 0 \pmod{x_2} \end{aligned}$$

□

Proposition 6. Let M be a finitely generated module over the noetherian local ring R. Then all maximal regular M-sequences have the same length, which is given by

$$\min\{i \mid \operatorname{Ext}^i_R(k, M) = 0\}$$

Proof. □

Definition 7. The **Krull dimension** of a ring R is given by

$$\dim(R) := \sup\{n \mid P_0 \subsetneq \dots P_n \text{ is a chain of primes in } R\}$$

The Krull dimension of an R-module M is the Krull dimension of $R/\mathrm{Ann}_R(M)$. The *codimension* of a prime ideal I is the Krull dimension of R_I. The codimension of an arbitrary ideal I is the infimum of codimensions of primes containing it.

Example 8.
- The Krull dimension of a field is 0.
- The Krull dimension of a DVR is 1.
- The Krull dimension of $k[x_1, \dots, x_n]$, where k is a field, is n.

Definition 9. Let R be a local ring. R is said to be **regular** if its maximal ideal can be generated by exactly $\dim R$ elements. A non-local ring R is regular if all its localizations at prime ideals are local.

Example 10.
- Any DVR is a regular local ring.
- Let k be a field. Then $k[x_1, \dots, x_n]$ is a regular ring; we shall see later that if R is regular, then $R[x]$ is regular, proving the regularity of polynomial rings over fields by induction.
- The power series ring in any number of variables over a field is a regular local ring.

Theorem 11. *A regular local ring R is an integral domain.*

Proof. Cf. [Eis95] Corollary 10.14. □

Proposition 12. Let R be a regular local ring with maximal ideal $\mathfrak{m}$ and a regular system of parameters $(x_1, \dots, x_n)$ - that is, the $x_i's$ are a basis of $\mathfrak{m}/\mathfrak{m}^2$ as an $R/\mathfrak{m}$ vector space. Then $(x_1, \dots, x_n)$ is a regular sequence.

Proof. For each i the ring $R/(x_1, \dots, x_i)$ is a regular local ring, and therefore an integral domain. The image of x_{i+1} is therefore not a zero divisor and non-zero, since that would contradict the minimality of the system of generators. □

Recall that given an R-module M we can construct the exterior algebra $\wedge M$ as the tensor algebra

$$\begin{aligned} \otimes M &:= \bigoplus_{i \geq 0} M^{\otimes i} \\ &= R \oplus M \oplus (M \otimes M) \oplus \dots \end{aligned}$$

modulo the ideal generated by $x \otimes x$ and $x \otimes y + y \otimes x$. $\wedge M$ is a graded algebra. We write $\wedge M = \oplus_{i \geq 0} \wedge^i M$, with $\wedge^i M$ being the part of degree i, which is generated as an R-module by the product of exactly i elements of M. Given two homogeneous elements a, b of degrees m, n respectively, we have

$$a \wedge b = (-1)^{mn} b \wedge a$$

and if $m = 1$, then $a \wedge a = 0$. This construction is functorial, maps of R-modules $f : M \to N$ lift to maps $\wedge f : \wedge M \to \wedge N$ sending $\bigwedge_{i \geq 0} a_i \mapsto \bigwedge_{i \geq 0} f(a_i)$. If M is free of rank n, then $\wedge^n M \cong R$, and if $f : M \to M$ is a morphism, then $\wedge^n f$ is multiplication by the determinant of any matrix representing f. Furthermore we have $\wedge^i M = 0$ for $i > n$. We can use the wedge product to define a very useful homological tool, called **the Koszul complex**: for any R-module M and element $x \in M$, the Kozsul complex is given by

$$K(x) := (\wedge^i M, d_x^i)_{i \geq -1} = 0 \to R \to M \to \wedge M \to \dots$$

where d_x sends an element a to $x \wedge a$. If M is free of rank n and $x = (x_1, \dots, x_n) \in R^n \cong M$, then we shall sometimes write $K(x_1, \dots, x_n)$ for the Koszul complex instead of $K(x)$. This construction is functorial: if $f : M \to N$ is an R-module homomorphism sending x to y, then the map $\wedge f$ preserves the differential, and is thus a map of complexes, exactly because it is a map of algebras. We are interested in computing the cohomology of the Koszul complex to prove certain theorems about depth, codimension, regularity, and Cohen-Macauly modules.

Proposition 13. For a free R-module M of rank n and $(x_1, \dots, x_n) \in R^n \cong M$, we have

$$H^n(K(x_1, \dots, x_n)) \cong R/(x_1, \dots, x_n)$$

Proof. We have a commutative diagram

$$\begin{array}{ccccc} \dots \wedge^{n-1} M & \longrightarrow & \wedge^n M & \longrightarrow & \wedge^{n+1} M \\ \downarrow & & \downarrow & & \downarrow \\ \dots R^n & \longrightarrow & R & \longrightarrow & 0 \end{array}$$

with the vertical maps being isomorphisms. The middle isomorphism identifies e_1 with $e_1 \wedge \dots \wedge e_n$. The leftmost isomorphism identifies the basis $e_1, \dots, e_n$ with the basis $(e_1, \dots, \hat{e_i}, \dots, e_n)_{1 \leq i \leq n}$ where $\hat{\bullet}$ refers to omitting the factor e_i. The map $\wedge^{n-1} M \to \wedge^n M$ sends the basis element $e_1, \dots, \hat{e_i}, \dots, e_n$ to $\pm x_i e_1, \dots, e_n$, so the image of the corresponding map in the bottom row is the ideal generated by $(x_1, \dots, x_n)$. The kernel of the map $R \to 0$ is clearly R, therefore

$$H^n(K(x_1, \dots, x_n)) \cong R/(x_1, \dots, x_n)$$

□

Theorem 14. *Let M be a finitely generated R-module and let $r \in \mathbb{Z}$ such that for $i < r$ we have*

$$H^i(M \otimes K(x_1, \dots, x_n)) = 0$$

and otherwise

$$H^i(M \otimes K(x_1, \dots, x_n)) \neq 0\,,$$

Then every maximal M-regular sequence contained in the ideal $(x_1, \dots, x_n)$ has length r.

Corollary 15. If $x = (x_1, \dots, x_n)$ is a regular M-sequence, then $M \otimes K(x)$ is exact except at the extreme right.

Proof. The length of a maximal regular M-sequence in the ideal $(x_1, \dots, x_n)$ is at least n, so theorem Theorem 14 gives us that

$$H^i(M \otimes K(x_1, \dots, x_n)) = 0 \text{ for } i < n,$$

which is the first assertion. The second assertion is a bit less straightforward. Firstly, note that $H^n(M \otimes K(x_1, \dots, x_n))$ is the homology of the complex

$$M \otimes \wedge^{n-1} R^n \to M \otimes \wedge^n R^n \to 0$$

So we have

$$\begin{aligned} H^n(M \otimes K(x_1, \dots, x_n)) &= \mathrm{Coker}(M \otimes \wedge^{n-1} R^n \to M \otimes \wedge^n R^n) \\ &\equiv M \otimes \mathrm{Coker}(\wedge^{n-1} R^n \to \wedge^n R^n) \\ &\equiv M \otimes R/(x_1, \dots, x_n) \\ &\equiv M/(x_1, \dots, x_n)M\,. \end{aligned}$$

□

The converse is in general false, but it holds in the local setting (cf. [Eis95] Example 17.3 and Corollary 17.12).

Definition 16. Let I be an ideal of R such that $IM \neq M$. The depth $\mathrm{depth}_R(I, M)$ of I on M over R is the length of any maximal regular M-sequence contained in I. If $M = R$, we shall have $\mathrm{depth}_R(I) := \mathrm{depth}_R(I, R)$.

Example 17. In $M = k[x, y]/(x)$, multiplication by y is injective: if $y \cdot \overline{f} = 0$ in M, then $yf \in (x)$ in $k[x, y]$. Since x, y is a regular sequence in $k[x, y]$, $x \mid f$, so $\overline{f} = 0$ in M. Hence y is M-regular. After modding out by y, we get

$$M/yM \cong k[x, y]/(x, y),$$

which is a field. Any element of $\mathfrak{m}$ kills this quotient, so no further M-regular element exists.

Thus the longest M-regular sequence in $\mathfrak{m}$ has length 1, and

$$\mathrm{depth}_{\mathfrak{m}}(M) = 1.$$

Proposition 18. Let I be an ideal of R and P be a prime ideal and assume that M is finitely generated.

- Regular sequences are stable under localization at prime ideals, meaning if P is a prime ideal in the support of M, then any M-regular sequence in P localizes to an M_P-regular sequence
- $\mathrm{depth}_R(I, M) \leq \mathrm{depth}_{R_P}(I_P, M_P)$
- There exists a maximal ideal $\mathfrak{m}$ in the support of M such that $\mathrm{depth}_R(I, M) = \mathrm{depth}_{R_{\mathfrak{m}}}(I_{\mathfrak{m}}, M_{\mathfrak{m}})$

Proof. For the first statement, note that quotients and non-zero divisors are stable under localization. Furthermore, Nakayama's lemma gaurantees that $I_\mathfrak{m} M_\mathfrak{m} \neq M_\mathfrak{m}$.
For the second statement, choose a maximal regular M-sequence $x_1, \dots, x_r$ contained in I. Since I consists of zero divisors on $M/(x_1, \dots x_r)M$, I is contained in the union of associated primes of $M/(x_1, \dots, x_r)M$, and since the set of associated primes is finite, prime avoidance shows that I is contained in one of them. Localizing at this prime, or at any prime containing it, will preserve the depth of I on M. □

Lemma 19. *If R is a local ring with maximal ideal $\mathfrak{m}$, M is a finitely generated R-module, I is an ideal of R, and $x \in R$. Then*

$$depth_R((I, x), M) \leq depth_R(I, M) + 1$$

Proof. Cf. [Eis95] Lemma 18.3. □

Proposition 20. Let R be a ring and let M and N be finitely generated R-modules. If $\mathrm{Ann}M + \mathrm{Ann}N = R$ the $\mathrm{Ext}^r_R(M, N) = 0$ for all r. Otherwise $\mathrm{depth}_R(\mathrm{Ann}M, N)$ is the smallest number r for which $\mathrm{Ext}^r_R(\mathrm{Ann}M, N)$ does not vanish.

Proof. Cf. [Eis95] Proposition 18.4. □

Corollary 21. $\mathrm{pd}_R(M) \geq \mathrm{depth}_R(annM, R)$.

Proof. Take $M = R$ in the previous proposition. □

Corollary 22. Let $0 \to N' \to N \to N" \to 0$ be an exact sequence of R-modules. Then:

1. $\mathrm{depth}_R(N'') \geq \min(\mathrm{depth}_R(N), \mathrm{depth}_R(N') - 1)$, and
2. $\mathrm{depth}_R(N') \geq \min(\mathrm{depth}_R(N), \mathrm{depth}_R(N'' + 1))$.

Definition 23. The **projective dimension** of an R-module M, denoted by $\mathrm{pd}_R(M)$ is the infimum of lengths of projective resolutions of M. The **injective dimension** $\mathrm{id}_R(M)$ is defined analogously. The **global dimension** of a ring R, denoted by $\mathrm{gldim}(R)$ is the supremum of the global dimensions of finitely generated modules over it.

Example 24. The projective dimension of $\mathbb{F}_p$ over $\mathbb{Z}_p$ is 1; we have a short exact sequence

$$0 \to \mathbb{Z}_p \xrightarrow{\cdot p} \mathbb{Z}_p \to \mathbb{F}_p \to 0$$

which is a projective resolution of $\mathbb{F}_p$ of length 1. Since $\mathbb{F}_p$ itself is not projective, we have $\mathrm{pd}_{\mathbb{Z}_p}(\mathbb{F}_p) = 1$. The global dimension of $\mathbb{Z}_p$ is also 1. This follows from the structure theorem of finitely generated modules over a PID; The free part is projective, and the torsion terms have exact sequences similar to the one above and can thus be shown to have projective dimension 1. Since projectivity is stable under direct sums, so is the projective dimension, hence the projective dimension of any module over Z_p is 1 and so is the global dimension. Indeed this proof works for any DVR.

Theorem 25. *For a ring R The following are equivalent:*

1. *$gldimR \leq n$,*
2. *$pd_R(R/I) \leq n$ for every ideal I,*

3. $id_R(M) \le n$ *for all R-modules M, and*

4. $Ext^i_R(M, N) = 0$ *for all R-modules M, N.*

Proof.

$1 \implies 2$ is trivial.

$2 \implies 3$: Suppose 2 holds and let

$$0 \to M \to E_0 \to \cdots \to E_{n-1} \to X \to 0$$

be an exact sequence with E_i injective. If we show that X must be injective, then we have found an injective resolution of M of length n, and we are done. Break the sequence into short exact sequences and consider the long exact sequences given by the derived functor $Ext^\bullet_R(R/I, -)$ to obtain

$$Ext^1_R(R/I, X) \cong Ext^{n+1}_R(R/I, M)$$

The latter being zero by evaluating through any projective resolution of length $\le n$. Computing $Ext^1_R(R/I, X)$ from a projective resolution of R/I, we see that this hypothesis is equivalent to saying that if $\psi : I \to X$ is any map, then there is a map $R \to X$ such that the composition $I \to R \to X$ is ψ. Baer's criterion then gives the injectivity of X.

$3 \implies 4$: Computing $Ext_R(M, N)$ from an injective resolution of M gives the result.

$4 \implies 1$: Suppose 4 holds, and let

$$0 \to X \to F_n \to \cdots \to F_0 \to M \to 0$$

be an exact sequence with F_i projective for all i. As in the proof of $2 \implies 3$, it suffices to show that X is projective. Splitting the sequence into short exact sequences and applying $Ext^\bullet_R(-, N)$ gives

$$Ext^1_R(X, N) \cong Ext^{n+1}_R(M, N) = 0.$$

To show that X is projective, consider a projective resolution of X

$$P : \cdots \to P_1 \to P_0 \to X \to 0.$$

Let N be the kernel of the map $P_0 \to X$. The map $P_1 \to P_0 \to X$ is a cycle of $Hom(P, N)$ and thus defines an element of $Ext^1_R(X, N)$; since this group vanishes, the element is a boundary so there is a map $P_0 \to N$ extending the map $P_1 \to X$. This map is a splitting of the inclusion $N \to P_0$, and thus Coker$(P_0 \to X)$ splits as well.

□

In the local case, the notions of free and projective modules coincide, leading us to think about "locally free" modules.

Definition 26. An R-module M is said to be **locally free** if $M_\mathfrak{m}$ is free for every maximal ideal (and therefore every prime ideal) $\mathfrak{m}$ of R.

Theorem 27. *Let M be a finitely generated module over a noetherian ring R. Then M is projective if and only if it is locally free.*

Proof. Cf. [Eis95, Theorem 19.7] □

The following corollary is important to show that every finitely generated module over a regular local ring has a finite free resolution of length (at most) n.

Corollary 28. If $x = (x_1, \dots, x_n)$ is a regular sequence, the $K(x)$ is a free resolution of $R/(x_1, \dots, x_n)$. In particular, if R is a regular local ring, and x gives a minimal set of generators for the maximal ideal of R, then the Koszul complex $K(x)$ is a finite free resolution of the residue class field of R.

Proof. The first statement is essentially Corollary (15). The second statement relies on the fact that a minimal set of generators for the maximal ideal forms a regular sequence, which is Proposition (12). □

The following theorem (the Jacobi criterion, which is essentially the inverse function theorem) shows the connection between regularity and smoothness

Theorem 29. *Let $S = k[x_1, \dots, x_n]$ where k is a field, $I = (f_1, \dots, f_n)$ be an ideal, and $R = S/I$. Let P be a prime ideal of S containing I and let $\kappa(P)$ be the residue class field at P. Let c be the codimension of I_P in S_P.*

1. *The Jacobian matrix*
$$J := (\delta f_i/\delta x_j)_{i,j}$$
has rank c when taken modulo P.

2. *If k has characteristic $p > 0$ and $\kappa(P)$ is seperable over k, then R_P is a regular local ring if and only if J modulo P has rank c.*

Proof. Cf. [Eis95] Theorem 16.19. □

Definition 30. A complex $X : \dots \to X_i \to X_{i+1} \to \dots$ over a local ring R with maximal ideal $\mathfrak{m}$ is said to be **minimal** if the maps in $X \otimes R/\mathfrak{m}$ are all zero.

Example 31. The complex $0 \to \mathbb{Z}_p \xrightarrow{\cdot p} \mathbb{Z}_p \to 0$ is minimal over $\mathbb{Z}_p$.

Lemma 32. *A free resolution $(F_i, d_i)_i$ over a regular local ring is a minimal complex if and only if a basis of F_i maps onto a minimal set of generators of coker d_{i+1}.*

Proof. Let R be a regular local ring with maximal ideal $\mathfrak{m}$, and let d_0 be the natural map $F_0 \to \mathrm{Coker}(d_1)$. For any ≥ 0, consider the induced surjective $R/\mathfrak{m}$-linear map

$$F_{n-1}/\mathfrak{m}F_{n-1} \to \mathrm{Coker}(d_n)/\mathfrak{m}\mathrm{Coker}(d_n).$$

Nakayama's lemma tells us that a basis for the vector space on the righ is a minimal set of generators of $\mathrm{Coker}(d_n)$. Thus the second condition of the lemma is satisfied if and only if this surjective linear map is an isomorphism. This is equivalent to the condition that the image of d_n is in $\mathfrak{m}F_{n-1}$, which is the condition of minimality. □

Corollary 33. If R is a local rings with residue class field k, and M is a finitely generated nonzero R-module, then the projective dimension of M is the length of every minimal free resolution of M. Furthermore, $\mathrm{pd}_R(M)$ is the smallest integer $i \geq 0$ such that $\mathrm{Tor}_{i+1}^R(k, M) = 0$. Thus the global dimension of R is equal to the projective dimension of the residue class field k.

Proof. Cf. [Eis95] Corollary 19.5. □

Theorem 34. *Let R be a local ring. Then R is a regular if and only if $gldim R = \dim R < \infty$.*

Proof. Let $x_1, \dots, x_n$ be generators of the maximal ideal of R. Then $K(x_1, \dots, x_n)$ is a minimal free resolution of length n of the residue class field of R, therefore $n = \mathrm{pd}_R(k) = \mathrm{gldim} R$. □

Proposition 35. Every finitely generated $k[x_1, \dots, x_n]$-module has a finite free resolution.

Proof. Cf. [Eis95] Corollary 19.8. □

Theorem 36. *A local ring has finite global dimension if and only if it is regular.*

Proof. Suppose R is a regular local ring and let $(x_1, \dots, x_n)$ be a minimal set of generators for the maximal ideal. Then the Koszul complex $K(x_1, \dots, x_n)$ gives a free resolution of the residue class field of R. For the other direction see [Eis95, Theorem 19.12]. □

Proposition 37. Every localization of a regular local ring is regular and every localization of a polynomial ring over a field is regular.

Proof. Cf. [Eis95, Corollary 19.14]. □

Proposition 38. A noetherian ring R is regular if and only if $R[x]$ is regular.

Theorem 39. *Let R be a local ring with maximal ideal $\mathfrak{m}$, and let M be a finitely generated R-module of finite projective dimension. Then*

$$pd_R M = depth_R(\mathfrak{m}, R) - depth_R(\mathfrak{m}, M)$$

Definition 40. A ring R is said to be **Cohen-Macauly** if $\mathrm{depth}_R(\mathfrak{m}) = codim(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of R. A module M over R is said to be Cohen-Macauly if $\mathrm{depth}_R(M) = dim_R(M)$. M is **maximal Cohen-Macauly (MCM)** if $\mathrm{depth}_R(M) = dim(R)$.

If a scheme is locally Cohen-Macauly at a point P, then P cannot lie in the intersection of components with different dimensions. This hints as to why Cohen-Macauly rings are important for the study of singularities; We will see later that for certain nice rings, the singularity category coincides with the stablization of the category of MCM modules over R.

Proposition 41. For any finitely generated R-module M we have $\mathrm{depth}_R(M) \leq \dim_R(M) \leq \dim(R)$

Proof. The second inequality is clear. To show that the first one holds, let $(x_1, \dots, x_n)$ be a regular M-sequence. We claim that

$$\dim M/(x_1, \dots, x_n)M = \dim M - n$$

□

Proposition 42. Let R be a local ring. If there exists a finitely generated module of projective dimension equal to the dimension of R, then R is Cohen-Macauly. If, on the other hand, R is Cohen-Macauly, then the projective dimension of a finite projective dimensional module M coincides with the dimension of R if and only if the maximal ideal is associated to M, i.e., it is the annihalator of an element of M

Definition 43. The stable category of maximal Cohen-Macauly R-modules $\underline{\mathrm{MCM}}(R)$ is the additive category whose objects are MCM R-modules, and whose hom-groups are given by $\mathrm{Hom}_R(M, M')/P$, where P is the subgroup of morphisms from M to M' which factor through a finitely generated projective R-module.

Definition 44. A ring is called **Gorenstein** if it has finite injective dimension as a module over itself.

Proposition 45. A local noetherian ring R with residue field k is Gorenstein if and only if there exists n such that $\mathrm{Ext}^i_R(k, R)$ vanishes for all $i > n$.

Proof. If R is Gorenstein, then calculating Ext from any finite injective resolution gives the result. The other direction follows from Theorem 25. □

Proposition 46. Suppose R is Gorenstein and M is a finitely generated R-module. Then $\mathrm{Ext}^r_R(M, R)$ vanishes for all $r > 0$ if and only if M is maximal Cohen-Macauly.

Proof. [Sym22] Lemma 5.2.13 □

3 Matrix Factorizations

Let R be a commutative ring with 1 and $\sigma \in R$.

Definition 47. The category of **matrix factorizations** of σ over R, denoted by $MF(R, \sigma)$, is the category whose objects are quadruples (F, G, ϕ, ψ) consisting of finitely generated free R-modules F, G and R-linear maps $\phi : F \to G$ and $\psi : G \to F$ such that $\phi\psi = \sigma \cdot id_G$ and $\psi\phi = \sigma \cdot id_F$. A morphism of matrix factorizations $f = (f_1, f_2) : (F, G, \phi, \psi) \to (F', G', \phi', \psi')$ is a pair of R-linear maps $f_1 : F \to F'$ and $f_2 : G \to G'$ such that the following diagram commutes

$$\begin{array}{ccccc} F & \xrightarrow{\phi} & G & \xrightarrow{\psi} & F \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} \\ F' & \xrightarrow{\phi'} & G' & \xrightarrow{\psi'} & F' \end{array}$$

The set of morphisms from M to M' is denoted by $\mathrm{Hom}_{MF}(M, M')$.

Proposition 48. The category $MF(R, \sigma)$ is R-linear. More precisely:

1. The operation $(f_1, f_2)+(g_1, g_2) = (f_1+g_1, f_2+g_2)$ makes $\mathrm{Hom}_{MF}(M, M')$ an abelian group.

2. The operation $a(f_1, f_2) = (af_1, af_2)$ makes $\mathrm{Hom}_{MF}(M, M')$ an R-module.

3. Composition of morphisms is bilinear, that is for all matrix factorizations M, M', M'' and all $g, h \in \mathrm{Hom}_{MF}(M, M')$ and $f \in \mathrm{Hom}_{MF}(M', M'')$ we have $f \circ (g + h) = f \circ g + f \circ h$.

4. The zero object is given by $0 = (0, 0, 0, 0)$.

5. Finite direct sums of matrix factorizations exist, and are given by the natural rule $(F, G, \phi, \psi) \oplus (F', G', \phi', \psi') = (F \oplus F', G \oplus G', \phi \oplus \phi', \psi \oplus \psi')$

Proof. 1. We can view $Hom_{MF}(M, M')$ as a subset of $Hom_R(F, F') \times Hom_R(G, G')$ by identifying a morphism (f_1, f_2) of matrix factorizations to the pair (f_1, f_2). In this way, we can check that $Hom_{MF}(M, M')$ is in fact a subgroup under addition. To do this, let $(f_1, f_2), (g_1, g_2) \in Hom_{MF}(M, M')$. Then we need to check that the diagram

$$\begin{CD} F @>{\phi}>> G @>{\psi}>> F \\ @V{f_1-g_1}VV @VV{f_2-g_2}V @VV{f_1-g_1}V \\ F' @>{\phi'}>> G' @>{\psi'}>> F' \end{CD}$$

commutes. Which is clear since

$$\begin{aligned} (f_2 - g_2)\phi &= f_2\phi - g_2\phi \\ &= \phi' f_1 - \phi' g_1 \quad \text{since } f, g \in Hom_{MF}(M, M') \\ &= \phi'(f_1 - g_1) \end{aligned}$$

an analagous calculation shows that the second square also commutes, thus we are done.

2. given a commutative diagram

$$\begin{CD} F @>{\phi}>> G @>{\psi}>> F \\ @V{f_1}VV @VV{f_2}V @VV{f_1}V \\ F' @>{\phi'}>> G' @>{\psi'}>> F' \end{CD}$$

the diagram

$$\begin{CD} F @>{\phi}>> G @>{\psi}>> F \\ @V{af_1}VV @VV{af_2}V @VV{af_1}V \\ F' @>{\phi'}>> G' @>{\psi'}>> F' \end{CD}$$

clearly commutes.

3. This follows from bilinearity of composition of R-module homomorphisms.

4. First, note that the zero quadruple is in fact a matrix factorization of σ since $0 = id_0 = \sigma id_0 = 0 \cdot 0$. It is clear that $Hom_{MF}(0, M) = 0$ and $Hom_{MF}(M, 0) = 0$, thus making 0 the zero object.

5. It follows from the existence of direct sums of R-modules that the direct sum of matrix factorizations exists as a quadruple. to check that it is a matrix factorization of σ, note that

$$(\phi \oplus \phi') \circ (\psi \oplus \psi') = (\phi \circ \psi) \oplus (\phi' \circ \psi') = \sigma id_G \oplus \sigma id_{G'} = \sigma id_{G \oplus G'}$$

and

$$(\psi \oplus \psi') \circ (\phi \oplus \phi') = (\psi \circ \phi) \oplus (\psi' \circ \phi') = \sigma id_F \oplus \sigma id_{F'} = \sigma id_{F \oplus F'}$$

□

Definition 49. Two morphisms $f = (f_1, f_2), g = (g_1, g_2)$ from $M = (F, G, \phi, \psi)$ to $M' = (F', G', \phi', \psi')$ are said to be **homotopic**, denoted by $f \sim g$ if a homotopy from f to g exists; that is a pair of morphisms $t : F \to G'$ and $s : G \to F'$ such that $f_1 - g_1 = s\phi + \psi' t$ and $f_2 - g_2 = t\psi + \phi' s$, depicted by the diagram:

$$
\begin{array}{ccccc}
F & \xrightarrow{\phi} & G & \xrightarrow{\psi} & F \\
{\scriptstyle f_1 - g_1}\downarrow & \overset{s}{\swarrow} & \downarrow{\scriptstyle f_2 - g_2} & \overset{t}{\swarrow} & \downarrow{\scriptstyle f_1 - g_1} \\
F' & \xrightarrow{\phi'} & G' & \xrightarrow{\psi'} & F'
\end{array}
$$

A morphism is said to be **nullhomotopic** if it is homotopic to the zero morphism. Two matrix factorizations M, M' are said to be homotopy equivalent, denoted by $M \sim M'$, if there are morphisms $f : M \to M'$ and $f' : M' \to M$ such that the compositions $f \circ f'$ and $f' \circ f$ are homotopic to the identity morphisms.

Note that two morphisms f, f' are homotopic if and only if $f - f'$ is nullhomotopic.

Proposition 50. Let $M = (F, G, \phi, \psi)$ and $M' = (F', G', \phi', \psi')$. The set of nullhomotopic morphisms forms an R-submodule of $\mathrm{Hom}_{MF}(M, M')$.

Proof. Let $f, g \in \mathrm{Hom}_{MF}(M, M')$ be nullhomotopic via the homotopies $(s, t), (s', t')$ from f and g to zero respectively and let $a, b \in R$. Then $(as - bs', at - bt')$ is a homotopy from $af - bg$ to 0 since

$$
\begin{aligned}
af_1 - bg_1 &= a(s\phi + \psi' t) - b(s'\phi + \psi' t') \\
&= (as - bs')\phi + \psi'(at - bt')
\end{aligned}
$$

and

$$
\begin{aligned}
f_2 - g_2 &= a(t\psi + \phi' s) - b(t'\psi + \phi' s') \\
&= (at - bt')\psi + \phi'(as - bs')
\end{aligned}
$$

□

Definition 51. The homotopy category of matrix factorizations $HMF(R, \sigma)$ is the category whose objects are matrix factorizations of σ over R and where the set of morphisms between two factorizations M, M' is given by $Hom_{HMF}(M, M') = Hom_{MF}(M, M')/\sim$.

We can equip $MF(R, \sigma)$ with an endofunctor Σ that sends an object (F, G, ϕ, ψ) to a "shift" $(G, F, -\psi, -\phi)$, and a morphism given by

$$
\begin{array}{ccccc}
F & \xrightarrow{\phi} & G & \xrightarrow{\psi} & F \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} \\
F' & \xrightarrow{\phi'} & G' & \xrightarrow{\psi'} & F'
\end{array}
$$

to the morphism given by

$$
\begin{array}{ccccc}
G & \xrightarrow{-\psi} & F & \xrightarrow{-\phi} & G \\
{\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\
G' & \xrightarrow{-\psi'} & F' & \xrightarrow{-\phi'} & G'
\end{array}
$$

Proposition 52. Σ, also called the **suspension**, is an additive autoequivalence of $MF(R, \sigma)$.

Proof. Σ is an autoequivalence as it clearly satisfies $\Sigma^2 = id_{MF}$. Furthermore, an equivalence of additive categories is automatically additive. $\square$

Remark 53. Σ clearly preserves nullhomotopic factorizations, thus it induces an additive autoequivalence on $\mathrm{HMF}(R, \sigma)$.

Proposition 54. Let R be an integral domain and let (F, G, ϕ, ψ) be a matrix factorization of a non-zero $\sigma \in R$. Then the ranks of F and G coincide. Therefore we can assume that the modules in a matrix factorization are free of the same finite rank, thus, we can define the rank of a matrix factorization over a domain to be the rank of those modules.

Proof. It clearly suffices to show that ϕ preserves linear independence. For by symmetry, the same holds for ψ, and thus the maps map bases onto linearly independent sets, thus showing that the ranks coincide. To that end, let $a, b \in F$ be linearly independent and $\alpha, \beta \in R$ such that

$$\alpha\phi(a) + \beta\phi(b) = 0.$$

Applying ψ to both sides gives

$$(\alpha\sigma)b + (\beta\sigma)b = 0.$$

Since a and b are independent, the coefficients are zero. Since R is a domain and σ is non-zero, then $\alpha = \beta = 0$. $\square$

Definition 55. Let R be an integral domain and $\sigma \in R$ a nonzero element. The **rank** of a matrix factorization $M = (F, G, \phi, \psi) \in MF(R, \sigma)$ is the rank of F (or equivalently, the rank of G) as a free R module.

From now on, we will assume that all matrix factorizations over an integral domain are of the form (R^m, R^m, ϕ, ψ) since for any $(F, G, \phi, \psi) \in MF(R, \sigma)$ we have an isomorphism given by

$$\begin{array}{ccccc} F & \xrightarrow{\phi} & G & \xrightarrow{\psi} & F \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ R^m & \xrightarrow{g\phi f^{-1}} & R^m & \xrightarrow{f\psi g^{-1}} & R^m \end{array}$$

where $f : F \to R^m$ and $g : G \to R^m$ are isomorphisms.

4 The Singularity Category

In this section, we recall the definitions of the singularity category and the equivalence with the stable category of MCM modules. Recall that an abelian category is a category whose hom-sets are abelian groups with $\mathbb{Z}$-bilinear composition of morphisms, has a zero object, admits finite products, kernels, and cokernels, and monomorphisms and epimorphisms are normal; that is, they are kernels and cokernels (respectively) of some morphisms. The bounded homotopy category $K^b(\mathfrak{A})$ of an abelian category A is the category whose objects are bounded complexes in $\mathfrak{A}$ and whose morphisms are morphisms of complexes up to homotopy equivalence. The bounded homotopy category is a triangulated category in which the acyclic complexes form a null system. The quotient of the bounded homotopy category by acyclic complexes gives the bounded derived category $D^b(\mathfrak{A})$ of the abelian category $\mathfrak{A}$. The bounded derived category of a ring R $D^b(R)$ is the defined to be the bounded derived category of the category of R-modules.

4.1 Abelian Categories

Abelian categories are essentially categories in which morphisms admit kernels and cokernels. This allows us to do homological algebra in a very natural way. In this part, we sketch the theory of abelian categories following [**Kash06**]. Let $\mathcal{A}$ be a category and R be a commutative ring with 1.

Definition 56. $\mathcal{A}$ is called an *R-linear category* if its Hom-sets are R-modules and composition is R-bilinear. If $R = \mathbb{Z}$, then we call $\mathcal{A}$ a *preadditive category*

Example 57. The cateory *Ab* of abelian groups is a preadditive category. The category of R-modules is R-linear.

Definition 58. An object $A \in \mathcal{A}$ is called an *initial object* if there exists a unique morphism $A \to X$ for any object $X \in \mathcal{A}$. A is called a *terminal object* if there exists a unique morphism $X \to A$ for any object $X \in \mathcal{A}$. A is called a *zero object* if it is both initial and terminal.

Example 59. The zero group is a zero object in *Ab*. The singletons are terminal objects in *Set*.

It is clear that initial, terminal, and zero objects are unique up to unique isomorphism.

Definition 60. Let A, B be objects of $\mathcal{A}$. A *product* of A and B is an object $A \times B$ and a pair of morphisms $A \times B \to A$ and $A \times B \to B$ such that for any pair of morphisms $T \to A$ and $T \to B$, there exists a unique morphism $T \to A \times B$ such that the diagram

$$\begin{array}{ccc} X & \longrightarrow & B \\ \downarrow & \searrow & \downarrow \\ A & \longrightarrow & A \times B \end{array}$$

commutes. The dual concept of a product is a *coproduct*.

The product (and the coproduct), if it exists, is unique up to unique isomorphism. So we can talk about *the* product and coproduct of objects.

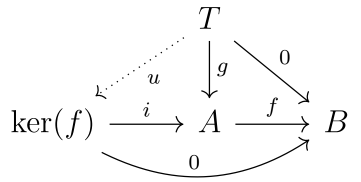
Example 61. In the category of sets, the cartesian product is the product of two sets, while the disjoint union gives the coproduct.

Definition 62. $\mathcal{A}$ is called *additive* if it is preadditive, has a zero object, and is closed under products. A functor $F : \mathcal{A} \to \mathcal{B}$ is between additive categories is called an *additive functor* if $F(f + g) = Ff + Fg$ for all morphisms f, g in $\mathcal{A}$, that is; the functor induces homomorphisms of groups between the Hom-sets.

Proposition 63. Finite products and coproducts coincide in additive categories.

Proof. See [tb11]. □

Definition 64. Let $f : A \to B$ be a morphism in an additive category $\mathcal{A}$. The *kernel* of f is a morphism $i : \ker(f) \to A$ such that $fi = 0$, and for any morphism $g : T \to A$ with the property that $fg = 0$, there exists a unique morphism $u : T \to \ker(f)$ such that the diagram



commutes. The dual concept is called a *cokernel*.

The (co)kernel, as with almost every object defined by universal properties, is unique up to unique isomorphism if it exists.

Proposition 65.

- The (co)kernel of f is the zero object if and only if f is a monomorphism (epimorphism).
- The map $\ker(f) \to A$ $(B \to \mathrm{Coker}(f))$ above is an isomorphism if and only if $f = 0$.

Definition 66. A category is said to be *abelian* if it's additive, admits all kernels and cokernels, all monomorphisms are kernels and all epimorphisms are cokernels of some morphisms.

Definition 67. A *cochain complex* (or simply: a complex) over an additive category $\mathcal{A}$ is a family $(X^\bullet, d_X^\bullet)$ indexed by $\mathbb{Z}$ where X^i are objects of $\mathcal{A}$, called the terms of the complex, and $d_X^i : X^i \to X^{i+1}$ are morphisms in $\mathcal{A}$, called the *differentials* of the complex, such that $d_X^{i+1} d_X^i = 0$ (or $d_X^2 = 0$ for short). We represent a complex X diagramatically as:

$$X : \ldots \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} \ldots$$

Example 68. Exact sequences of modules over a ring are examples of complexes.

Definition 69. A complex $X^\bullet$ is said to be *bounded below* if there exists $n \in \mathbb{Z}$ such that $X^i = 0$ for all $i < n$. It is *bounded above* if $X^i = 0$ for $i > n$ for some $n \in \mathbb{Z}$. It is simply *bounded* if it is bounded below and above.

Definition 70. A morphism of complexes $f : X^\bullet \to Y^\bullet$ is a family of morphisms $(f^i : X^i \to Y^i)_{i \in \mathbb{Z}}$ such that the diagrams

$$\begin{array}{ccc} X^i & \xrightarrow{d_X^i} & X^{i+1} \\ \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} \\ Y^i & \xrightarrow{d_Y^i} & Y^{i+1} \end{array}$$

commute.

We denote by $C(\mathcal{A})$ the category of complexes over $\mathcal{A}$. We denote by $C^b(\mathcal{A})$ the full subcategory of bounded complexes. It is not hard to see that $C(\mathcal{A})$ is again an additive category. In fact, if $\mathcal{A}$ is abelian, then so is $C(\mathcal{A})$.

Definition 71. A *homotopy* between two morphisms of complexes $f, g : X^\bullet \to Y^\bullet$ is a family of morphisms $(s^i : X^i \to Y^{i-1})_{i \in \mathbb{Z}}$ such that

$$f^i - g^i = s^{i+1} d_X^i + d_Y^{i-1} s^i.$$

In this case, f and g are said to be *homotopic.* If $g = 0$, f is said to be *nullhomotopic.* A morphism $f : X^\bullet \to Y^\bullet$ is said to be a homotopy equivalence if there exists a morphism $g : Y^\bullet \to X^\bullet$ such that fg and gf are homotopic to the identities on their domains.

It's not difficult to see that the null homotopic morphisms form a subgroup of the Hom-sets of $C(\mathcal{A})$, which motivates the following definition:

Definition 72. The *homotopy category* $K(\mathcal{A})$ of an additive category $\mathcal{A}$ is the category whose objects are complexes over $\mathcal{A}$ and whose Hom-sets are the Hom-sets of $C(\mathcal{A})$ modulo nullhomotopic morphisms.

Definition 73. The *i-th cohomology group* of a complex $X = (X^\bullet, d_X^\bullet)$ is given by

$$H^i(X) := \frac{\ker d_X^i}{\mathrm{im} d_X^{i-1}}.$$

A complex is said to be exact at the i-th term if the i-th cohomology group vanishes. If all cohomology groups of a complex vanish, the complex is called *acyclic.*

Some diagram chasing enables us to see that morphisms of complexes induce maps on cohomology groups.

Proposition 74. Homotopic morphisms induce the same map on cohomology groups, in particular, homotopy equivalences induce isomorphisms on cohomology groups.

Definition 75. A morphism of complexes $f : X \to Y$ is called a *quasi-isomorphism* if it induces isomorphisms on cohomology groups.

4.2 Triangulated Categories

An issue that arises when dealing with complexes is that $K(\mathcal{A})$, although additive, is not necessarily an abelian category even when $\mathcal{A}$ is. However, it has the slightly more complicated structure of a triangulated category, which we describe here.

Definition 76. Let $\mathcal{A}$ be an additive category with an additive autoequivalence $[1] : \mathcal{A} \to \mathcal{A}$, sending an object A to $A[1]$. We call such a pair $(A, [1])$ an additive *category with translation.* An additive functor of additive categories with translation $F : \mathcal{A} \to \mathcal{A}'$ is one that commutes with the translations; that is it satisfies $F \circ [1] \simeq [1] \circ F$.

Definition 77. Let $F, F' : \mathcal{A} \to \mathcal{A}'$ be functors of additive categories with translation. A morphism of functors of additive categories with translation is a morphism of functors $\theta : F \to F'$ such that the diagram

$$\begin{array}{ccc} F[1] & \xrightarrow{\theta[1]} & F'[1] \\ \downarrow & & \downarrow \\ [1]F & \xrightarrow{[1]\theta} & [1]F' \end{array}$$

where the downwards arrows are the isomorphisms from the previous definition.

Definition 78. A *triangle* in an additive category with translation is a sequence

$$X \to Y \to Z \to X[1]\,.$$

A morphism of triangles is a commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f[1]} \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]
\end{array}$$

Definition 79. An additive category with translation $(\mathcal{A}, [1])$ is called a *triangulated category* if it contains a family of triangles, called exact (or distinguished) triangles satisfying the following axioms:

TR0 A triangle isomorphic to an exact triangle is exact.

TR1 For any X in $\mathcal{A}$, the triangle $X \xrightarrow{id_X} X \to 0 \to X[1]$ is an exact triangle.

TR2 For any morphism $X \to Y$ in $\mathcal{A}$ there exists an exact triangle $X \to Y \to Z \to X[1]$.

TR3 The triangle $X \xrightarrow{f} Y \to Z \to X[1]$ is exact if and only if $Y \to Z \to X[1] \xrightarrow{f[1]} Y[1]$ is exact.

TR4 A commutative diagram

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow{\scriptstyle f} & & \downarrow & & & & \downarrow{\scriptstyle f[1]} \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & & X'[1]
\end{array}$$

where the rows are exact triangles can be extended to a morphism of triangles

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f[1]} \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]
\end{array}$$

TR5 Given exact triangles

$$X \xrightarrow{f} Y \xrightarrow{h} Z' \longrightarrow X[1]$$

$$Y \xrightarrow{g} Z \xrightarrow{k} X' \longrightarrow Y[1]$$

$$X \xrightarrow{g \circ f} Z \xrightarrow{l} Y' \longrightarrow X[1]$$

There exists an exact triangle

$$Z' \xrightarrow{u} Y' \xrightarrow{v} X' \xrightarrow{w} Z'[1]$$

such that we have a commutative diagram

$$\begin{CD}
X @>{f}>> Y @>{g}>> Z' @>>> X[1] \\
@V{id}VV @V{h}VV @V{u}VV @V{id}VV \\
Y @>{g}>> Z @>{k}>> X' @>>> Y[1] \\
@V{f}VV @V{id}VV @V{v}VV @V{f[1]}VV \\
X @>{g\circ f}>> Z @>{l}>> Y' @>>> X[1] \\
@V{h}VV @V{l}VV @V{id}VV @V{h[1]}VV \\
Z' @>{u}>> Y' @>{v}>> X' @>{w}>> Z'[1]
\end{CD}$$

Definition 80. 1. A *triangulated functor* between triangulated categories is an additive functor that preserves exact triangles. An equivalence of triangulated categories is a triangulated functor which is also an equivalence of categories.

2. A morphism of triangulated functors is a morphism of additive categories with translation.

3. A triangulated subcategory of a triangulated category is one for which the inclusion functor is triangulated.

Definition 81. Let $\mathcal{T}$ be a triangulated category and let $\mathcal{A}$ be an abelian category. A functor from $F : \mathcal{T} \to \mathcal{A}$ is called a *cohomological functor* or an *exact functor* if for any exact triangle $X \to Y \to Z \to X[1]$, the sequence $F(X) \to F(Y) \to F(Z)$ is exact in $\mathcal{A}$.

Note that together with axiom (TR3), a cohomological functor H turns exact triangles $X \to Y \to Z \to X[1]$ to long exact sequences

$$\cdots \to H(X[i]) \to H(Y[i]) \to H(Z[i]) \to H(X[i+1]) \to \dots$$

where $[i] := [1]^i$.

Definition 82. Let $\mathcal{T}$ be a triangulated category. A **null system** in $\mathcal{T}$ is a collection $\mathcal{N}$ of objects of $\mathcal{T}$ satisfying:

N1 $0 \in \mathcal{N}$

N2 $\mathcal{N}$ is closed under translation and its inverse.

N3 if $X \to Y \to Z \to X[1]$ is an exact triangle with $X, Y \in \mathcal{N}$, then $Z \in \mathcal{N}$.

There is a special collection of morphisms associated to a null system defined as follows:

Definition 83. In the above situation. We define the *multiplicative system* associated to $\mathcal{N}$ as

$$S(\mathcal{N}) := \{(f : X \to Y) \mid \exists \text{ exact triangle } X \xrightarrow{f} Y \to Z \to X[1] \text{ with } Z \in \mathcal{N}\}$$

The reason this is called a multiplicative system is that it satisfies certain axioms that make them easy to invert in some bigger category. Those axioms are discussed in detail in [**stacks-project**] and will not be repeated here as we will not be using them in the main part of this thesis.

4.3 The Derived Category and The Singularity Category

The derived category of an abelian category is essentially the homotopy category with quasi-isomorphisms inverted; i.e. they become isomorphisms in the derived category. To make this concept rigorous, localization of categories was introduced. We recall those concepts in this part.

Definition 84. Let $\mathcal{C}$ be a category and S be a collection of morphisms of $\mathcal{C}$. A *localization* of $\mathcal{C}$ by S is a category $\mathcal{C}[S^{-1}]$ and a functor $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ such that

- For all $s \in S$, $Q(s)$ is an isomorphism.
- Any functor $F : \mathcal{C} \to \mathcal{T}$ such that $F(s)$ is an isomorphism for all $s \in S$ factors through Q
- The functor $(- \circ Q) : \mathrm{Func}(\mathcal{C}[S^{-1}], \mathcal{T}) \to \mathrm{Func}(\mathcal{C}, \mathcal{T})$ is fully faithful.

It is known that localizations always exist, but there are some set theoretic issues that might arise with them; the general localization procedure yields proper classes that might not be sets. However, those issues vanish in the case where the category is triangulated, and S is the multiplicative set associated to a null system, in which case we view the localization as a quotient of categories by the null system or the subcategory. for more details, refer to [Sta25, Tag 05R1].

Theorem 85. *Let $\mathcal{A}$ be an abelian category. Then $K(\mathcal{A})$ with $X[1]^i := X^{i-1}$ is a triangulated category where the acyclic complexes form a null system.*

Proof. Cf. [**stacks-project**] for the first statement. For the second statement it is clear that 0 is acyclic and that shifts of acyclic complexes are acyclic. (N3) follows from the fact that the functor $H^0(-)$ is cohomological, which gives a long exact cohomology sequence, allowing us to easily compute the cohomology groups of Z to be 0. □

Definition 86. Let $* \in \{b, \}$. The derived category of an abelian category $\mathcal{A}$, $D^*(\mathcal{A})$, is the quotient of $K^*(\mathcal{A})$ by the collection of acyclic complexes. In the case where $\mathcal{A} = (R - mod)$ (finitely generated R-modules), we shall simply write $D^*(R)$.

The most important example for us is the singularity category. Which, following [Orl06, Definition 1.7], can be defined as follows:

Definition 87. Let $\mathcal{T}$ be a triangulated category. An object T is said to be *homologically finite* if for any object $S \in \mathcal{T}$, all $\mathrm{Hom}(T, S[i])$ are trivial except for a finite number of $i \in \mathbb{Z}$. Those objects form a triangulated subcategory, which we denote by $\mathcal{T}_{hf}$.

Definition 88. We define $\mathcal{T}_{sg}$, the *singularity category* of $\mathcal{T}$ to be the quotient of $\mathcal{T}$ by $\mathcal{T}_{hf}$.

Definition 89. A complex of R-modules is said to be *perfect* if it is quasi-isomorphic to a bounded complex of finitely generated projective modules.

In [Orl06, Proposition 1.11], it was shown that the perfect complexes are exactly the homologically finite objects of $D^b(R)$ (Although Orlov did this for the more general case of a certain class of schemes). Which finally allows us to define the singularity category of a ring.

Definition 90. The *singularity category* of a ring R is the quotient of the derived category of the category of finitely generated R-modules $D^b(R)$ by perfect complexes.

It was shown in [**Sym22**] that the singularity category of a Gorenstein ring is equivalent to the stable category of MCM modules over it. Furthermore, note that given a matrix factorization (F, G, ϕ, ψ) we have an exact sequence

$$0 \to F \xrightarrow{\phi} G \to \operatorname{Coker}(\phi) \to 0\,.$$

Since F and G are free modules, this is a projective resolution of $\operatorname{Coker}(\phi)$ of length 1, so that the projective dimension $\operatorname{Coker}(\phi)$ is 1, making is maximal Cohen-Macaulay by an easy calculation using the Auslander-Buchsbaum formula. This gives a functor from the category of matrix factorizations to the category of MCM modules over R, which can be shown to send nullhomotopic factorizations to projective modules, inducing a functor from the homotopy category of matrix factorizations to the stablized category of MCM modules, which can be shown to be an equivalence of categories.

Theorem 91. *The functor*

$$HMF(R,\sigma) \to \underline{MCM(R/\sigma)}\,;\ (F,G,\phi,\psi) \mapsto Coker(\phi)$$

is an equivalence of categories. Thus, the homotopy category of matrix factorizations of σ over R is equivalent to the singularity category of R/σ

Proof. Omitted. See [Sym22, Theorem 5.3.20] for a sketch of the proof or [Lan16, Theorem §5.2]. □

5 Matrix Factorizations over PIDs

We now study matrix factorizations in the interesting case where R is a principal ideal domain and $\sigma \in R/\{0\}$.

Proposition 92. Let R be a PID and $M = (R^m, R^m, \phi, \psi) \neq 0$ be a matrix factorization of $\sigma \in R$ and represent ϕ and ψ with matrices in $\operatorname{Mat}_m(R)$. Then M is isomorphic to a matrix factorization $M' = (R^m, R^m, \phi', \psi')$ where ϕ' and ψ' are diagonal matrices.

Proof. Since R is a PID, we can find invertible matrices S and T such that $D_\phi := S\phi T$ is a diagonal matrix (the Smith normal form of ϕ). Furthermore, we have the adjoint matrix of D_ϕ, denoted by $adj(D_\phi)$, which satisfies $adj(D_\phi)D_\phi = det(D_\phi)E_m$, where E_m is the $m \times m$ identity matrix. Putting all of this together we get

$$\begin{aligned} \phi\psi &= \sigma E_m \\ S^{-1}S\phi TT^{-1}\psi &= \sigma E_m \\ S^{-1}D_\phi T^{-1}\psi &= \sigma E_m \\ adj(D_\phi)D_\phi T^{-1}\psi &= \sigma adj(D_\phi)S \\ det(D_\phi)T^{-1}\psi S^{-1} &= \sigma adj(D_\phi) \end{aligned}$$

The adjoint matrix of a diagonal matrix is diagonal, which means that $det(D_\phi)T^{-1}\psi S^{-1}$ is a diagonal matrix. Since R is a domain and $det(D_\phi) \neq 0$, $D_\psi := T^{-1}\psi S^{-1}$ is diagonal. The pair (T^{-1}, S) gives an isomorphism of matrix factorizations from M to $(R^m, R^m, D_\phi, D_\psi)$ with inverse (T, S^{-1}). □

Corollary 93. Every matrix factorization of $\sigma \in R$ is isomorphic to a direct sum of factorizations of rank 1.

Proof. Given a matrix factorization $M = (R^m, R^m, \phi, \psi)$ whose maps are diagonal matrices, we have

$$M = \bigoplus_{i=1}^{m} (R, R, \phi_i, \psi_i)$$

where ϕ_i and ψ_i are the i^{th} diagonal entries in ϕ and ψ. The proposition tells us that every matrix factorization is isomorphic to such an M. □

We now study matrix factorizations of rank 1 over a PID more closely. For the rest of this discussion, let $M_1 = (R, R, \alpha_1, \beta_1)$ and $M_2 = (R, R, \alpha_2, \beta_2)$ be rank 1 matrix factorizations of σ. We can identify morphisms in $Hom_{MF}(M_1, M_2)$ with elements of R in the following way: a morphism (f_1, f_2) depicted by the diagram

$$\begin{array}{ccccc} R & \xrightarrow{\alpha_1} & R & \xrightarrow{\beta_1} & R \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} \\ R & \xrightarrow{\alpha_2} & R & \xrightarrow{\beta_2} & R \end{array}$$

must satisfy $\alpha_2 f_1 = f_2 \alpha_1$. Viewing this as an equality in R we can see that there exists $f \in R$ such that $f_1 = \frac{\alpha_1}{d} f$ and $f_2 = \frac{\alpha_2}{d} f$ where d is the greatest common factor of α_1 and α_2. So we obtain a map $R \to Hom_{MF}(M_1, M_2)$ sending f to the morphism $(\frac{\alpha_1}{d} f, \frac{\alpha_2}{d} f)$.

Proposition 94. The map $R \to Hom_{MF}(M_1, M_2)$ described above is an isomorphism.

Proof. We first show that the map is a well defined group homomorphism. For $f \in R$, the map $(\frac{\alpha_1}{d} f, \frac{\alpha_2}{d} f)$ gives rise to a diagram

$$\begin{array}{ccccc} R & \xrightarrow{\alpha_1} & R & \xrightarrow{\beta_1} & R \\ \downarrow{\scriptstyle \frac{\alpha_1}{d} f} & & \downarrow{\scriptstyle \frac{\alpha_2}{d} f} & & \downarrow{\scriptstyle \frac{\alpha_1}{d} f} \\ R & \xrightarrow{\alpha_2} & R & \xrightarrow{\beta_2} & R \end{array}$$

The left square is clearly commutative. The right square is commutative since $\alpha_1\beta_1 = \sigma = \alpha_2\beta_2$, hence the map is well defined. To see that it is a group homomorphism take $f, g \in R$. Then $(\frac{\alpha_1}{d} f, \frac{\alpha_2}{d} f) + (\frac{\alpha_1}{d} g, \frac{\alpha_2}{d} g) = (\frac{\alpha_1}{d}(f+g), \frac{\alpha_2}{d}(f+g))$. The map is injective since $(\frac{\alpha_1}{d} f, \frac{\alpha_2}{d} f) = 0$ if and only if $f = 0$, $\alpha_1 = 0$ or $\alpha_2 = 0$, but M_1, M_2 are of rank 1, so the $\alpha' s$ cannot be 0. Hence $f = 0$. It is surjective by the discussion above. □

From now on, we will identify morphisms in $Hom_{MF}(M_1, M_2)$ with elements $f \in R$, and we set $d := (\alpha_1, \alpha_2)$, $l = lcm(\alpha_1, \alpha_2)$ where $(-,-)$ stands for the greatest common divisor -we choose d and l as elements in R in such a way that $dl = \alpha_1\alpha_2$-. We can compose the isomorphism $R \to Hom_{MF}(M_1, M_2)$ with the canonical projection to $Hom_{HMF}(M_1, M_2)$. The kernel of this composition consists of elements of R which give rise to nullhomotopic morphisms from M_1 to M_2. We can describe the kernel as follows.

Proposition 95. In the situation above f gives rise to a nullhomotopic morphism if and only if $(\frac{\sigma}{l}, d)|f$. So $Hom(M_1, M_2) \cong R/(\frac{\sigma}{l}, d)$

Proof. $(\frac{\alpha_1}{d}f, \frac{\alpha_2}{d}f)$ is homotopic to zero if and only if there are elements $s, t \in R$ such that $\frac{\alpha_2}{d}f = s\beta_1 + t\alpha_2$ and $\frac{\alpha_1}{d}f = s\beta_2 + t\alpha_1$. By substituting $\frac{\sigma}{\alpha_1} = \beta_1$ and $\frac{\sigma}{\alpha_2} = \beta_2$ in the first and second equations respectively, we get

$$\frac{\alpha_2}{d}f = s\frac{\sigma}{\alpha_1} + t\alpha_2$$
$$\frac{\alpha_2\alpha_1}{d}f = s\sigma + t\alpha_2\alpha_1$$

and

$$\frac{\alpha_1}{d}f = s\frac{\sigma}{\alpha_2} + t\alpha_1$$
$$\frac{\alpha_1\alpha_2}{d}f = s\sigma + t\alpha_1\alpha_2$$

so the equations are equivalent. By rearranging either one we obtain

$$f = \frac{d\sigma}{\alpha_1\alpha_2}s + dt = \frac{\sigma}{l}s + dt$$

which has a solution (s, t) if and only if $(\frac{\sigma}{l}, d)$ divides f. On the other hand, let $f \in R$ such that $(\frac{\sigma}{l}, d)|f$. Bezout's identity gives us $s, t \in R$ such that

$$f = \frac{\sigma}{l}s + dt = \frac{d\sigma}{\alpha_1\alpha_2} + dt$$

replacing σ by $\alpha_1\beta_1$ and $\alpha_2\beta_2$, simplifying, and rearranging we get

$$\frac{\alpha_2}{d}f = s\beta_1 + t\alpha_2$$
$$\text{and } \frac{\alpha_1}{d}f = s\beta_2 + t\alpha_1$$

which shows that f gives rise to a nullhomotopic morphism $(\frac{\alpha_1}{d}f, \frac{\alpha_2}{d}f)$. □

Corollary 96. Let $M = (R, R, \alpha, \beta)$ be a matrix factorization

1. $\mathrm{End}_{HMF}(M)$ is isomorphic to $R/(\alpha, \beta)$
2. $Hom_{HMF}(M, \Sigma M)$ is isomorphic to $R/(\alpha, \beta)$
3. M is homotopy equivalent to zero if and only if α and β are coprime.

Proof. 1. The greatest common divisor and the least common multiple are both α, therefore $Hom_{HMF}(M, M) \cong R/(\frac{\sigma}{\alpha}, \alpha) = R/(\beta, \alpha)$

2. $Hom_{HMF}(M, \Sigma M) \cong R/(\frac{\sigma}{l}, (\alpha, \beta)) = R/(\frac{(\alpha,\beta)\alpha\beta}{\alpha\beta}, (\alpha, \beta)) = R/(\alpha, \beta)$

3. If $(\alpha, \beta) = 1$, then $End_{HMF}(M) \cong R/(\alpha, \beta)$ vanishes, and so the composition of the zero maps $M \to 0$ and $0 \to M$ is homotopic to the identity in both directions. Going backwards, if M is homotopy equivalent to zero, then the identity is nullhomotopic, and thus $(\alpha, \beta)|1$, which means that α and β are coprime.

□

Proposition 97. M_1 and M_2 are homotopy equivalent if and only if there are $f, g \in R$ such that $L|\frac{l}{d}fg - 1$ where $L = \text{lcm}((\alpha_1, \beta_1), (\alpha_2, \beta_2))$

Proof. Two elements $f, g \in R$ define morphisms between M_1 and M_2 described by the commutative diagram:

$$\begin{array}{ccccc} R & \xrightarrow{\alpha_1} & R & \xrightarrow{\beta_1} & R \\ \downarrow{\scriptstyle \frac{\alpha_1}{d} f} & & \downarrow{\scriptstyle \frac{\alpha_2}{d} f} & & \downarrow{\scriptstyle \frac{\alpha_1}{d} f} \\ R & \xrightarrow{\alpha_2} & R & \xrightarrow{\beta_2} & R \\ \downarrow{\scriptstyle \frac{\alpha_2}{d} g} & & \downarrow{\scriptstyle \frac{\alpha_1}{d} g} & & \downarrow{\scriptstyle \frac{\alpha_2}{d} g} \\ R & \xrightarrow{\alpha_1} & R & \xrightarrow{\beta_1} & R \end{array}$$

These morphisms constitute a homotopy equivalence if the compositions in either direction are homotopic to 1. The composition in either direction is given by the element $\frac{l}{d}fg$. So f and g define a homotopy equivalence if and only if (α_1, β_1) and (α_2, β_2) divide $\frac{l}{d}fg - 1$, which happens exactly when $L|\frac{l}{d}fg - 1$. □

Corollary 98. Two factorizations M_1 and M_2 of rank 1 are homotopy equivalent if and only if $(L, \frac{l}{d}) = 1$.

Proof. if $(L, \frac{l}{d}) \neq 1$, then L and $\frac{l}{d}$ share a non unit common factor q, which does not divide $\frac{l}{d}fg - 1$, so they are not homotopy equivalent. On the other hand, suppose $(L, \frac{l}{d}) = 1$. Then, by Bezout's identity over PIDs, we can find $h, -fg \in R$ such that $Lh + \frac{l}{d}(-fg) = -1$. Rearranging, we obtain $Lh = \frac{l}{d}fg - 1$, which means exactly that $L|\frac{l}{d}fg - 1$, by the previous proposition, we are done. □

Corollary 99. Let $p \in R$ be irreducible. Then the only matrix factorizations of $p^n \in R$ of rank 1 up to homotopy equivalence are:

$$R \xrightarrow{1} R \xrightarrow{p^n} R$$

$$R \xrightarrow{p} R \xrightarrow{p^{n-1}} R$$

$$\dots$$

$$R \xrightarrow{p^{n-1}} R \xrightarrow{p} R$$

Proof. Suppose $M_i = (R, R, p^i, p^{n-i})$ and $M_j = (R, R, p^j, p^{n-j})$ are homotopy equivalent. By symmetry, we can assume that $i \leq j \leq n-j \leq n-i$. In this case, $L = p^j$ and $\frac{l}{d} = p^{j-i}$ are coprime if and only if $i = j$. □

Example 100.
- For a field k, $R = k[x]$, and $\sigma = x^2$, we have precisely one indecomposible object $M = (R, R, x, x)$ with $\text{End}_{\text{HMF}}(M) \cong k[x]/(x, x) = k$, from which it is clear that $D_{sg}(k[x]/(x^2)) \cong (k - mod)$ via the functor mapping $M \mapsto k$ and extended to the rest of the category through commutativity with direct sums.

- The indecomposible objects of $\underline{\text{MCM}}(R/(x^n))$ (R as above) are $\{R/x^i \,|\, 1 \le i \le n-1\}$ (cf. [Sym22] Proposition 5.4.2), corresponding to the matrix factorizations (R, R, x^i, x^{n-1}) for $1 \le i \le n-1$.

Let $\sigma = \prod_p p^{v_p(\sigma)}$ be the prime factorization of σ where $v_p(.)$ is the p-adic valuation. We now give a condition equivalent to homotopy equivalence that describes the relationship between the factorizations which is captured by homotopy equivalence.

Proposition 101. M_1 and M_2 are homotopy equivalent if and only if every prime p that divides both elements in one of the factorizations, divides corresponding elements equally, that is, if we set $d_i = (\alpha_i, \beta_i)$, we get

$$p|d_1 d_2 \implies v_p(\frac{\alpha_1}{\alpha_2}) = v_p(\frac{\beta_1}{\beta_2}) = 0$$

Proof. An easy calculation shows that

$$\frac{l}{d} = \frac{\text{lcm}(\alpha_1, \alpha_2)}{(\alpha_1, \alpha_2)} = \prod_p p^{|v_p(\alpha_1) - v_p(\alpha_2)|}$$

By corollary (98), we have the following equivalences:

$$\begin{aligned}
&M_1 \text{ and } M_2 \text{ are homotopy equivalent}\\
\iff&(L, \frac{l}{d}) = 1\\
\iff&p|L \implies p \nmid \frac{l}{d}\\
\iff&p|d_1 d_2 \implies v_p(\frac{l}{d}) = 0\\
\iff&p|d_1 d_2 \implies |v_p(\alpha_1) - v_p(\alpha_2)| = 0\\
\iff&p|d_1 d_2 \implies v_p(\alpha_1) = v_p(\alpha_2)\\
\iff&p|d_1 d_2 \implies v_p(\frac{\alpha_1}{\alpha_2}) = 0
\end{aligned}$$

□

Corollary 102. For $\sigma = \alpha\beta$, set n_σ and n_d to be the number of distinct primes dividing σ and $d = (\alpha, \beta)$ respectively (up to multiplication by a unit). Then the number of isomorphism classes of rank 1 factorizations which are homotopy equivalent to (R, R, α, β) is $2^{n_\sigma - n_d}$.

Proof. To build a new factorization equivalent to $\sigma = \alpha\beta$ in light of the proposition, we have to move the full prime power factors of α and β around, keeping those in common between α and β fixed in place, this is the same as choosing a subset of distinct primes dividing σ but not dividing d to go into the first factor, sending the rest to the second factor, which can be done in $2^{n_\sigma - n_d}$ ways. □

Corollary 103. The number of homotopy classes of rank 1 factorizations of $\sigma = \prod_{i=1}^{n_\sigma} p_i^{v_{p_i}(\sigma)}$ is given by

$$\sum_{A \subseteq \{1, \ldots, n_\sigma\}} \prod_{i \in A} (v_{p_i}(\sigma) - 1)$$

with the empty product being 1. In particular, $HMF(R, \sigma)$ vanishes if and only if σ is square free.

Proof. Choosing a homotopy class is the same as choosing the greatest common factor between the factors in the matrix factorization of σ, which can be done by selecting a subset of distinct primes factors of σ, then choosing the powers of those primes to be strictly between 0 and the valuation of σ at those prime, which can be done in $v_p(\sigma) - 1$ ways for each prime p. σ is square free if and only if $v_p(\sigma) - 1 = 0$ for all primes dividing σ, hence the only non-zero term in the sum is the empty product, which is 1, so $HMF(R, \sigma)$ contains one object only, which must be the zero class. □

The last statement of the corollary corresponds to the fact that $R/(\sigma)$ is regular if and only if σ is square free (cf. [**Sym22**]).

6 Conclusion

We have reviewed the definition of the singularity category of a ring and mentioned its charecterizations as the homotopy category of matrix factorizations and the stable category of MCM modules. We have thoroughly studied, for the first time, the singularity category of hypersurface rings over a PID. We have tried to use a similar approach to study the case $R[x]/(f)$ where R is a PID, the particular case $k[x, y]/(xy)$ was of interest. Although the singularity category is known to be the direct sum of two copies of $(k - mod)$ (cf. [Sym22, Proposition 5.4.13]), it is extremely difficult to show without some theory of MCM modules (as well as more commutative algebra, see [Yos90, Page 75-76]). The reason is the lack of something similar to the Smith normal form in this case.

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Eidesstattliche Versicherung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Masterarbeit unter Betreuung durch Prof. Dr. Jan Kohlhaase und Prof. Dr. Ulrich Görtz selbstständig verfasst und Zitate kenntlich gemacht habe. Andere als die angegebenen Quellen und Hilfsmittel wurden von mir nicht benutzt. Diese Masterarbeit hat in dieser oder ähnlicher Form noch keiner anderen Prüfungsbehörde vorgelegen.
Essen, den 09.09.2025

Hasan K. Hasan