Admissible φ-modules and p-adic unitary representations

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Abstract. Generalizing results of J-M. Fontaine and A. Scholl, we describe the p-adic unitary, not necessarily finite dimensional representations of the absolute Galois group of certain fields in terms of *admissible φ-modules* and *admissible (φ, Γ)-modules.* If K is a nonarchimedean local field and if $h \geq 1$ is an integer then several prominent compact subgroups of $\mathrm{GL}_h(K)$ appear as quotients of these Galois groups. This allows us to give a functorial description of their p-adic unitary representations in terms of admissible φ-modules and admissible (φ, Γ)-modules over certain coefficient rings.

Contents

0 Introduction

Let p be a prime number, and let $\mathbb{Q}_p$ be the field of p-adic numbers. If H is a topological group then a *p-adic unitary representation of H* is a $\mathbb{Q}_p$-Banach space V carrying a continuous $\mathbb{Q}_p$-linear action of H such that the topology of V can be defined by a norm with respect to which H acts by isometries. Let us mention that the p-adic unitary representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ play a fundamental role in the p-adic local Langlands correspondence of P. Colmez (cf. [4]).

As of now, the general structure theory of p-adic unitary representations is only poorly understood, and p-adic unitary representations are difficult to construct. The aim of the present article is to show that in a particular case these problems can be addressed by adapting a strategy of J-M. Fontaine, developed to study p-adic representations of local Galois groups.

Namely, let K be a nonarchimedean local field with residue class field k of characteristic p, let $\mathfrak{o}$ denote the valuation ring of K, and set $G_K := \mathrm{Gal}(K^{\mathrm{sep}}|K)$. Depending on the characteristic of K, one of the most fundamental results of Fontaine's theory gives a functorial description of a finite dimensional continuous $\mathbb{Q}_p$-linear representation of G_K in terms of a so-called *étale φ-module* or an *étale (φ, Γ_0)-module* over a certain coefficient ring (cf. [6], Proposition 1.2.6 and Théorème 3.4.3).

Let h be an integer with $h \geq 1$, and let P be the subgroup of $\mathrm{GL}_h(K)$ consisting of all matrices $g = (g_{ij})_{i,j}$ such that $\det(g) \in \mathfrak{o}^\times$ and such that $g_{i1} = 0$ for all indices i with $2 \leq i \leq h$. Set $G_0 := \mathrm{GL}_h(\mathfrak{o})$ and $P_0 := P \cap G_0$. In this article we give a description of the categories of p-adic unitary representations of P_0 and of related groups in terms of certain types of φ-modules and (φ, Γ)-modules by interpreting P_0 as the Galois group of a suitable field extension and by then following Fontaine's strategy.

In fact, the group P_0 appears as the Galois group of a field extension $L_\infty|L_0$, where L_0 is a complete discretely valued field of the same characteristic as K. This is seen by considering a certain fiber of the moduli tower of Lubin-Tate whose decomposition group was determined by M. Strauch (cf. Theorem 1.3). Trying to apply a variant of Fontaine's strategy to this situation, one is confronted with two obvious problems.

First of all, the topologically irreducible p-adic unitary representations of the Galois group P_0 are generally *not* finite dimensional (cf. [13], section 4, noting that P_0 admits $\mathrm{GL}_{h-1}(\mathfrak{o})$ as a quotient). Fortunately, if E is any field of characteristic p and if $\mathfrak{o}_{\breve{\mathcal{E}}}$ is a Cohen ring of E^{sep} together with its natural action of $H := \mathrm{Gal}(E^{\mathrm{sep}}|E)$, then Fontaine's description of the continuous H-representations of finite type over $\mathbb{Z}_p$ in terms of étale φ-modules over $\mathfrak{o}_{\mathcal{E}} := \mathfrak{o}_{\breve{\mathcal{E}}}^H$, formally extends to an equivalence of categories

$$\mathbb{D} : \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(H) \longrightarrow \Phi_{\mathfrak{o}_{\mathcal{E}}}^{\mathrm{adm}},$$

given by $\mathbb{D}(V) := (\mathfrak{o}_{\breve{\mathcal{E}}} \widehat{\otimes}_{\mathbb{Z}_p} V)^H$ (cf. Theorem 3.6). Here $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(H)$ denotes the category of continuous representations of H on arbitrary p-adically separated and complete $\mathbb{Z}_p$-modules V, and $\mathfrak{o}_{\breve{\mathcal{E}}} \widehat{\otimes}_{\mathbb{Z}_p} V$ denotes the p-adic completion of the algebraic tensor product $\mathfrak{o}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V$. Further, $\Phi_{\mathfrak{o}_{\mathcal{E}}}^{\mathrm{adm}}$ denotes the category of what we call *admissible φ-modules over* $\mathfrak{o}_{\mathcal{E}}$ (cf. Definition 3.4). Denoting by $\mathcal{E}$ the quotient field of $\mathfrak{o}_{\mathcal{E}}$, we deduce that the categories $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{uni}}(H)$ of p-adic unitary representations of H over $\mathbb{Q}_p$ and $\Phi_{\mathcal{E}}^{\mathrm{adm}}$ of admissible φ-modules over $\mathcal{E}$ are equivalent (cf. Theorem 3.8). Since H is compact, the former is simply the category of all continuous $\mathbb{Q}_p$-linear representations of H on Banach spaces over $\mathbb{Q}_p$ (cf. Remark 3.7).

If the characteristic of K is zero, then the second and more serious problem is that the theory of classical (φ, Γ_0)-modules relies on J-M. Fontaine's and J-P. Wintenberger's theory of norm fields for complete discretely valued fields of characteristic zero with *perfect* residue class fields. Unless $h = 1$, the latter is not applicable to the field L_0. A generalization of this theory which matches our needs was given in [15] by A. Scholl. We recall some of his results in section 2. For the sake of greater applicability, we work in a rather general setup.

Namely, let $\overline{F}_0$ be any perfect field of characteristic p, let $\mathfrak{o}_{F_0} := W(\overline{F}_0)$ be its ring of Witt vectors, and let K_0 be the quotient field of the complete local ring of the power series ring $\mathfrak{o}_{F_0}[[t_1, \ldots, t_{h-1}]]$ at the prime ideal generated by p. We consider a generalized cyclotomic tower $K_\infty|K_0$ with Galois group $\mathbb{Z}_p^{h-1} \rtimes \mathbb{Z}_p^\times$, in which the quotient $\mathbb{Z}_p^\times$ corresponds to the classical case $h = 1$ (cf. Lemma 2.1). By the work of A. Scholl, the group $\mathrm{Gal}(K_0^{\mathrm{sep}}|K_\infty)$ is functorially isomorphic to the absolute Galois group of a complete discretely valued field X_{K_∞} of characteristic p. The action of $\mathrm{Gal}(K_0^{\mathrm{sep}}|K_\infty)$ on its separable closure extends to an action of the whole Galois group $\mathrm{Gal}(K_0^{\mathrm{sep}}|K_0)$. One of A. Scholl's major achievements is the construction of a Cohen ring $\mathfrak{o}_{\breve{\mathcal{E}}}$ of $X_{K_\infty}^{\mathrm{sep}}$ to which this action lifts and which commutes with a suitable lift of the Frobenius endomorphism of $X_{K_\infty}^{\mathrm{sep}}$.

If $F_0 := W(\overline{F}_0)[1/p]$ denotes the quotient field of $\mathfrak{o}_{F_0}$, and if $F|F_0$ is a finite totally ramified extension, then we let $G := \mathrm{Gal}(K_0^{\mathrm{sep}}|FK_0)$, as well as $H := \mathrm{Gal}(K_0^{\mathrm{sep}}|FK_\infty)$, and define $\Gamma := G/H$. Applying Theorem 3.6 to the field X_{FK_∞} of characteristic p and setting $\mathfrak{o}_{\mathcal{E}} := \mathfrak{o}_{\breve{\mathcal{E}}}^H$, we conclude that the functor

$$\mathbb{D} : \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(G) \longrightarrow \Phi\Gamma^{\mathrm{adm}}_{\mathfrak{o}_{\mathcal{E}}},$$

given by $\mathbb{D}(V) := (\mathfrak{o}_{\breve{\mathcal{E}}} \widehat{\otimes}_{\mathbb{Z}_p} V)^H$, is an equivalence of categories (cf. Theorem 4.5). Here $\Phi\Gamma^{\mathrm{adm}}_{\mathfrak{o}_{\mathcal{E}}}$ denotes the category of what we call *admissible* (φ, Γ)*-modules over* $\mathfrak{o}_{\mathcal{E}}$ (cf. Definition 4.3). There is also a variant for the category $\mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(G)$ of p-adic unitary representations of G over $\mathbb{Q}_p$.

Still assuming the characteristic of K to be zero and specializing to the case $h = 1$, $\overline{F}_0 := k$ and $F := K$, these theorems generalize Fontaine's afore mentioned result [6], Théorème 3.4.3, and give a description of the continuous representations of G_K on arbitrary p-adically separated and complete $\mathbb{Z}_p$-modules in terms of admissible (φ, Γ_0)-modules. Specializing to the case of an arbitrary integer $h \geq 1$, $\overline{F}_0 := k^{\mathrm{sep}}$ and $F := \breve{K} := KF_0$, they also apply to the absolute Galois group of the field $L_0 = \breve{K}K_0$.

In section 5 we apply the above results to the categories of continuous p-adic representations of several compact groups.

If the characteristic of K is arbitrary, then the residue class field $\overline{L}_0$ of L_0 admits a Galois extension with Galois group $\mathrm{GL}_{h-1}(\mathfrak{o})$. Denoting by $\mathfrak{o}_{\mathcal{E}}$ a Cohen ring of $\overline{L}_0$, we obtain an embedding of categories

$$\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(\mathrm{GL}_{h-1}(\mathfrak{o})) \longrightarrow \Phi^{\mathrm{adm}}_{\mathfrak{o}_{\mathcal{E}}}, \tag{1}$$

admitting a variant for the category $\mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(\mathrm{GL}_{h-1}(\mathfrak{o}))$ (cf. Theorem 5.1).

The theory is most flexible if the characteristic of K is p. In this case, we obtain embeddings of the categories $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(P_0)$ and $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(G_0)$ into the categories of admissible φ-modules over Cohen rings of

$$L_0 \simeq \mathrm{Quot}(k^{\mathrm{sep}}[[\pi, t_1, \ldots, t_{h-1}]]\widehat{\ }_{(\pi)}) \text{ and } Q_0 \simeq \mathrm{Quot}(k^{\mathrm{sep}}[[\pi, t_1, \ldots, t_{h-1}]]),$$

respectively (cf. Theorem 5.2 and Theorem 5.3). We study how the functors of restriction and induction between the categories $\mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(P_0)$ and $\mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(G_0)$ translate into functors between the corresponding categories of admissible φ-modules (cf. Proposition 5.4). Further, the Cohen ring $\mathfrak{o}_{\mathcal{E}}$ of L_0 contains a Cohen ring $\mathfrak{o}_{\mathcal{E}_0}$ of $\breve{K}$. If $I_K := \mathrm{Gal}(K^{\mathrm{sep}}|K^{\mathrm{unr}})$ denotes the inertia group of K then we obtain two functors

$$\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(P_0) \longrightarrow \Phi^{\mathrm{adm}}_{\mathfrak{o}_{\mathcal{E}}} \longleftarrow \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(I_K).$$

Proposition 5.5 shows, however, that this gives only a weak link between the two categories of representations involved.

Now assume the field K to be of characteristic zero. The group P_0 appears as a quotient of the absolute Galois group of the field L_0, so that we obtain an embedding of categories $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(P_0) \to \Phi\Gamma^{\mathrm{adm}}_{\mathfrak{o}_{\mathcal{E}}}$ (cf. Theorem 5.6 and the subsequent discussion). Here $\mathfrak{o}_{\mathcal{E}}$ is a Cohen ring of a finite separable extension of $X_{K_\infty} \simeq \mathrm{Quot}(k^{\mathrm{sep}}[[t_1,\dots,t_{h-1}]])((\pi))$ – the trivial one if $K|\mathbb{Q}_p$ is unramified – and the actions of φ and $\mathbb{Z}_p^{h-1} \rtimes \mathbb{Z}_p^\times$ on a suitable Cohen ring of X_{K_∞} are very explicit. The above embedding also admits a variant for the category of p-adic unitary representations of P_0 over $\mathbb{Q}_p$.

In this situation we have $\Gamma \simeq \Delta \rtimes \Gamma_0$, where $\Delta \simeq \mathbb{Z}_p^{h-1}$ and where Γ_0 is the group appearing in Fontaine's classical theory for G_K. If at least one of q or h is different from 2, then we show that the towers $L_\infty|L_0$ and $\breve{K}K_\infty|\breve{K}K_0$ are linearly disjoint up to the classical cyclotomic extension (cf. Theorem 2.2). As a consequence, if an admissible (φ,Γ)-module over $\mathfrak{o}_{\mathcal{E}}$ comes from an object of $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(P_0)$ then one may forget the action of the subgroup Δ without losing any information. Our main result in characteristic zero is Theorem 5.7, stating that under the above hypotheses we obtain embeddings of categories

$$\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(P_0) \longrightarrow (\Phi\Gamma_0)^{\mathrm{adm}}_{\mathfrak{o}_{\mathcal{E}}} \quad \text{and} \quad \mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(P_0) \longrightarrow (\Phi\Gamma_0)^{\mathrm{adm}}_{\mathcal{E}}.$$

Since the embedding (1) is compatible with reduction modulo p^n for any integer $n \geq 0$, one can describe the irreducible smooth representations of $\mathrm{GL}_{h-1}(\mathfrak{o})$ on $\mathbb{F}_p$-vector spaces in terms of simple étale φ-modules over a field of characteristic p. We work out explicit examples in section 6.

The last part of our work concerns p-adic unitary representations of the non-compact group P and of its quotient $\mathrm{GL}_{h-1}(K)$.

By their modular construction, the actions of P_0 and $\mathrm{GL}_{h-1}(\mathfrak{o})$ on L_∞ and $\overline{L}_\infty$ extend to actions of P and $\mathrm{GL}_{h-1}(K)$, as follows from Proposition 1.2. The field $\overline{L}_\infty$ is perfect (cf. Corollary 1.5) so that there is a natural action of $\mathrm{GL}_{h-1}(K)$ on its Cohen ring $\mathfrak{o}_{\breve{\mathcal{E}}} := W(\overline{L}_\infty)$. Letting G denote the subgroup of $\mathrm{GL}_{h-1}(K)$ consisting of all elements g with $\det(g) \in \mathfrak{o}^\times$, we use the G-action on $\mathfrak{o}_{\breve{\mathcal{E}}}$ to show that the category $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(G)$ is equivalent to a certain category of locally trivial cohomological coefficient systems of admissible φ-modules on the Bruhat-Tits building of $\mathrm{GL}_{h-1}(K)$ (cf. Theorem 7.1).

If the characteristic of K is p then the action of P on L_∞ induces an action of P on the Cohen ring $\mathfrak{o}_{\breve{\mathcal{E}}} := W(L_\infty^{\mathrm{rad}})$ of the perfect closure of L_∞. If $\breve{\mathcal{E}}$ denotes the quotient field of $\mathfrak{o}_{\check{\mathcal{E}}}$ then $\check{\mathcal{E}}$ is a p-adic unitary representation of P whose restriction to P_0 trivializes all objects of $\mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(P_0)$. Since $\check{\mathcal{E}}^P = W(\check{K}^{\mathrm{rad}})[1/p]$, as follows from Theorem 1.7, we obtain two functors

$$\mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(P) \underset{\mathbb{V}}{\overset{\mathbb{D}}{\rightleftarrows}} \Phi^{\mathrm{uni}}_{W(\check{K}^{\mathrm{rad}})[1/p]}.$$

The category on the right contains the category of admissible φ-modules over $W(\check{K}^{\mathrm{rad}})[1/p]$, which is equivalent to the category $\mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(I_K)$ of p-adic unitary representations of the inertia group I_K of K. At the present stage, however, the relation between the functors $\mathbb{D}$ and $\mathbb{V}$ remains unclear.

Conventions and notation. If F is a field we fix an algebraic closure F^{alg} and a separable closure $F^{\mathrm{sep}} \subseteq F^{\mathrm{alg}}$ of F. If the characteristic of F is nonzero then we let $F^{\mathrm{rad}} \subseteq F^{\mathrm{alg}}$ be a fixed perfect closure of F.
If F is a field which is endowed with a nontrivial nonarchimedean valuation then we denote by $\mathfrak{o}_F$ and by $\overline{F}$ the valuation ring and the residue class field of F, respectively. If F is complete then we let F^{unr} be the maximal unramified extension of F and denote by $\check{F}$ the completion of F^{unr}.
Throughout this article we let K be a nonarchimedean local field, i.e. a field which is complete with respect to a nontrivial discrete valuation such that the residue class field of K is finite. We set $\mathfrak{o} := \mathfrak{o}_K$, $k := \overline{K}$ and $\check{\mathfrak{o}} := \mathfrak{o}_{\check{K}}$, and denote by q and p the cardinality and the characteristic of k, respectively. Let π be a fixed uniformizer of K.
If F is a field of positive characteristic p then by a *Cohen ring of F* we mean a complete discrete valuation ring of characteristic zero with residue class field F which is *totally unramified*, i.e. whose maximal ideal is generated by p.
If R is a commutative unital ring then we denote by $W(R)$ the ring of Witt vectors with coefficients in R. If R is a commutative integral domain then we denote by $\mathrm{Quot}(R)$ its quotient field. Finally, we fix an integer h with $h \geq 1$.

1 A fiber of the Lubin-Tate tower

Let $\mathcal{C}$ be the category of unital commutative complete noetherian local $\check{\mathfrak{o}}$-algebras $R = (R, \mathfrak{m}_R)$ with residue class field $R/\mathfrak{m}_R \simeq k^{\mathrm{sep}}$. We fix a one dimensional formal $\mathfrak{o}$-module $\mathbb{H}$ of height h over k^{sep}, defined over k. If H is a one dimensional formal $\mathfrak{o}$-module defined over some object R of $\mathcal{C}$, if X is a coordinate of H and if $\alpha \in \mathfrak{o}$ then we denote by $[\alpha]_H = [\alpha]_H(X) \in R[[X]]$ the corresponding endomorphism of H.

Let $m \geq 0$ be an integer and consider the set valued functor $\mathfrak{Y}_m$ on $\mathcal{C}$ associating with an object R of $\mathcal{C}$ the set of isomorphism classes $[(H, \rho, \varphi)]$ of triples (H, ρ, φ) consisting of the following data. H is a one dimensional formal $\mathfrak{o}$-module of height h over R, $\varphi : (\pi^{-m}\mathfrak{o}/\mathfrak{o})^h \to (\mathfrak{m}_R, +_H)$ is a homomorphism of abstract $\mathfrak{o}$-modules such that the power series $\prod_{\alpha \in (\pi^{-m}\mathfrak{o}/\mathfrak{o})^h}(X - \varphi(\alpha))$ divides the power series $[\pi^m]_H$ in $R[[X]]$ (a so-called *level-m structure*), and $\rho : \mathbb{H} \to H$ mod $\mathfrak{m}_R$ is an $\mathfrak{o}$-linear isomorphism.

If m' is an integer with $m' \geq m$ then restricting level-m' structures to the submodule $(\pi^{-m}\mathfrak{o}/\mathfrak{o})^h$ of $(\pi^{-m'}\mathfrak{o}/\mathfrak{o})^h$ gives rise to a natural transformation $\mathfrak{Y}_{m'} \to \mathfrak{Y}_m$. The following fundamental results are due to V. Drinfeld (cf. [5], Proposition 4.2 and Proposition 4.3).

Theorem 1.1 (Drinfeld). *(i) For any integer $m \geq 0$ the functor $\mathfrak{Y}_m$ is representable by an object R_m of $\mathcal{C}$. The local ring R_m is regular.*

(ii) Assume $m \geq 1$. If $[(H_m^u, \rho_m^u, \varphi_m^u)]$ denotes the universal isomorphism class for $\mathfrak{Y}_m$, and if $(e_m^i)_{1\leq i\leq h}$ is an $\mathfrak{o}/\pi^m\mathfrak{o}$-basis of $(\pi^{-m}\mathfrak{o}/\mathfrak{o})^h$, then the family $(\varphi_m^u(e_m^i))_i$ is a system of regular coordinates of R_m.

(iii) If m' and m are integers with $m' \geq m \geq 0$ then the ring homomorphism $R_m \to R_{m'}$ induced by the natural transformation $\mathfrak{Y}_{m'} \to \mathfrak{Y}_m$ is finite and flat.

(iv) The ring R_0 is noncanonically isomorphic to the ring $\breve{\mathfrak{o}}[[t_1, \ldots, t_{h-1}]]$ of formal power series in $h-1$ indeterminates $t_1, \ldots, t_{h-1}$ over $\breve{\mathfrak{o}}$. □

In the following we shall always work with the standard basis $(e_m^i)_i$ of $(\pi^{-m}\mathfrak{o}/\mathfrak{o})^h$. For any integer $m \geq 0$ the natural action of $G_0 := \mathrm{GL}_h(\mathfrak{o})$ on $(\pi^{-m}\mathfrak{o}/\mathfrak{o})^h$ induces an $\breve{\mathfrak{o}}$-linear action of G_0 on R_m, compatible with the transition maps $R_m \to R_{m'}$. We let $Q_m := \mathrm{Quot}(R_m)$ be the quotient field of R_m. According to a result of M. Strauch, the above action makes Q_m a Galois extension of Q_0 with Galois group $\mathrm{GL}_h(\mathfrak{o}/\pi^m\mathfrak{o})$ (cf. [17], Theorem 2.1.2). We let

$$R_\infty := \varinjlim_m R_m \quad \text{and} \quad Q_\infty := \varinjlim_m Q_m = \mathrm{Quot}(R_\infty).$$

The extension $Q_\infty|Q_0$ is Galois with Galois group G_0, and the G_0-actions on R_∞ and Q_∞ extend to $\breve{\mathfrak{o}}$-linear actions of the subgroup of $\mathrm{GL}_h(K)$ consisting of all elements $g \in \mathrm{GL}_h(K)$ with $\det(g) \in \mathfrak{o}^\times$ (cf. [17], section 2.2.2 for details).

According to [8], Proposition 5.7, the regular coordinates $t_1, \ldots, t_{h-1}$ of R_0 and the coordinate X of H_0^u can be chosen in such a way that for all $0 \leq i \leq h$

$$[\pi]_{H_0^u}(X) \equiv t_i X^{q^i} \mod (t_0, \ldots, t_{i-1}, X^{q^i+1})$$

with $t_0 := \pi$ and $t_h := 1$. Following [18], we let $\mathfrak{p}_i := \mathfrak{p}_{i,0}$ be the prime ideal of R_0 generated by $t_0, \ldots, t_{i-1}$ for any $0 \leq i \leq h$. For any integer $m \geq 1$ we let $\mathfrak{p}_{i,m}$ be the prime ideal of R_m generated by $\varphi_m^u(e_m^1), \ldots, \varphi_m^u(e_m^i)$.

For any integer $0 \leq i \leq h$ we let $P^{(i)}$ be the subgroup of $\mathrm{GL}_h(K)$ consisting of matrices of the form

$$g = \begin{pmatrix} A & * \\ 0 & B \end{pmatrix}$$

with $A \in \mathrm{GL}_i(K)$, $B \in \mathrm{GL}_{h-i}(K)$ and $\det(g) \in \mathfrak{o}^\times$. We need the following slight generalization of a result of M. Strauch (cf. [18], Proposition 4.1).

Proposition 1.2. *For any integer $0 \leq i \leq h$ and any two integers m' and m with $m' \geq m \geq 0$ we have $\mathfrak{p}_{i,m'} \cap R_m = \mathfrak{p}_{i,m}$. The prime ideal $\mathfrak{p}_{i,\infty} := \varinjlim_m \mathfrak{p}_{i,m}$ of R_∞ is stable under the action of $P^{(i)}$.*

Proof: For $m = 0$ the relation $\mathfrak{p}_{i,m'} \cap R_0 = \mathfrak{p}_i$ was shown in [18], Proposition 4.1. We may therefore assume $m \geq 1$. For any index k with $1 \leq k \leq i$ there is an element $\alpha \in \mathfrak{o}$ such that $e^i_m = \alpha \cdot e^i_{m'}$ via the inclusion $(\pi^{-m}\mathfrak{o}/\mathfrak{o})^h \subseteq (\pi^{-m'}\mathfrak{o}/\mathfrak{o})^h$. Pushing out along the ring homomorphism $R_m \to R_{m'}$ we identify $H^u := H^u_0 = H^u_m = H^u_{m'}$, and obtain a commutative diagram of $\mathfrak{o}$-modules

$$\begin{array}{ccc}
(\pi^{-m'}\mathfrak{o}/\mathfrak{o})^h & \xrightarrow{\varphi^u_{m'}} & (\mathfrak{m}_{R_{m'}}, +_{H^u}) \\
\uparrow & & \uparrow \\
(\pi^{-m}\mathfrak{o}/\mathfrak{o})^h & \xrightarrow{\varphi^u_{m}} & (\mathfrak{m}_{R_{m}}, +_{H^u}).
\end{array}$$

Therefore, $\varphi^u_m(e^k_m) = [\alpha]_{H^u}(\varphi^u_{m'}(e^k_{m'})) \in \varphi^u_{m'}(e^k_{m'}) \cdot R_{m'}$ and $\mathfrak{p}_{i,m} \subseteq \mathfrak{p}_{i,m'} \cap R_m$. By Theorem 1.1, the prime ideal $\mathfrak{p}_{i,m}$ has height i, as does $\mathfrak{p}_{i,m'} \cap R_m$ because R_m is integrally closed and $R_m \to R_{m'}$ is an integral ring extension (cf. [3], VIII.2.3 Théorème 2). It follows that $\mathfrak{p}_{i,m'} \cap R_m = \mathfrak{p}_{i,m}$.

Now let $g \in P^{(i)}$ and choose an element $r \in \mathbb{Z}$ such that the matrix $\pi^r g^{-1}$ has coefficients in $\mathfrak{o}$. Given an integer $m \geq 1$ we choose an integer $m' \geq m$ such that $\pi^{-r} g \cdot \mathfrak{o}^h \subseteq \pi^{-(m'-m)}\mathfrak{o}^h$. The element g defines a natural transformation $g : \mathfrak{Y}_{m'} \to \mathfrak{Y}_m$, corresponding to a local homomorphism $g : R_m \to R_{m'}$. By representability, the universal pair $(H^u_{m'}, \varphi^u_{m'})$ is sent to $(H^u_m \times_{R_m, g} R_{m'}, g|_{\mathfrak{m}_{R_m}} \circ \varphi^u_m)$. On the other hand, the transformation g is constructed as follows.

By the choice of r we have $\mathfrak{o}^h \subseteq \pi^{-r} g \cdot \mathfrak{o}^h$. Moreover, $\pi^{-r} g \cdot \mathfrak{o}^h/\mathfrak{o}^h$ is an $\mathfrak{o}$-submodule of $(\pi^{-m'}\mathfrak{o}/\mathfrak{o})^h$. According to [5], Proposition 4.4, the quotient $H' := H^u_{m'}/\varphi^u_{m'}(\pi^{-r} g\mathfrak{o}^h/\mathfrak{o}^h)$ exists as a formal $\mathfrak{o}$-module over $R_{m'}$. Further, multiplication with $\pi^{-r}g$ induces an injection

$$(\pi^{-m}\mathfrak{o}/\mathfrak{o})^h \xhookrightarrow{\pi^{-r}g} \pi^{-m'}\mathfrak{o}^h/\pi^{-r}g\mathfrak{o}^h,$$

whose composition with the map $\pi^{-m'}\mathfrak{o}^h/\pi^{-r}g\mathfrak{o}^h \to (\mathfrak{m}_{R_{m'}}, +_{H'})$ induced by $\varphi^u_{m'}$ is denoted by φ' and which is a level-m structure of H' (cf. [loc.cit.]). One sets $(H^u_{m'}, \varphi^u_{m'}) \cdot g := (H', \varphi')$.

Now consider the commutative diagram of $\mathfrak{o}$-modules

$$\begin{array}{ccccc}
(\pi^{-m}\mathfrak{o}/\mathfrak{o})^h & \xhookrightarrow{\pi^{-r}g} & \pi^{-m'}\mathfrak{o}^h/\pi^{-r}g\mathfrak{o}^h & \xleftarrow{\mathrm{pr}_1} & (\pi^{-m'}\mathfrak{o}/\mathfrak{o})^h \\
\downarrow \varphi^u_m & \searrow^{\varphi'} & \downarrow & & \downarrow \varphi^u_{m'} \\
(\mathfrak{m}_{R_m}, +_{H^u}) & \xrightarrow[g]{} & (\mathfrak{m}_{R_{m'}}, +_{H'}) & \xleftarrow[\mathrm{pr}_2]{} & (\mathfrak{m}_{R_{m'}}, +_{H^u})
\end{array}$$

in which the right horizontal arrows are the natural projections.

We need to show that $g(\varphi^u_m(e^k_m)) \in \mathfrak{p}_{i,m'}$ for all $1 \leq k \leq i$. Since $\pi^{-r}g(e^k_m) \in \mathrm{pr}_1(\sum_{j=1}^i \mathfrak{o} \cdot e^j_{m'})$ by assumption on g, it suffices to see that $(\mathrm{pr}_2 \circ \varphi^u_{m'})(\beta) \in \mathfrak{p}_{i,m'}$ for all $\beta \in \sum_{j=1}^i \mathfrak{o} \cdot e^j_{m'}$.

Coming from a homomorphism of formal groups, the map pr_2 is given by a power series $\mathrm{pr}_2(X) \in X \cdot R_{m'}[[X]]$. In fact,

$$\mathrm{pr}_2(X) = \prod_{\alpha \in \pi^{-r} g \mathfrak{o}^h / \mathfrak{o}^h} H^u_{m'}(X, \varphi^u_{m'}(\alpha)). \tag{2}$$

Therefore, it suffices to see that $\varphi^u_{m'}(\beta) \in \mathfrak{p}_{i,m'}$ for all β as above. Now $H^u_{m'}(X,0) = X$, $H^u_{m'}(0,Y) = Y$ and $[\alpha]_{H^u_{m'}}(0) = 0$ for all $\alpha \in \mathfrak{o}$. Since $\varphi^u_{m'}$ is a homomorphism of $\mathfrak{o}$-modules, we are further reduced to the case $\beta = e^j_{m'}$ for some $1 \leq j \leq i$. In this case, the assertion is true by definition. □

From now on we will concentrate on the case $i = 1$ in the above proposition and set $\pi_0 := \pi$, as well as $\pi_m := \varphi^u_m(e^1_m)$ for any integer $m \geq 1$. For any integer $m \geq 0$ we let

$$\mathfrak{o}_{L_m} := (R_m)^{\widehat{}}_{(\pi_m)}$$

be the complete local ring of R_m at the prime ideal $\mathfrak{p}_{1,m} = (\pi_m)$, which is a complete discrete valuation ring (cf. [3], VI.1.4 Proposition 2 and VI.5.3 Proposition 5). We denote by $L_m := \mathrm{Quot}(\mathfrak{o}_{L_m})$ its quotient field and set

$$\mathfrak{o}_{L_\infty} := \varinjlim_m \mathfrak{o}_{L_m} \quad \text{and} \quad L_\infty := \varinjlim_m L_m = \mathrm{Quot}(\mathfrak{o}_{L_\infty}).$$

According to Proposition 1.2 there are actions of the group $P := P^{(1)}$ on $\mathfrak{o}_{L_\infty}$ and on L_∞. Their restrictions to $P_0 := P \cap G_0$ have the following properties.

Theorem 1.3 (Strauch)**.** *The above action of P_0 makes $L_\infty|L_0$ a Galois extension with $\mathrm{Gal}(L_\infty|L_0) = P_0$. Denoting by $\mathbf{1} \in \mathrm{GL}_{h-1}(\mathfrak{o})$ the identity matrix, its inertia group is the subgroup M_0 of P_0 given by*

$$M_0 := \{\begin{pmatrix} \alpha & \beta \\ 0 & \mathbf{1} \end{pmatrix} \mid \alpha \in \mathfrak{o}^\times \text{ and } \beta \in \mathfrak{o}^{h-1}\}.$$

In particular, the ring extension $\mathfrak{o}^{M_0}_{L_\infty}|\mathfrak{o}_{L_0}$ is étale and Galois with Galois group $P_0/M_0 \simeq \mathrm{GL}_{h-1}(\mathfrak{o})$.

Proof: If $m \geq 0$ is an integer, then the extension $Q_m|Q_0$ is Galois with Galois group $\mathrm{GL}_h(\mathfrak{o}/\pi^m\mathfrak{o})$. The decomposition and inertia groups of the prime ideals $\mathfrak{p}_{1,m}$ were computed in [18], Proposition 4.1. They are the images of P_0 and M_0 in $\mathrm{GL}_h(\mathfrak{o}/\pi^m\mathfrak{o})$ under the natural reduction map $G_0 \to \mathrm{GL}_h(\mathfrak{o}/\pi^m\mathfrak{o})$. Now one uses [10], Corollaire II.3.4. □

Consider the residue class field $\overline{L}_\infty$ of L_∞. According to Proposition 1.2 and Theorem 1.3 it carries an action of P whose kernel contains the smallest normal subgroup of P containing M_0. The latter is the subgroup

$$M := \{\begin{pmatrix} \alpha & \beta \\ 0 & \mathbf{1} \end{pmatrix} \mid \alpha \in \mathfrak{o}^\times \text{ and } \beta \in K^{h-1}\},$$

for which one has $P/M \simeq \mathrm{GL}_{h-1}(K)$.

Theorem 1.4. *We have $\overline{L}^P_\infty = k^{\mathrm{sep}}$.*

Proof: Any P-invariant element of $\overline{L}_\infty$ is contained in a subfield $\overline{L}_m$ for some integer $m \geq 1$. For $2 \leq i \leq h$ we denote the image of $\varphi_m^u(e_m^i)$ in $R_m/(\pi_m)$ by τ_i. By Theorem 1.1 and Proposition 1.2, $R_m/(\pi_m)$ is a complete noetherian regular local $k^{\rm sep}$-algebra with residue class field $k^{\rm sep}$ for which $\tau_2, \ldots, \tau_h$ is a system of regular coordinates. Thus, $R_m/(\pi_m) \simeq k^{\rm sep}[[\tau_2, \ldots, \tau_h]]$ (cf. [3], VIII.5.2 Corollaire 3). Note that $\overline{L}_m$ is the quotient field of $R_m/(\pi_m)$.

Consider the $k^{\rm sep}$-linear endomorphism of $\varinjlim_n R_n/(\pi_n)$ defined by the diagonal matrix $g := \mathrm{diag}(\pi^{-(h-1)}\ \pi\ \ldots\ \pi)$ in P. Setting $m' := m+h$ and $r := 1$ we carry out the constructions of the proof of Proposition 1.2 and obtain that the power series $\mathrm{pr}_2(X)$ for g defined in (2) satisfies

$$\begin{aligned} \mathrm{pr}_2(X) &= \prod_{\alpha \in \pi^{-1} g\mathfrak{o}^h/\mathfrak{o}^h} H_{m'}^u(X, \varphi_{m'}^u(\alpha)) = \prod_{\alpha \in \mathfrak{o}/\pi^h \mathfrak{o}} H_{m'}^u(X, \varphi_{m'}^u(\alpha \pi^m e_{m'}^1)) \\ &\equiv X^{q^h} \mod \pi_{m'} R_{m'}[[X]], \end{aligned}$$

because $\varphi_{m'}^u(\alpha\pi^m e_{m'}^1) = [\alpha\pi^m]_{H_{m'}^u}(\pi_{m'}) \in \pi_{m'} R_{m'}$ and since $H_{m'}^u(X,Y) \equiv X \bmod Y R_{m'}[[X,Y]]$. In particular, we have $g(\tau_i) = \tau_i^{q^h}$ for all i. Thus, the field $\overline{L}_m \simeq \mathrm{Quot}(k^{\rm sep}[[\tau_2, \ldots, \tau_h]])$ is actually stable under g, and the latter acts on it by raising the variables τ_i to the power of q^h. It is elementary to see that any element $f \in \overline{L}_m$ fixed under this operation has to lie in the subfield $k^{\rm sep}$. □

Corollary 1.5. *The field $\overline{L}_\infty$ is perfect.*

Proof: Let $m \geq 1$ and $n \geq 1$ be integers. With the notation of the proof of Theorem 1.4 we have

$$\overline{L}_m^{q^{-hn}} \simeq \mathrm{Quot}(k^{\rm sep}[[\tau_2, \ldots, \tau_h]])^{q^{-hn}} = \mathrm{Quot}(k^{\rm sep}[[\tau_2^{q^{-hn}}, \ldots, \tau_h^{q^{-hn}}]]),$$

because $k^{\rm sep}$ is perfect. However, $\tau_i^{q^{nh}} = g^n(\tau_i)$, and the field endomorphism g^n of $\overline{L}_\infty$ is bijective (its inverse is given by $g^{-n} \in P$). Since $g^{-n}(\tau_i)^{q^{nh}} = g^{-n}(\tau_i^{q^{nh}}) = \tau_i$, we have $\tau_i^{q^{-nh}} = g^{-n}(\tau_i) \in \overline{L}_\infty$. □

Corollary 1.6. *The P-action on $\overline{L}_\infty$ factors through a faithful action of $P/M \simeq \mathrm{GL}_{h-1}(K)$.*

Proof: We have already remarked that M is contained in the kernel of the P-action on $\overline{L}_\infty$. The restriction of the $\mathrm{GL}_{h-1}(K)$-action to $\mathrm{GL}_{h-1}(\mathfrak{o})$ makes the separable closure of $\overline{L}_0$ in $\overline{L}_\infty$ a Galois extension of $\overline{L}_0$ with Galois group $\mathrm{GL}_{h-1}(\mathfrak{o})$ (cf. Theorem 1.3). Thus, unless $h \leq 2$, the kernel of the $\mathrm{GL}_{h-1}(K)$-action cannot contain $\mathrm{SL}_{h-1}(K)$, hence has to be a central subgroup in all cases. Let $\alpha \in K^\times$ act trivially on $\overline{L}_\infty$ and write $\alpha = \alpha_0 \pi^n$ for some integer n and some element $\alpha_0 \in \mathfrak{o}^\times$. Replacing α by α^{-1} if necessary, we may assume $n \geq 0$. In this case the action of π^n is given by g^n with g as in the proof of Theorem 1.4. As seen there, the automorphism g^n of $\overline{L}_\infty$ stabilizes any of the subfields $\overline{L}_m$. However, in contrast to α_0, its restriction to $\overline{L}_m$ is no longer bijective unless $n = 0$ or unless we are in the trivial case $h = 1$. Thus, $\alpha = \alpha_0 \in \mathfrak{o}^\times$ and $\alpha = 1$ by Theorem 1.3. □

Theorem 1.7. *We have $L_\infty^P = \breve{K}$.*

Proof: We have $L_\infty^P \subseteq L_\infty^{P_0} = L_0$. Let $\alpha \in L_0$ be P-invariant. Since $\pi \in L_\infty^P \cap L_0$ we may assume $\alpha \in \mathfrak{o}_{L_0}$. Its image in $\overline{L}_0$ is P-invariant so that there is an element $\alpha_0 \in \breve{\mathfrak{o}}$ with $\alpha - \alpha_0 \in \pi\mathfrak{o}_{L_0}$ by Theorem 1.4. Setting $\beta := (\alpha-\alpha_0)/\pi \in \mathfrak{o}_{L_0}$, the same procedure yields an element $\alpha_1 \in \breve{\mathfrak{o}}$ such that $\beta - \alpha_1 \in \pi\mathfrak{o}_{L_0}$, i.e. $\alpha - \alpha_0 - \pi\alpha_1 \in \pi^2\mathfrak{o}_{L_0}$. Proceeding inductively, one constructs a sequence $(\alpha_n)_{n\geq 0}$ of elements of $\breve{\mathfrak{o}}$ such that $\alpha - \sum_{i=0}^n \alpha_i\pi^i \in \pi^{n+1}\mathfrak{o}_{L_0}$ for all $n \geq 0$. Since $\mathfrak{o}_{L_0}$ is π-adically separated and $\breve{\mathfrak{o}}$ is π-adically complete, we obtain $\alpha = \sum_{i=0}^\infty \alpha_i\pi^i \in \breve{\mathfrak{o}}$. □

2 Higher fields of norms

Let $\overline{F}_0$ be any perfect field of characteristic p, let $\mathfrak{o}_{F_0} := W(\overline{F}_0)$, and let F_0 be the quotient field of $\mathfrak{o}_{F_0}$.

Let $t_1, \ldots, t_{h-1}$ be indeterminates, and let $\mathfrak{o}_{K_0}$ be the complete local ring of $\mathfrak{o}_{F_0}[[t_1,\ldots,t_{h-1}]]$ at the prime ideal generated by p. For any integer $m \geq 1$ we let K_m be the extension of $K_0 := \mathrm{Quot}(\mathfrak{o}_{K_0})$ obtained by adjoining the group μ_{p^m} of p^m-th roots of unity, as well as fixed p^m-th roots $t_1^{1/p^m}, \ldots, t_{h-1}^{1/p^m}$ of the elements $t_1, \ldots, t_{h-1}$. We choose the latter in such a way that $(t_i^{1/p^{m+1}})^p = t_i^{1/p^m}$ for all $m \geq 1$ and all i. Similarly, we choose primitive p^m-th roots of unity ζ_{p^m} satisfying $\zeta_{p^{m+1}}^p = \zeta_{p^m}$ for all integers $m \geq 1$.

Being finite over K_0, the fields K_m are complete discretely valued fields whose valuation rings will be denoted by $\mathfrak{o}_{K_m}$. We have $K_m \subseteq K_{m'}$ for any two integers m and m' with $m' \geq m \geq 0$, and we let

$$\mathfrak{o}_{K_\infty} := \varinjlim_m \mathfrak{o}_{K_m} \quad \text{and} \quad K_\infty := \varinjlim_m K_m = \mathrm{Quot}(\mathfrak{o}_{K_\infty}).$$

We shall need the following standard facts whose proofs are elementary.

Lemma 2.1. *Let $m \geq 1$ be an integer.*

(i) $K_\infty | K_0$ is a Galois extension with $\mathrm{Gal}(K_\infty|K_0) \simeq \mathbb{Z}_p^{h-1} \rtimes \mathbb{Z}_p^\times$.

(ii) We have $\mathfrak{o}_{K_m} = \mathfrak{o}_{K_{m-1}}[\zeta_{p^m}, t_1^{1/p^m}, \ldots, t_{h-1}^{1/p^m}]$. Further, $\zeta_{p^m} - 1$ is a uniformizer of K_m, and the residue class field of K_m is given by $\overline{K}_m = \overline{K}_0^{p^{-m}}$. The minimal polynomial of $\zeta_{p^{m+1}}$ over K_m is $X^p - \zeta_{p^m}$, the minimal polynomial of t_i^{1/p^m} over $K_{m-1}(\mu_{p^m}, t_1^{1/p^m}, \ldots, t_{i-1}^{1/p^m})$ is $X^p - t_i^{1/p^{m-1}}$. We have $[K_{m+1} : K_m] = p^h$ and $[\overline{K}_m : \overline{K}_m^p] = p^{h-1}$.

(iii) There is an isomorphism $\Omega^1_{\mathfrak{o}_{K_{m+1}}|\mathfrak{o}_{K_m}} \simeq (\mathfrak{o}_{K_{m+1}}/p \cdot \mathfrak{o}_{K_{m+1}})^h$ of $\mathfrak{o}_{K_{m+1}}$-modules. □

Assume the characteristic of K to be zero. Let $\overline{F}_0 := k^{\mathrm{sep}}$ and construct the field K_0 correspondingly. We have $\mathfrak{o}_{F_0} = \breve{\mathbb{Z}}_p$, and Theorem 1.1 allows us to view L_0 as a finite extension of K_0. Let $N_{K|\mathbb{Q}_p} : K \to \mathbb{Q}_p$ be the norm map and let P_0^N be the kernel of the homomorphism $N_{K|\mathbb{Q}_p} \circ \det : P_0 \to \mathbb{Z}_p^\times$. Consider the composite extension $L_\infty K_\infty$ of $L_0 = \breve{K} K_0$.

Theorem 2.2. *Assume the characteristic of K to be zero and at least one of h or q to be different from 2. Let $\mathfrak{P} := \mathrm{Gal}(L_\infty K_\infty|L_0)$ and consider the short exact sequences of Galois groups*

$$0 \longrightarrow \mathrm{Gal}(L_\infty K_\infty|L_0 K_\infty) \longrightarrow \mathfrak{P} \longrightarrow \mathrm{Gal}(L_0 K_\infty|L_0) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Gal}(L_\infty K_\infty|L_\infty) \longrightarrow \mathfrak{P} \longrightarrow \mathrm{Gal}(L_\infty|L_0) \longrightarrow 0.$$

We have $L_\infty \cap L_0 K_\infty = L_0(\mu_{p^\infty})$. As a consequence, there are isomorphisms $\mathrm{Gal}(L_\infty K_\infty|L_0 K_\infty) \simeq P_0^N$ and

$$\mathrm{Gal}(L_\infty K_\infty|L_\infty) \simeq \mathrm{Gal}(\breve{K} K_\infty|\breve{K} K_0(\mu_{p^\infty}))) \simeq \mathrm{Gal}(K_\infty|K_0(\mu_{p^\infty})) \simeq \mathbb{Z}_p^{h-1}.$$

If $\Gamma := \mathrm{Gal}(L_0 K_\infty|L_0)$ then $\Gamma \simeq \Delta \rtimes \Gamma_0$ with $\Delta := \mathrm{Gal}(\breve{K} K_\infty|\breve{K} K_0(\mu_{p^\infty}))$ and $\Gamma_0 := \mathrm{Gal}(\breve{K} K_0(\mu_{p^\infty})|\breve{K} K_0) \simeq \mathrm{Gal}(\breve{K}(\mu_{p^\infty})|\breve{K})$, which is isomorphic to an open subgroup of $\mathbb{Z}_p^\times$.

Proof: For any integer $m \geq 0$ let $\breve{K}_m$ be the finite extension of $\breve{K}$ obtained by adjoining to $\breve{K}$ the π^m-torsion points of a formal Lubin-Tate module of height one over $\mathfrak{o}$. According to [16], Corollary 3.4, we have $\breve{K}_m \subseteq L_m$. In particular, L_∞ contains the maximal abelian extension of K, whence $\breve{K}(\mu_{p^\infty}) \subseteq L_\infty$. By the proof of [16], Theorem 4.4, and by local class field theory, the action of P_0 on $L_0(\mu_{p^\infty})$ factors through $N_{K|\mathbb{Q}_p} \circ \det$ and $\mathrm{Gal}(L_\infty|L_0(\mu_{p^\infty})) = P_0^N$.

Let $\breve{K}' := \varinjlim_m \breve{K}_m$. Since $\mu_{p^\infty} \subset \breve{K}'$, the extension $\breve{K}' K_\infty|\breve{K}' K_0$ is abelian by Kummer theory and so is the extension $L_\infty \cap \breve{K}' K_\infty|\breve{K}' K_0$. It follows from the results of M. Strauch quoted above that $\mathrm{Gal}(L_\infty|\breve{K}' L_0) = P_0 \cap \mathrm{SL}_h(\mathfrak{o})$, of which $\mathrm{Gal}(L_\infty \cap \breve{K}' K_\infty|\breve{K}' L_0)$ is an abelian quotient. Except for the case $q = h = 2$, which we excluded, the commutator subgroup of $P_0 \cap \mathrm{SL}_h(\mathfrak{o})$ contains the unipotent radical N_0 of P_0. Therefore, $L_\infty \cap \breve{K}' K_\infty \subseteq L_\infty^{N_0}$. On the other hand, $N_0 = M_0 \cap \mathrm{SL}_h(\mathfrak{o})$, so that the extension $\mathfrak{o}_{L_\infty^{N_0}}|\mathfrak{o}_{\breve{K}' L_0}$ is étale (cf. Theorem 1.3). In particular, the natural map

$$\mathrm{Gal}(L_\infty \cap \breve{K}' K_\infty|\breve{K}' L_0) \longrightarrow \mathrm{Gal}(\overline{L_\infty \cap \breve{K}' K_\infty}|\overline{\breve{K}' L_0}) \tag{3}$$

is bijective, and the residue class extension on the right is separable. We omit the proof of the following elementary lemma.

Lemma 2.3. *If F is an algebraic extension of $\breve{\mathbb{Q}}_p(\mu_{p^\infty})$ then $K_\infty \cap F K_0 = K_0(\mu_{p^\infty})$ and $\breve{K} K_\infty \cap \breve{K} F K_0 = \breve{K} K_0(\mu_{p^\infty})$. Further, the inclusion $K_\infty \subseteq F K_\infty$ induces an isomorphism on the residue class fields.* □

By Lemma 2.1 and Lemma 2.3, the residue class extension $\overline{L_\infty \cap \breve{K}' K_\infty}|\overline{\breve{K}' L_0}$ is purely inseparable. Being also separable, it has to be trivial, and we obtain $L_\infty \cap \breve{K}' K_\infty = \breve{K}' L_0 = \breve{K}' K_0$ by the bijectivity of (3). Lemma 2.3 then implies that $L_\infty \cap \breve{K} K_\infty = \breve{K}' K_0 \cap \breve{K} K_\infty = \breve{K} K_0(\mu_{p^\infty}) = L_0(\mu_{p^\infty})$. As a consequence,

$$\mathrm{Gal}(L_\infty K_\infty|L_0 K_\infty) \simeq \mathrm{Gal}(L_\infty|L_\infty \cap L_0 K_\infty) \simeq \mathrm{Gal}(L_\infty|L_0(\mu_{p^\infty})) \simeq P_0^N.$$

Further, Lemma 2.1 and Lemma 2.3 imply that

$$\begin{aligned} \mathrm{Gal}(\breve{K} K_\infty|\breve{K} K_0(\mu_{p^\infty})) &\simeq \mathrm{Gal}(K_\infty|K_\infty \cap \breve{K} K_0(\mu_{p^\infty})) \\ &\simeq \mathrm{Gal}(K_\infty|K_0(\mu_{p^\infty})) \simeq \mathbb{Z}_p^{h-1}. \end{aligned}$$

Finally, since K_0 is totally unramified, any Eisenstein polynomial over $\breve{\mathbb{Z}}_p$ remains irreducible over K_0. By a degree argument, $FK_0 \cap F^{\mathrm{alg}} = F$ for any algebraic extension F of $\breve{\mathbb{Q}}_p$. Now consider the commutative diagram of restriction maps

$$\begin{array}{ccc} \mathrm{Gal}(\breve{K}K_0(\mu_{p^\infty})|\breve{K}K_0) & \longrightarrow & \mathrm{Gal}(\breve{K}(\mu_{p^\infty})|\breve{K}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Gal}(K_0(\mu_{p^\infty})|K_0) & \longrightarrow & \mathrm{Gal}(\breve{\mathbb{Q}}_p(\mu_{p^\infty})|\breve{\mathbb{Q}}_p) \simeq \mathbb{Z}_p^\times, \end{array}$$

in which the vertical arrows are well defined by the preceding remark, and in which the lower horizontal arrow is an isomorphism according to Lemma 2.1. Moreover, $K_0(\mu_{p^\infty}) \cap \breve{K}K_0 \cap \breve{\mathbb{Q}}_p(\mu_{p^\infty}) = \breve{K} \cap \breve{\mathbb{Q}}_p(\mu_{p^\infty})$, implying the upper horizontal arrow to be an isomorphism, as well. □

A generalization of J-M. Fontaine's and J-P. Wintenberger's theory of norm fields to higher dimensional local fields was first given by V. Abrashkin (cf. [1]). Building on results of G. Faltings, different and still more general approaches were developed by A. Scholl (cf. [15]) and F. Andreatta (cf. [2]). We are going to follow the ideas of Scholl.

Let again $\overline{F}_0$ be an arbitrary perfect field of characteristic p, and let K_0 be as at the beginning of this section. Let $M|K_0$ be a finite extension of fields and set $M_\infty := MK_\infty$. According to Lemma 2.1 and [15], Theorem 1.3.3, the tower $(MK_m)_{m\geq 0}$ is strictly deeply ramified in the sense of [loc.cit.], page 692. More precisely, Lemma 2.1, [15], Proposition 1.2.1, and the proof of [15], Theorem 1.3.3, show that if $M_m := MK_m$ and if ζ_p denotes a primitive p-th root of unity, then there is an integer $n_0 \geq 1$ with the property that for all integers $n \geq n_0$ the endomorphism $(x \mapsto x^p)$ of $\mathfrak{o}_{M_{n+1}}/(\zeta_p - 1)\mathfrak{o}_{M_{n+1}}$ factors as

$$(4) \quad \mathfrak{o}_{M_{n+1}}/(\zeta_p - 1)\mathfrak{o}_{M_{n+1}} \twoheadrightarrow \mathfrak{o}_{M_n}/(\zeta_p - 1)\mathfrak{o}_{M_n} \hookrightarrow \mathfrak{o}_{M_{n+1}}/(\zeta_p - 1)\mathfrak{o}_{M_{n+1}}.$$

Further, the ramification index of the extension $M_{n+1}|M_n$ is p. In particular, one can choose uniformizers π_{M_n} of M_n sucht that $\pi_{M_{n+1}}^p \equiv \pi_{M_n} \mod (\zeta_p - 1)\mathfrak{o}_{M_{n+1}}$ for all $n \geq n_0$ and set

$$\Pi_{M_\infty} := (\pi_{M_n})_{n\geq n_0} \in X_{M_\infty}^+ := \varprojlim_{n\geq n_0} \mathfrak{o}_{M_n}/(\zeta_p - 1)\mathfrak{o}_{M_n}.$$

In each step, the transition map in this inverse limit is the left arrow in (4). The first main result of Scholl's theory is the following theorem (cf. [15], Theorem 1.3.2).

Theorem 2.4 (Scholl)**.** *The ring $X_{M_\infty}^+$ is a complete discrete valuation ring of characteristic p with uniformizer Π_{M_∞}. For any integer $n \geq n_0$ we have $\overline{M}_{n+1}^p = \overline{M}_n$, and the natural map $X_{M_\infty}^+ \to \mathfrak{o}_{M_n}/(\zeta_p - 1)\mathfrak{o}_{M_n} \to \overline{M}_n$ induces an isomorphism $X_{M_\infty}^+/\Pi_{M_\infty}X_{M_\infty}^+ \simeq \overline{M}_n$.* □

Up to isomorphism, the quotient field X_{M_∞} of $X_{M_\infty}^+$ depends only on M_∞, rather than on M, and only on a certain equivalence class of the tower $(M_n)_{n\geq 0}$. Due to this observation, the construction of X_{M_∞} is actually functorial in M_∞. Scholl's second main result is the following theorem (cf. [15], Theorem 1.3.5).

Theorem 2.5 (Scholl)**.** *The functor $(M_\infty \mapsto X_{M_\infty})$ is an equivalence between the category of finite field extensions of K_∞ (with K_∞-linear homomorphisms of fields) and the category of finite separable extensions of X_{K_∞} (with X_{K_∞}-linear homomorphisms of fields).* □

Let $\tilde{X}$ be the completion of $X_{K_\infty}^{\mathrm{alg}}$ with respect to the valuation induced from X_{K_∞}. If $M|K_0$ is a Galois extension, then so is $MK_m|K_0$ for any integer $m \geq 0$. Since the ideal of $\mathfrak{o}_{MK_m}$ generated by $\zeta_p - 1$ is stable under the action of $\mathrm{Gal}(MK_\infty|K_0)$, it follows that the action of $\mathrm{Gal}(K_0^{\mathrm{sep}}|K_\infty)$ on $X_{K_\infty}^{\mathrm{alg}}$ and $\tilde{X}$ extends to an action of $\mathrm{Gal}(K_0^{\mathrm{sep}}|K_0)$.

There is a totally unramified complete discrete valuation ring $\mathfrak{o}_{\breve{\mathcal{E}}} \subseteq W(\tilde{X})$ with residue class field $X_{K_\infty}^{\mathrm{sep}}$ which is stable under the Frobenius endomorphism $\varphi := W(x \mapsto x^p)$ of $W(\tilde{X})$ and under the natural action of $\mathrm{Gal}(K_0^{\mathrm{sep}}|K_0)$ (cf. [15], section 2.1 and section 2.3). In fact, in the notation of [15] we may set $\mathfrak{o}_{\breve{\mathcal{E}}} := \mathbf{A}$.

3 Admissible φ-modules

Let H be a topological group. We let $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(H)$ denote the category of p-adically separated and complete $\mathbb{Z}_p$-modules V carrying a $\mathbb{Z}_p$-linear action of H such that the structure map $H \times V \to V$ is continuous for the p-adic topology on V and the product topology on the left hand side. Note that if H acts $\mathbb{Z}_p$-linearly on a p-adically separated and complete $\mathbb{Z}_p$-module V then the structure map $H \times V \to V$ is continuous if and only if for every integer $n \geq 0$ the induced H-action on V/p^nV is *smooth* in the sense that the stabilizer of every element of V/p^nV is open in H.

If R is a commutative unital ring and if A and B are two R-modules, then we denote by

$$A\widehat{\otimes}_R B := \varprojlim_{n\geq 0}(A \otimes_R B)/p^n(A \otimes_R B)$$

the p-adic completion of the R-module $A \otimes_R B$.

Lemma 3.1. *Let R be a commutative unital ring, and let A and B be two p-adically separated and complete R-modules carrying continuous R-linear actions of H. The diagonal R-linear action of H on $A \otimes_R B$ induces a continuous R-linear action of H on $A\widehat{\otimes}_R B$.*

Proof: For every integer $n \geq 0$ there are canonical isomorphisms

$$(5) \qquad (A \otimes_R B)/p^n(A \otimes_R B) \simeq A \otimes_R (B/p^nB) \simeq (A/p^nA) \otimes_R (B/p^nB).$$

The H-representations A and B being continuous, the induced R-linear actions on A/p^nA and B/p^nB are smooth, and hence so is the diagonal H-action on the right hand side of (5). □

We need to slightly generalize the formalism of [6], section A.1.2, developed by J-M. Fontaine.

Let E be a field of characteristic p, let $\mathfrak{o}_{\mathcal{E}}$ be a Cohen ring of E, and set $\mathcal{E} := \mathrm{Quot}(\mathfrak{o}_{\mathcal{E}})$. The group $H := \mathrm{Gal}(E^{\mathrm{sep}}|E) \simeq \mathrm{Gal}(\mathcal{E}^{\mathrm{unr}}|\mathcal{E})$ acts on $\breve{\mathcal{E}}$ and $\mathfrak{o}_{\breve{\mathcal{E}}}$ in such a way that the induced action on the residue class field E^{sep} of $\mathfrak{o}_{\breve{\mathcal{E}}}$ is the natural one. Note that this makes $\mathfrak{o}_{\breve{\mathcal{E}}}$ an object of $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(H)$ because for any integer $n \geq 0$ we have $\mathfrak{o}_{\breve{\mathcal{E}}}/p^n\mathfrak{o}_{\breve{\mathcal{E}}} \simeq \mathfrak{o}_{\mathcal{E}^{\mathrm{unr}}}/p^n\mathfrak{o}_{\mathcal{E}^{\mathrm{unr}}}$, and the action of H on $\mathfrak{o}_{\mathcal{E}^{\mathrm{unr}}}$ is smooth.

We choose a Frobenius endomorphism φ of $\mathfrak{o}_{\mathcal{E}}$, i.e. a ring endomorphism whose reduction modulo p is the endomorphism $(x \mapsto x^p)$ of E. The endomorphism φ extends uniquely to Frobenius endomorphisms φ of $\mathfrak{o}_{\mathcal{E}^{\mathrm{unr}}}$ and $\mathfrak{o}_{\breve{\mathcal{E}}}$, commuting with the action of H.

If H' is a closed subgroup of H we set $\mathfrak{o}_{\mathcal{E}_{H'}} := \mathfrak{o}_{\breve{\mathcal{E}}}^{H'}$ and let $\mathcal{E}_{H'}$ be its quotient field. The ring $\mathfrak{o}_{\mathcal{E}_{H'}}$ is the p-adic completion of $\mathfrak{o}_{\mathcal{E}^{\mathrm{unr}}}^{H'}$ and is a Cohen ring of $(E^{\mathrm{sep}})^{H'}$.

Now assume H' to be a closed subgroup of H which is also normal. If V is an object of $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(H/H')$ then we set

$$\mathbb{D}_{H'}(V) := (\mathfrak{o}_{\mathcal{E}_{H'}} \widehat{\otimes}_{\mathbb{Z}_p} V)^H = (\mathfrak{o}_{\mathcal{E}_{H'}} \widehat{\otimes}_{\mathbb{Z}_p} V)^{H/H'}.$$

If $H' = \{1\}$ we simply write $\mathbb{D}(V) := \mathbb{D}_{\{1\}}(V)$.

Theorem 3.2. *Assume H' to be a closed normal subgroup of $H := \mathrm{Gal}(E^{\mathrm{sep}}|E)$. For every object V of $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(H/H')$ the natural map*

$$\mathfrak{o}_{\mathcal{E}_{H'}} \widehat{\otimes}_{\mathfrak{o}_{\mathcal{E}}} \mathbb{D}_{H'}(V) \longrightarrow \mathfrak{o}_{\mathcal{E}_{H'}} \widehat{\otimes}_{\mathbb{Z}_p} V \tag{6}$$

is a topological isomorphism of p-adically separated and complete $\mathfrak{o}_{\mathcal{E}_{H'}}$-modules. The functor $\mathbb{D}_{H'}$ from $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(H/H')$ into the category of p-adically separated and complete $\mathfrak{o}_{\mathcal{E}}$-modules is faithful and transforms strict exact sequences into strict exact sequences.

Proof: Let us first assume that $p^nV = 0$ for some integer $n \geq 0$. Then all complete tensor products in (6) can be replaced by the algebraic ones. Moreover, since the ring extensions $\mathfrak{o}_{\mathcal{E}_{H'}}|\mathfrak{o}_{\mathcal{E}}$ and $\mathfrak{o}_{\mathcal{E}_{H'}}|\mathbb{Z}_p$ are flat, the algebraic tensor products involved, as well as the functor $(\,\cdot\,)^H$, commute with filtered unions. Now V, being a smooth representation of the compact group H, is the filtered union of its H-subrepresentations whose underlying $\mathbb{Z}_p$-modules are finitely generated. Therefore, in order to show the bijectivity of (6), we may assume V to be of finite type over $\mathbb{Z}_p$. This case is proved in [6], Proposition 1.2.4, and also gives that $\mathbb{D}_{H'}$ is exact on exact sequences of p-torsion representations of finite type over $\mathbb{Z}_p$. Since the direct limit is an exact functor on filtered inductive limits of $\mathfrak{o}_{\mathcal{E}}$-modules, the latter result implies $\mathbb{D}_{H'}$ to be exact on exact sequences of objects of $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(H/H')$ all of which are annihilated by p^n.

For arbitrary V the bijectivity of (6) follows from the same argument of *passage to the limit* used in the proof of [6], Proposition 1.2.4. One also obtains a natural isomorphism

$$\mathbb{D}_{H'}(V/p^nV) \simeq \mathbb{D}_{H'}(V)/p^n\mathbb{D}_{H'}(V). \tag{7}$$

If $V \in \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(H/H')$ is such that $\mathbb{D}_{H'}(V) = 0$ then also $\mathbb{D}_{H'}(V)/p^n\mathbb{D}_{H'}(V) \simeq \mathbb{D}_{H'}(V/p^nV) = 0$ for all integers $n \geq 0$. Consider the isomorphism (6) for V/p^nV, in which the complete tensor products can be replaced by the algebraic ones. Since $\mathfrak{o}_{\mathcal{E}_{H'}}$ is faithfully flat over $\mathbb{Z}_p$, we obtain $V/p^nV = 0$ for all n and thus $V = 0$.

Recall that a homomorphism $\rho : M \to N$ of abelian groups is called *strict* for the p-adic topology if $p^n\mathrm{im}(\rho) = \mathrm{im}(\rho) \cap p^nN$ for all integers $n \geq 0$. Consider a strict exact sequence $0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$ in $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(H/H')$. By strictness, the induced sequence modulo p^n is still exact for any integer $n \geq 0$. Applying $\mathbb{D}_{H'}$ to the latter, we obtain an exact sequence of $\mathfrak{o}_{\mathcal{E}}$-modules because of the exactness result already proved. We also know the natural maps $\mathbb{D}_{H'}(V_1/p^{n+1}V_1) \to \mathbb{D}_{H'}(V_1/p^nV_1)$ to be surjective, so that the Mittag-Leffler criterion implies the sequence $0 \longrightarrow \mathbb{D}_{H'}(V_1) \longrightarrow \mathbb{D}_{H'}(V_2) \longrightarrow \mathbb{D}_{H'}(V_3) \longrightarrow 0$ to be exact. It is strict because it remains exact after reduction modulo p^n for any integer $n \geq 0$ (cf. (7)). □

Corollary 3.3. *If H' is a closed and normal subgroup of $H = \mathrm{Gal}(E^{\mathrm{sep}}|E)$, and if V is an object of $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(H/H')$, also viewed as an object of $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(H)$ via the projection $H \to H/H'$, then the natural $\mathfrak{o}_{\mathcal{E}}$-linear map $\mathbb{D}_{H'}(V) \to \mathbb{D}(V)$ is a topological isomorphism for the p-adic topologies.*

Proof: Let $n \geq 0$ be an integer. The diagram of homomorphisms of $\mathfrak{o}_{\mathcal{E}}$-modules

$$\begin{array}{ccc} \mathbb{D}_{H'}(V)/p^n\mathbb{D}_{H'}(V) & \longrightarrow & \mathbb{D}(V)/p^n\mathbb{D}(V) \\ \downarrow & & \downarrow \\ \mathbb{D}_{H'}(V/p^nV) & \longrightarrow & \mathbb{D}(V/p^nV) \end{array}$$

is commutative and the vertical arrows are isomorphisms by (7). We may therefore assume V to be annihilated by some power p^n of p. Consider the commutative diagram of $\mathfrak{o}_{\breve{\mathcal{E}}}$-modules

$$\begin{array}{ccc} \mathfrak{o}_{\breve{\mathcal{E}}} \otimes_{\mathfrak{o}_{\mathcal{E}}} \mathbb{D}_{H'}(V) & \longrightarrow & \mathfrak{o}_{\breve{\mathcal{E}}} \otimes_{\mathfrak{o}_{\mathcal{E}}} \mathbb{D}(V) \\ \downarrow & & \downarrow \\ \mathfrak{o}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V & = & \mathfrak{o}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V. \end{array}$$

The vertical arrows are isomorphisms by Theorem 3.2, so that the claim follows from $\mathfrak{o}_{\breve{\mathcal{E}}}$ being faithfully flat over $\mathfrak{o}_{\mathcal{E}}$. □

We call *(complete) φ-module over $\mathfrak{o}_{\mathcal{E}}$* any ($p$-adically separated and complete) $\mathfrak{o}_{\mathcal{E}}$-module M together with a φ-semilinear endomorphism φ_M. The category of complete φ-modules over $\mathfrak{o}_{\mathcal{E}}$ will be denoted by $\Phi^{\mathrm{cpl}}_{\mathfrak{o}_{\mathcal{E}}}$.

As usual, a φ-module (M, φ_M) over $\mathfrak{o}_{\mathcal{E}}$ is called *étale* if M is finitely generated over $\mathfrak{o}_{\mathcal{E}}$ and if the $\mathfrak{o}_{\mathcal{E}}$-linear homomorphism

$$\Phi_M : \mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{o}_{\mathcal{E}},\varphi} M \to M, \quad \alpha \otimes m \mapsto \alpha \cdot \varphi_M(m),$$

is bijective. Note that any finitely generated $\mathfrak{o}_{\mathcal{E}}$-module is automatically p-adically separated and complete.

Definition 3.4. A complete φ-module (M, φ_M) over $\mathfrak{o}_{\mathcal{E}}$ is called *admissible* if for any integer $n \geq 0$ the φ-module $(M/p^n M, \varphi_M \bmod p^n)$ over $\mathfrak{o}_{\mathcal{E}}$ is the union of its étale φ-submodules. The category of admissible φ-modules over $\mathfrak{o}_{\mathcal{E}}$ is denoted by $\Phi^{\mathrm{adm}}_{\mathfrak{o}_{\mathcal{E}}}$.

Note that if a φ-module over $\mathfrak{o}_{\mathcal{E}}$ is the union of its étale φ-submodules then it is also the *filtered* union of its étale φ-submodules (cf. [6], Proposition 1.1.5).

It follows from (5) that if (M, φ_M) is an admissible φ-module over $\mathfrak{o}_{\mathcal{E}}$ then the natural map $\Phi_M : \mathfrak{o}_{\mathcal{E}} \widehat{\otimes}_{\mathfrak{o}_{\mathcal{E}},\varphi} M \to M$ is bijective. If M is finitely generated over $\mathfrak{o}_{\mathcal{E}}$ then the natural map

$$\mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{o}_{\mathcal{E}},\varphi} M \to \mathfrak{o}_{\mathcal{E}} \widehat{\otimes}_{\mathfrak{o}_{\mathcal{E}},\varphi} M \tag{8}$$

is bijective and M is admissible if and only if it is étale. Note also that if E admits a finite p-basis then φ makes $\mathfrak{o}_{\mathcal{E}}$ a finitely generated and free module over itself and the natural homomorphism (8) is bijective for any p-adically separated and complete $\mathfrak{o}_{\mathcal{E}}$-module M.

Lemma 3.5. *If* $0 \to M_1 \to M_2 \to M_3 \to 0$ *is a strict exact sequence of complete* φ*-modules over* $\mathfrak{o}_{\mathcal{E}}$ *then* M_2 *is admissible if and only if* M_1 *and* M_3 *are.*

Proof: By strictness, the sequence remains exact after reduction modulo p^n, and we may assume $p^n M_2 = 0$ for some integer $n \geq 0$. It follows from Definition 3.4 and [6], Proposition 1.1.6, that together with M_2 also M_1 and M_3 are admissible.

Conversely, assume M_1 and M_3 to be admissible, let $m \in M_2$, and let N_3 be an étale φ-submodule of M_3 containing the image of m. Choose a finitely generated $\mathfrak{o}_{\mathcal{E}}$-submodule N_2' of M_2 mapping onto N_3 and containing m. Let $(n_i)_i$ be a finite set of generators of N_2'. For every index i there is an element $n_i' \in N_2'$ such that $\varphi_{M_2}(n_i) - n_i' \in M_1$. Let N_1 be an étale φ-submodule of M_1 containing all elements $\varphi_{M_2}(n_i) - n_i'$, as well as $N_2' \cap M_1$. Setting $N_2 := N_1 + N_2'$, we obtain a short exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$ of φ-modules over $\mathfrak{o}_{\mathcal{E}}$ in which N_1 and N_3, and hence N_2, are étale (cf. [6], Proposition 1.1.6). $\square$

If as above $H = \mathrm{Gal}(E^{\mathrm{sep}}|E)$, and if V is an object of $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(H)$, then the restriction of $\varphi \otimes \mathrm{id}_V$ from $\mathfrak{o}_{\breve{\mathcal{E}}} \widehat{\otimes}_{\mathbb{Z}_p} V$ to $\mathbb{D}(V)$ makes the latter a complete φ-module over $\mathfrak{o}_{\mathcal{E}}$. If (M, φ_M) is an object of $\Phi^{\mathrm{cpl}}_{\mathfrak{o}_{\mathcal{E}}}$ then the φ-semilinear endomorphism $\varphi \otimes \varphi_M$ of $\mathfrak{o}_{\breve{\mathcal{E}}} \otimes_{\mathfrak{o}_{\mathcal{E}}} M$ extends to an endomorphism of $\mathfrak{o}_{\breve{\mathcal{E}}} \widehat{\otimes}_{\mathfrak{o}_{\mathcal{E}}} M$. We denote this extension by φ_M, again. Note that φ_M commutes with the action of H coming from the natural action on $\mathfrak{o}_{\breve{\mathcal{E}}}$ and the trivial action on M.

Theorem 3.6. *(i) For any object* V *of* $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(H)$*, the complete* φ*-module* $\mathbb{D}(V)$ *over* $\mathfrak{o}_{\mathcal{E}}$ *is admissible.* V *is of finite type over* $\mathbb{Z}_p$ *if and only if* $\mathbb{D}(V)$ *is of finite type over* $\mathfrak{o}_{\mathcal{E}}$*.*

(ii) The functor $\mathbb{D} : \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(H) \to \Phi^{\mathrm{adm}}_{\mathfrak{o}_{\mathcal{E}}}$ *is an equivalence of categories transforming strict exact sequences into strict exact sequences.*

(iii) The functor $\mathbb{V} : \Phi^{\mathrm{adm}}_{\mathfrak{o}_{\mathcal{E}}} \to \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(H)$ *given by* $\mathbb{V}(M) := (\mathfrak{o}_{\breve{\mathcal{E}}} \widehat{\otimes}_{\mathfrak{o}_{\mathcal{E}}} M)^{\varphi_M = 1}$ *is quasi inverse to* $\mathbb{D}$*.*

Proof: Let V be an object of $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(H)$. For any integer $n \geq 0$ we have $\mathbb{D}(V)/p^n\mathbb{D}(V) \simeq \mathbb{D}(V/p^nV)$ as φ-modules over $\mathfrak{o}_{\mathcal{E}}$. Writing V/p^nV as the filtered union of its H-subrepresentations of finite type over $\mathbb{Z}_p$, it follows from [6], Proposition 1.2.6, that the φ-module $\mathbb{D}(V)$ is admissible.

Let M be an object of $\Phi^{\mathrm{adm}}_{\mathfrak{o}_{\mathcal{E}}}$. If $p^nM = 0$ then $\mathbb{V}(M)$, being an H-subrepresentation of the smooth H-representation $\mathfrak{o}_{\tilde{\mathcal{E}}}\widehat{\otimes}_{\mathfrak{o}_{\mathcal{E}}}M$, is smooth, as well.

Now assume $p^nV = 0$ and $p^nM = 0$. By flatness, the algebraic tensor products involved, as well as kernels and the functor $(\,\cdot\,)^H$, commute with filtered unions. In order to show that the natural maps $M \to \mathbb{D}(\mathbb{V}(M))$ and $V \to \mathbb{V}(\mathbb{D}(V))$ are bijective, we may therefore assume V to be of finite type over $\mathbb{Z}_p$ and M to be étale. In this case, the assertion was proved by J-M. Fontaine (cf. [7], Theorem 2.32). This establishes (ii) and (iii) for any of the full subcategories of objects annihilated by some fixed power of p. In particular, the functor $\mathbb{V}$ is exact here, as follows formally from Theorem 3.2.

Now let M be arbitrary and note that $\mathbb{V}(M) \simeq \varprojlim_n \mathbb{V}(M/p^nM)$. The results we have proved so far imply that $\mathbb{V}(M)/p^n\mathbb{V}(M) \simeq \mathbb{V}(M/p^nM)$ for all integers $n \geq 0$. In particular, the $\mathbb{Z}_p$-module $\mathbb{V}(M)$ is p-adically separated and complete, and the H-representation $\mathbb{V}(M)$ is continuous. We have

$$\begin{aligned}\mathbb{D}(\mathbb{V}(M)) &\simeq \varprojlim_n \mathbb{D}(\mathbb{V}(M)/p^n\mathbb{V}(M)) \simeq \varprojlim_n \mathbb{D}(\mathbb{V}(M/p^nM)) \\ &\simeq \varprojlim_n M/p^nM \simeq M,\end{aligned}$$

naturally in M, and likewise $\mathbb{V}(\mathbb{D}(V)) \simeq V$ by the torsion case already treated.

The final assertion in (i) follows from the fact that if V (resp. M) is finitely generated over $\mathbb{Z}_p$ (resp. over $\mathfrak{o}_{\mathcal{E}}$) then the functor $\mathbb{D}$ (resp. $\mathbb{V}$) coincides with that of Fontaine. $\square$

If H is an arbitrary topological group then we let $\mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(H)$ be the category of *unitary representations of H over $\mathbb{Q}_p$*, i.e. that of all $\mathbb{Q}_p$-Banach spaces V carrying a continuous $\mathbb{Q}_p$-linear action of H and admitting a bounded open H-stable $\mathbb{Z}_p$-lattice $\mathring{V}$. As morphisms we consider all continuous H-equivariant $\mathbb{Q}_p$-linear maps.

We let $\Phi^{\mathrm{uni}}_{\mathcal{E}}$ be the category of *unitary Banach φ-modules over $\mathcal{E}$*, i.e. that of $\mathcal{E}$-Banach spaces M carrying a φ-semilinear endomorphism φ_M and admitting a bounded open and φ_M-stable $\mathfrak{o}_{\mathcal{E}}$-lattice $\mathring{M}$. As morphisms we consider all continuous $\mathcal{E}$-linear maps which commute with the given Frobenii. We let $\Phi^{\mathrm{adm}}_{\mathcal{E}}$ be the full subcategory of $\Phi^{\mathrm{uni}}_{\mathcal{E}}$ consisting of those objects for which the bounded open φ_M-stable $\mathfrak{o}_{\mathcal{E}}$-lattice $\mathring{M}$ can be chosen in such a way that the complete φ-module $(\mathring{M}, \varphi_M|\mathring{M})$ over $\mathfrak{o}_{\mathcal{E}}$ is admissible in the sense of Definition 3.4. The objects of $\Phi^{\mathrm{adm}}_{\mathcal{E}}$ will be called *admissible φ-modules over $\mathcal{E}$*.

Remark 3.7. If V is a $\mathbb{Q}_p$-Banach space carrying a continuous $\mathbb{Q}_p$-linear action of a topological group H then the condition of V being unitary in the above sense is equivalent to the existence of a norm on V, defining its topology, for which the action of H is by isometries. Note that if H is compact then any continuous

$\mathbb{Q}_p$-linear representation of H on a $\mathbb{Q}_p$-Banach space is automatically unitary (cf. [11], Remark 8.2).
Similarly, if M is an $\mathcal{E}$-Banach space carrying a φ-semilinear endomorphism φ_M, then the existence of a φ_M-stable bounded open $\mathfrak{o}_{\mathcal{E}}$-lattice $\mathring{M}$ of M is equivalent to the existence of a norm on M, defining its topology, for which φ_M is norm decreasing. Indeed, φ_M is norm decreasing for the gauge norm associated with $\mathring{M}$ because $\varphi : \mathcal{E} \to \mathcal{E}$ is an isometry. The converse is clear.

If F is a complete valued nonarchimedean field with valuation ring $\mathfrak{o}_F$, and if V and W are two F-Banach spaces, then we denote by $V\widehat{\otimes}_F W$ the completion of $V \otimes_F W$ with respect to the tensor product norm. It is again an F-Banach space. If F is discretely valued, if π_F denotes a uniformizer and if $\mathring{V}$ and $\mathring{W}$ are bounded open $\mathfrak{o}_F$-lattices in V and W, respectively, then the π_F-adic completion $\mathring{V}\widehat{\otimes}_{\mathfrak{o}_F}\mathring{W}$ of $\mathring{V} \otimes_{\mathfrak{o}_F} \mathring{W}$ injects as a bounded open $\mathfrak{o}_F$-lattice into $V\widehat{\otimes}_F W$. In particular, there is an isomorphism $F \otimes_{\mathfrak{o}_F} (\mathring{V}\widehat{\otimes}_{\mathfrak{o}_F}\mathring{W}) \simeq V\widehat{\otimes}_F W$ of F-vector spaces. If V and W carry unitary F-linear actions of H, this implies that the diagonal H-action on $V \otimes_F W$ extends to a unitary action on $V\widehat{\otimes}_F W$ (cf. Lemma 3.1).

Likewise, if (M, φ_M) is a unitary Banach φ-module over $\mathcal{E}$ and if $\mathring{M}$ is a φ_M-stable bounded open $\mathfrak{o}_{\mathcal{E}}$-lattice of M, then the φ-linear endomorphism $\varphi \otimes \varphi_M$ of $\breve{\mathcal{E}} \otimes_{\mathcal{E}} M$ extends to $\breve{\mathcal{E}}\widehat{\otimes}_{\mathcal{E}} M$ by continuity and stabilizes the bounded open $\mathfrak{o}_{\breve{\mathcal{E}}}$-lattice $\mathfrak{o}_{\breve{\mathcal{E}}}\widehat{\otimes}_{\mathfrak{o}_{\mathcal{E}}}\mathring{M}$. We denote this extension by φ_M again. The following theorem is a direct consequence of Theorem 3.6.

Theorem 3.8. *Let H' be a closed normal subgroup of $H := \mathrm{Gal}(E^{\mathrm{sep}}|E)$ and set $\mathcal{E}_{H'} := \breve{\mathcal{E}}^{H'}$. Let V be an object of $\mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(H/H')$, viewed also as an object of $\mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(H)$ via the projection $H \to H/H'$.*

(i) The restriction of $\varphi\widehat{\otimes} id_V$ from $\mathcal{E}_{H'}\widehat{\otimes}_{\mathbb{Q}_p} V$ to $\mathbb{D}_{H'}(V) := (\mathcal{E}_{H'}\widehat{\otimes}_{\mathbb{Q}_p} V)^H$ gives the latter the structure of an admissible φ-module over $\mathcal{E}$. The natural map $\mathbb{D}_{H'}(V) \to \mathbb{D}(V) := \mathbb{D}_{\{1\}}(V)$ is a topological isomorphism of $\mathcal{E}$-Banach spaces.

(ii) The functor $\mathbb{D} : \mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(H) \to \Phi^{\mathrm{adm}}_{\mathcal{E}}$ is an equivalence of categories transforming strict exact sequences into strict exact sequences. A quasi inverse $\mathbb{V}$ is given by $\mathbb{V}(M) := (\breve{\mathcal{E}}\widehat{\otimes}_{\mathcal{E}} M)^{\varphi_M = 1}$. An object V of $\mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(H)$ is finite dimensional over $\mathbb{Q}_p$ if and only if $\mathbb{D}(V)$ is finite dimensional over $\mathcal{E}$. □

4 Admissible (φ, Γ)-modules

Let F_0 be as in section 2, let $t_1, \ldots, t_{h-1}$ be indeterminates, and denote by K_0 the quotient field of $\mathfrak{o}_{K_0} := \mathfrak{o}_{F_0}[[t_1, \ldots, t_{h-1}]]^{\widehat{}}_{(p)}$. We let K_∞ and $\tilde{X}$ be as above.

As a set, the ring $W(\tilde{X})$ of Witt vectors over $\tilde{X}$ is in bijection with $\prod_{n\geq 0} \tilde{X}$. Endowing the latter with the product topology of the valuation topologies on each factor, one obtains the so-called *weak topology* on $W(\tilde{X})$. It makes $W(\tilde{X})$ a topological ring. Note that for any integer $n \geq 0$ the projection $W(\tilde{X}) \to \tilde{X}^n$ onto the first n components induces a bijection $W(\tilde{X})/p^n W(\tilde{X}) \simeq \tilde{X}^n$. Endowing each of these quotients with the quotient topology (which is the product topology of the valuation topologies on each factor), the bijection

$$W(\tilde{X}) \simeq \varprojlim_{n\geq 0} W(\tilde{X})/p^n W(\tilde{X})$$

is a topological isomorphism for the weak topology on the left hand side and for the inverse limit topology on the right hand side.

Recall further that there is an action of $\mathrm{Gal}(K_0^{\mathrm{sep}}|K_0)$ on $\tilde{X}$, as well as a totally unramified complete discrete valuation ring $\mathfrak{o}_{\breve{\mathcal{E}}} \subseteq W(\tilde{X})$ which is stable under the action of $\mathrm{Gal}(K_0^{\mathrm{sep}}|K_0)$ and under the Frobenius endomorphism of $W(\tilde{X})$. Its residue class field is a separable closure of the norm field X_{K_∞}.

If H is a fixed closed subgroup of $\mathrm{Gal}(K_0^{\mathrm{sep}}|K_\infty)$ we endow the ring $\mathfrak{o}_{\mathcal{E}} := \mathfrak{o}_{\breve{\mathcal{E}}}^H$ with the subspace topology of $W(\tilde{X})$. In this way, it becomes a topological ring.

If M is a finitely generated $\mathfrak{o}_{\mathcal{E}}$-module then its *weak* $\mathfrak{o}_{\mathcal{E}}$*-topology* is defined to be the quotient topology of $\mathfrak{o}_{\mathcal{E}}^d/\ker\rho$ for any $\mathfrak{o}_{\mathcal{E}}$-linear surjection $\rho : \mathfrak{o}_{\mathcal{E}}^d \to M$. If M is any p-adically separated and complete $\mathfrak{o}_{\mathcal{E}}$-module then we define its *weak* $\mathfrak{o}_{\mathcal{E}}$*-topology* to be the inverse limit topology of $M \simeq \varprojlim_{n\geq 0} M/p^n M$ where each $\mathfrak{o}_{\mathcal{E}}$-module M/p^nM is the topological inductive limit (in the category of topological spaces) of its finitely generated $\mathfrak{o}_{\mathcal{E}}$-submodules endowed with their weak $\mathfrak{o}_{\mathcal{E}}$-topologies.

Remark 4.1. Note that if M is a finitely generated $\mathfrak{o}_{\mathcal{E}}$-module then it is p-adically separated and complete, and the two notions of weak $\mathfrak{o}_{\mathcal{E}}$-topology given above coincide. Namely, endow M and any of the modules M/p^nM with the quotient topologies coming from suitable presentations ρ as above. The projection $M \to M/p^nM$ is then continuous and open, and it suffices to see that the continuous bijection $M \to \varprojlim_n M/p^nM$ is a topological isomorphism. This in turn is due to the fact that M admits a basis of open neighborhoods of zero consisting of additive subgroups, each of which contains some submodule p^nM.

Note also that if M and N are two p-adically separated and complete $\mathfrak{o}_{\mathcal{E}}$-modules then any $\mathfrak{o}_{\mathcal{E}}$-linear map $M \to N$ is automatically continuous for the weak $\mathfrak{o}_{\mathcal{E}}$-topologies.

We need to supplement the results of [15], section 2.1, by a few topological considerations. The necessity of doing so was pointed out in the diploma thesis [14] of T. Schoeneberg.

Lemma 4.2. *Let $F|F_0$ be a finite totally ramified extension and set $G := \mathrm{Gal}(K_0^{\mathrm{sep}}|FK_0)$, $H := G \cap \mathrm{Gal}(K_0^{\mathrm{sep}}|K_\infty) \simeq \mathrm{Gal}(K_0^{\mathrm{sep}}|FK_\infty)$ and $\mathfrak{o}_{\mathcal{E}} := \mathfrak{o}_{\breve{\mathcal{E}}}^H$. If V is an object of $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(G)$ then the diagonal action of G on $\mathfrak{o}_{\breve{\mathcal{E}}}\widehat{\otimes}_{\mathbb{Z}_p}V$ is continuous for the weak $\mathfrak{o}_{\breve{\mathcal{E}}}$-topology. The action of $\Gamma := G/H$ on $\mathbb{D}(V) := (\mathfrak{o}_{\breve{\mathcal{E}}}\widehat{\otimes}_{\mathbb{Z}_p}V)^H$ is continuous for the weak $\mathfrak{o}_{\mathcal{E}}$-topology.*

Proof: The first assertion is easily established. As for the second assertion, we follow the arguments of [14], Proposition 3.2.21, and have the results of Theorem 3.2 at our disposal. Thus, we may assume $p^nV = 0$ for some integer $n \geq 0$ and V to be finitely generated over $\mathbb{Z}_p$. The natural map

$$\mathfrak{o}_{\breve{\mathcal{E}}} \otimes_{\mathfrak{o}_{\mathcal{E}}} \mathbb{D}(V) \longrightarrow \mathfrak{o}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V$$

is an $\mathfrak{o}_{\breve{\mathcal{E}}}$-linear isomorphism, hence a topological isomorphism for the weak $\mathfrak{o}_{\breve{\mathcal{E}}}$-topologies. Since it is also G-equivariant, our above result reduces us to showing that the natural map $\mathbb{D}(V) \to \mathfrak{o}_{\breve{\mathcal{E}}} \otimes_{\mathfrak{o}_{\mathcal{E}}} \mathbb{D}(V)$ is a topological embedding if the left (resp. the right) hand side is endowed with its weak $\mathfrak{o}_{\mathcal{E}}$-topology (resp. with its weak $\mathfrak{o}_{\breve{\mathcal{E}}}$-topology). Since $\mathbb{D}(V)$ is a finitely generated torsion module over the totally unramified discrete valuation ring $\mathfrak{o}_{\mathcal{E}}$ of characteristic zero, we may assume $\mathbb{D}(V) \simeq \mathfrak{o}_{\mathcal{E}}/p^m\mathfrak{o}_{\mathcal{E}}$ for some integer $m \geq 0$.

The homomorphisms $\mathfrak{o}_{\mathcal{E}} \hookrightarrow \mathfrak{o}_{\breve{\mathcal{E}}} \hookrightarrow W(\tilde{X})$ being continuous, the induced homomorphisms modulo p^m are continuous for the quotient topologies, and it suffices to see that the homomorphism $\mathfrak{o}_{\mathcal{E}}/p^m\mathfrak{o}_{\mathcal{E}} \hookrightarrow W(\tilde{X})/p^m W(\tilde{X})$ is a topological embedding.

For the moment, consider the ring $\mathfrak{o}_{\mathcal{E}(K_0)} := \mathfrak{o}_{\breve{\mathcal{E}}}^{\mathrm{Gal}(K_0^{\mathrm{sep}}|K_\infty)}$, corresponding to the case $F = F_0$. According to A. Scholl's construction in [15], section 2.1 and section 2.3, the ring $\mathfrak{o}_{\mathcal{E}(K_0)}$ contains a G-stable and φ-stable two dimensional noetherian regular local subring $\mathfrak{o}^+_{\mathcal{E}(K_0)}$ such that $\mathfrak{o}_{\mathcal{E}(K_0)} \simeq (\mathfrak{o}^+_{\mathcal{E}(K_0)})\widehat{_{(p)}}$ and

$$\mathfrak{o}^+_{\mathcal{E}(K_0)}/p\mathfrak{o}^+_{\mathcal{E}(K_0)} = \mathfrak{o}_{X_{K_\infty}} \subset X_{K_\infty} = \mathfrak{o}_{\mathcal{E}(K_0)}/p\mathfrak{o}_{\mathcal{E}(K_0)}.$$

Choosing a Cohen ring $C \subseteq \mathfrak{o}^+_{\mathcal{E}(K_0)}$ of $\overline{X}_{K_\infty}$, there is a noncanonical isomorphism $\mathfrak{o}^+_{\mathcal{E}(K_0)} \simeq C[[t]]$, sending the formal variable t to a lift of a uniformizer of $\mathfrak{o}_{X_{K_\infty}}$. Further, $\mathfrak{o}_{\mathcal{E}(K_0)}$ is isomorphic to the p-adic completion $C[[t]][t^{-1}]\widehat{\ }$ of $C[[t]][t^{-1}]$, which is also the complete local ring of $\mathfrak{o}^+_{\mathcal{E}(K_0)}$ at (p).

The choice of φ on $\mathfrak{o}_{\mathcal{E}(K_0)}$ – viewed abstractly as a Cohen ring of X_{K_∞} – is related to the embedding $\mathfrak{o}_{\mathcal{E}(K_0)} \to W(\tilde{X})$ through a factorization

$$\mathfrak{o}_{\mathcal{E}(K_0)} \overset{\mu}{\hookrightarrow} W(\mathfrak{o}_{\mathcal{E}(K_0)}) \longrightarrow W(\mathfrak{o}_{\mathcal{E}(K_0)}/p\mathfrak{o}_{\mathcal{E}(K_0)}) \longrightarrow W(\tilde{X}).$$

The two arrows on the right are the natural ones, whereas μ is determined by the condition that for any element $a \in \mathfrak{o}_{\mathcal{E}(K_0)}$ and any integer $n \geq 0$ the n-th ghost component of $\mu(a)$ be $\varphi^n(a)$. Since the subring $\mathfrak{o}^+_{\mathcal{E}(K_0)}$ of $\mathfrak{o}_{\mathcal{E}(K_0)}$ is φ-stable, the embedding $\mathfrak{o}_{\mathcal{E}(K_0)} \subseteq W(\tilde{X})$ restricts to an embedding $\mathfrak{o}^+_{\mathcal{E}(K_0)} \subseteq W(\mathfrak{o}_{\tilde{X}})$.

Note that the weak topology of $W(\tilde{X})$ can be defined by the basis of open additive subgroups $p^i W(\tilde{X}) + t^j W(\mathfrak{o}_{\tilde{X}})$ with integers $i \geq 0$ and $j \geq 0$. We retopologize $\mathfrak{o}_{\mathcal{E}(K_0)}$ by defining a basis of open neighborhoods of zero through the $\mathfrak{o}^+_{\mathcal{E}(K_0)}$-submodules $p^i\mathfrak{o}_{\mathcal{E}(K_0)} + t^j\mathfrak{o}^+_{\mathcal{E}(K_0)}$. This topology is finer than the weak $\mathfrak{o}_{\mathcal{E}(K_0)}$-topology and induces a continuous embedding $\mathfrak{o}_{\mathcal{E}(K_0)}/p^m\mathfrak{o}_{\mathcal{E}(K_0)} \to W(\tilde{X})/p^m W(\tilde{X})$. It suffices to see that the latter is topological. For this it suffices to see that

$$\mathfrak{o}^+_{\mathcal{E}(K_0)}/p^m\mathfrak{o}^+_{\mathcal{E}(K_0)} = \mathfrak{o}_{\mathcal{E}(K_0)}/p^m\mathfrak{o}_{\mathcal{E}(K_0)} \cap W(\mathfrak{o}_{\tilde{X}})/p^m W(\mathfrak{o}_{\tilde{X}})$$

in $W(\tilde{X})/p^m W(\tilde{X})$. The analogous result for $\mathfrak{o}_{\breve{\mathcal{E}}}$ was shown in [6], Proposition B.1.8.3 (iv).

For brevity, we set $R_m := R/p^m R$ for any of the rings R we are working with. We have $\mathfrak{o}^+_{\mathcal{E}(K_0),m} \simeq C_m[[t]]$, where C is a complete discrete valuation ring of characteristic zero with maximal ideal $p \cdot C$. Any element $f \in \mathfrak{o}^+_{\mathcal{E}(K_0),m}$ is of the form $p^r t^s e$ with a unit $e \in (\mathfrak{o}^+_{\mathcal{E}(K_0),m})^\times$ and integers $0 \le r < m$ and $s \ge 0$. Since $\mathfrak{o}_{\mathcal{E}(K_0),m} = \cup_{j \ge 0} t^{-j} \mathfrak{o}^+_{\mathcal{E}(K_0),m}$ and since $\mathfrak{o}_{\mathcal{E}(K_0),m} \cap W_m(\mathfrak{o}_{\tilde{X}})$ is a ring containing $\mathfrak{o}^+_{\mathcal{E}(K_0),m}$, the assumption $\mathfrak{o}^+_{\mathcal{E}(K_0),m} \neq \mathfrak{o}_{\mathcal{E}(K_0),m} \cap W_m(\mathfrak{o}_{\tilde{X}})$ would imply $p^r t^{-s} \in W_m(\mathfrak{o}_{\tilde{X}})$ for certain integers $0 \le r < m$ and $s > 0$. This is absurd because the r-th component of the Witt vector $p^r t^{-s}$ is the image of $t^{-sp^r} \in \mathfrak{o}_{\mathcal{E}(K_0)}$ in $X_{K_\infty} = \mathfrak{o}_{\mathcal{E}(K_0)}/p\mathfrak{o}_{\mathcal{E}(K_0)}$, which is contained in $\tilde{X} \setminus \mathfrak{o}_{\tilde{X}}$.

Let us return to the general case. The extension $FK_m|K_m$ is totally ramified for any integer $m \ge 0$. According to [15], Theorem 1.3.4, so is the extension $X_{FK_\infty}|X_{K_\infty}$. In particular, any uniformizer of X_{FK_∞} is the root of an Eisenstein polynomial over $\mathfrak{o}_{X_{K_\infty}}$ and generates $\mathfrak{o}_{X_{FK_\infty}}$. We lift such a polynomial to a monic polynomial over $\mathfrak{o}^+_{\mathcal{E}(K_0)}$. It is Eisenstein with respect to the maximal ideal of $\mathfrak{o}^+_{\mathcal{E}(K_0)}$ and, by Hensel's lemma, admits a root $t' \in \mathfrak{o}_{\mathcal{E}}$. By the corollary to [3], VIII.5.4 Proposition 4, the subring $\mathfrak{o}^+_{\mathcal{E}} := \mathfrak{o}^+_{\mathcal{E}(K_0)}[t']$ of $\mathfrak{o}_{\mathcal{E}}$ is regular with maximal ideal generated by p and t'.

Since $\mathfrak{o}^+_{\mathcal{E}}$ and $\mathfrak{o}^+_{\mathcal{E}(K_0)}$ have the same residue class field, there is a noncanonical isomorphism $\mathfrak{o}^+_{\mathcal{E}} \simeq C[[t']]$, where C is the subring of $\mathfrak{o}^+_{\mathcal{E}(K_0)}$ we considered before (cf. [3], VIII.5.5 Théorème 2). Further, the natural homomorphism $(\mathfrak{o}^+_{\mathcal{E}})\widehat{\ }_{(p)} \to \mathfrak{o}_{\mathcal{E}}$ is an isomorphism because it is so modulo p.

Although we cannot guarantee that $t' \in W(\mathfrak{o}_{\tilde{X}})$, we have $C \subseteq W(\mathfrak{o}_{\tilde{X}})$. Further, the fact that the image of t' in $\mathfrak{o}_{\tilde{X}}$ has positive valuation implies that for any integer $m \ge 0$ there is an integer $k_0 \ge 0$ such that $(t')^k \in p^m W(\tilde{X}) + W(\mathfrak{o}_{\tilde{X}})$ for all integers $k \ge k_0$. The idea of [14], Proposition 3.2.21, is to consider the subring $B := W(\mathfrak{o}_{\tilde{X}})[t']$ of $W(\tilde{X})$. By the preceding remark, the weak topology on $W(\tilde{X})$ admits the fundamental system $p^i W(\tilde{X}) + (t')^j B$ of open neighborhoods of zero with i and j running through all positive integers.

As above, we retopologize $\mathfrak{o}_{\mathcal{E}}$ by means of the fundamental system $p^i \mathfrak{o}_{\mathcal{E}} + (t')^j \mathfrak{o}^+_{\mathcal{E}}$ of open neighborhoods of zero. We need to show that the corresponding quotient topology on $\mathfrak{o}_{\mathcal{E},m}$ coincides with the coarser topology induced from $W_m(\tilde{X})$. Denoting by B_m the image of B in $W_m(\tilde{X})$, it suffices to prove that the set $\mathfrak{o}^+_{\mathcal{E},m}$ contains the subset $\mathfrak{o}_{\mathcal{E},m} \cap (t')^j B_m = (t')^j(\mathfrak{o}_{\mathcal{E},m} \cap B_m)$ for some integer $j \ge 0$. If this is not true then the above argument implies the existence of an integer r with $0 \le r < m$ and a strictly increasing infinite sequence of positive integers $(j_i)_i$ such that $p^r (t')^{-j_i} \in B_m$ for all i. This would imply $p^r \in \cap_{i \ge 0} (t')^{j_i} B_m = \{0\}$, i.e. $p^r = 0$ in $\mathfrak{o}_{\mathcal{E},m}$ – a contradiction.

As a consequence, the weak topology of $\mathfrak{o}_{\mathcal{E}}$ admits the fundamental system of open neighborhoods of zero given by the sets $p^i \mathfrak{o}_{\mathcal{E}} + t^j(\mathfrak{o}_{\mathcal{E}} \cap W(\mathfrak{o}_{\tilde{X}}))$. □

Let G, H, $\Gamma = G/H$ and $\mathfrak{o}_{\mathcal{E}}$ be as in Lemma 4.2.

Definition 4.3. An *admissible* (φ,Γ)*-module over* $\mathfrak{o}_{\mathcal{E}}$ is an object (M,φ_M) of $\Phi^{\mathrm{cpl}}_{\mathfrak{o}_{\mathcal{E}}}$, carrying a semilinear action of Γ, such that for all integers $n\geq 0$ the φ-module $(M/p^nM,\varphi_M \bmod p^n)$ over $\mathfrak{o}_{\mathcal{E}}$ is the union of its Γ-stable étale φ-submodules and such that on each of the latter, the action of Γ is continuous with respect to the weak $\mathfrak{o}_{\mathcal{E}}$-topology. The category of admissible (φ,Γ)-modules over $\mathfrak{o}_{\mathcal{E}}$ is denoted by $\Phi\Gamma^{\mathrm{adm}}_{\mathfrak{o}_{\mathcal{E}}}$.

Note that the Γ-action on an admissible (φ,Γ)-module over $\mathfrak{o}_{\mathcal{E}}$ is continuous for the weak $\mathfrak{o}_{\mathcal{E}}$-topology. Moreover, Theorem 3.6 and the proof of Lemma 4.2 show that if V is an object of $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(G)$ then the complete φ-module $\mathbb{D}(V)$ over $\mathfrak{o}_{\mathcal{E}}$ carries the structure of an admissible (φ,Γ)-module over $\mathfrak{o}_{\mathcal{E}}$.

Lemma 4.4. *With the notation of Lemma 4.2, if* (M,φ_M) *is an object of* $\Phi\Gamma^{\mathrm{adm}}_{\mathfrak{o}_{\mathcal{E}}}$ *then the diagonal action of* G *on* $\mathfrak{o}_{\breve{\mathcal{E}}}\widehat{\otimes}_{\mathfrak{o}_{\mathcal{E}}}M$ *is continuous for the weak* $\mathfrak{o}_{\breve{\mathcal{E}}}$*-topology. The induced action of* G *on* $\mathbb{V}(M):=(\mathfrak{o}_{\breve{\mathcal{E}}}\widehat{\otimes}_{\mathfrak{o}_{\mathcal{E}}}M)^{\varphi_M=1}$ *is continuous for the* p*-adic topology.*

Proof: For the first assertion we may assume M to be a finitely generated p-torsion module. Let $\mathfrak{o}^+_{\breve{\mathcal{E}}}:=\mathfrak{o}_{\breve{\mathcal{E}}}\cap W(\mathfrak{o}_{\breve{X}})$ and $\breve{M}:=\mathfrak{o}_{\breve{\mathcal{E}}}\widehat{\otimes}_{\mathfrak{o}_{\mathcal{E}}}M\simeq\mathfrak{o}_{\breve{\mathcal{E}}}\otimes_{\mathfrak{o}_{\mathcal{E}}}M$. Let the element $t\in\mathfrak{o}^+_{\breve{\mathcal{E}}}$ be as in the proof of Lemma 4.2. By the usual reasoning, one is reduced to the claim that the map $G\times\breve{M}\to\breve{M}$ is continuous at $(1,0)$. Let $(e_i)_i$ be a set of generators of M over $\mathfrak{o}_{\mathcal{E}}$, and let $m\geq 0$ be an integer. Then $\breve{U}:=\sum_i t^m\mathfrak{o}^+_{\breve{\mathcal{E}}}e_i\subseteq\breve{M}$ is a basic open neighborhood of zero in $\breve{M}$ whose intersection with M contains the basic open neighborhood $U:=\sum_i t^m(\mathfrak{o}_{\mathcal{E}}\cap W(\mathfrak{o}_{\breve{X}}))e_i$. Since the G-action on M is continuous, there is an integer n and an open subgroup G_0 of G such that $G_0\cdot\sum_i t^n(\mathfrak{o}_{\mathcal{E}}\cap W(\mathfrak{o}_{\breve{X}}))e_i\subseteq U$. Since $\mathfrak{o}^+_{\breve{\mathcal{E}}}$ is G-stable, this implies $G_0\cdot\sum_i t^n\mathfrak{o}^+_{\breve{\mathcal{E}}}e_i\subseteq\mathfrak{o}^+_{\breve{\mathcal{E}}}\cdot U=\breve{U}$.

As in the proof of Lemma 4.2, the second assertion is reduced to the claim that for any integer $n\geq 0$ the natural map $\mathbb{Z}_p/p^n\mathbb{Z}_p\to\mathfrak{o}_{\breve{\mathcal{E}}}/p^n\mathfrak{o}_{\breve{\mathcal{E}}}$ is a topological embedding for the p-adic (i.e. the discrete) topology on the left and the weak $\mathfrak{o}_{\breve{\mathcal{E}}}$-topology on the right hand side. The latter, however, is Hausdorff, and the discrete topology is the only Hausdorff topology on the finite set $\mathbb{Z}_p/p^n\mathbb{Z}_p$.□

Summarizing Theorem 3.6, the remark after Definition 4.3, and Lemma 4.4, we find the following generalization of the results [6], Théorème 3.4.3, of J-M. Fontaine and [15], Theorem 2.1.3, of A. Scholl.

Theorem 4.5. *Let* $F|F_0$ *be a finite totally ramified extension and set* $G:=\mathrm{Gal}(K_0^{\mathrm{sep}}|FK_0)$*,* $H:=G\cap\mathrm{Gal}(K_0^{\mathrm{sep}}|K_\infty)\simeq\mathrm{Gal}(K_0^{\mathrm{sep}}|FK_\infty)$*,* $\mathfrak{o}_{\mathcal{E}}:=\mathfrak{o}^H_{\breve{\mathcal{E}}}$ *and* $\Gamma:=G/H$*. The functor*

$$\mathbb{D}:\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(G)\longrightarrow\Phi\Gamma^{\mathrm{adm}}_{\mathfrak{o}_{\mathcal{E}}},$$

given by $\mathbb{D}(V):=(\mathfrak{o}_{\breve{\mathcal{E}}}\widehat{\otimes}_{\mathbb{Z}_p}V)^H$*, is an equivalence of categories transforming strict exact sequences into strict exact sequences. A quasi inverse is given by the functor* $\mathbb{V}$ *with* $\mathbb{V}(M):=(\mathfrak{o}_{\breve{\mathcal{E}}}\widehat{\otimes}_{\mathfrak{o}_{\mathcal{E}}}M)^{\varphi_M=1}$. □

An *admissible* (φ,Γ)*-module over* $\mathcal{E}:=\mathrm{Quot}(\mathfrak{o}_{\mathcal{E}})$ is a unitary Banach φ-module (M,φ_M) over $\mathcal{E}$ carrying a semilinear action of Γ such that M admits a bounded open $\mathfrak{o}_{\mathcal{E}}$-lattice $\mathring{M}$ which is stable under the actions of Γ and φ_M and which, with respect to the induced structures, is an admissible (φ, Γ)-module over $\mathfrak{o}_{\mathcal{E}}$ in the sense of Definition 4.3. We denote by $\Phi\Gamma_{\mathcal{E}}^{\mathrm{adm}}$ the category of admissible (φ, Γ)-modules over $\mathcal{E}$. Theorem 4.5 implies that the functor

$$\mathbb{D} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{uni}}(G) \longrightarrow \Phi\Gamma_{\mathcal{E}}^{\mathrm{adm}},$$

given by $\mathbb{D}(V) := (\breve{\mathcal{E}} \widehat{\otimes}_{\mathbb{Q}_p} V)^H$, is a well-defined equivalence of categories transforming strict exact sequences into strict exact sequences. A quasi inverse is given by the functor $\mathbb{V}$ with $\mathbb{V}(M) := (\breve{\mathcal{E}} \widehat{\otimes}_{\mathcal{E}} M)^{\varphi_M = 1}$.

5 Unitary representations of compact groups

Let the characteristic of K be arbitrary. Let $t_1, \ldots, t_{h-1}$ be indeterminates, set $E := \mathrm{Quot}(k^{\mathrm{sep}}[[t_1, \ldots, t_{h-1}]])$, and let $\mathfrak{o}_{\mathcal{E}}$ be the complete local ring of $W(k^{\mathrm{sep}})[[t_1, \ldots, t_{h-1}]]$ at the prime ideal generated by p, viewed as a Cohen ring of E. Choose any Frobenius endomorphism φ of $\mathfrak{o}_{\mathcal{E}}$, and let $\mathcal{E}$ be the quotient field of $\mathfrak{o}_{\mathcal{E}}$.

Theorem 5.1. *With the above notation, there is a closed subgroup H' of $H := \mathrm{Gal}(E^{\mathrm{sep}}|E)$ such that $H/H' \simeq \mathrm{GL}_{h-1}(\mathfrak{o})$. Let $\mathfrak{o}_{\mathcal{E}_{H'}} := \mathfrak{o}_{\breve{\mathcal{E}}}^{H'}$ and $\mathcal{E}_{H'} := \breve{\mathcal{E}}^{H'}$.*

(i) The functor $\mathbb{D}_{H'} : \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(\mathrm{GL}_{h-1}(\mathfrak{o})) \to \Phi_{\mathfrak{o}_{\mathcal{E}}}^{\mathrm{adm}}$ defined through $\mathbb{D}_{H'}(V) := (\mathfrak{o}_{\mathcal{E}_{H'}} \widehat{\otimes}_{\mathbb{Z}_p} V)^{\mathrm{GL}_{h-1}(\mathfrak{o})}$ is an embedding of categories transforming strict exact sequences into strict exact sequences.

(ii) The functor $\mathbb{D}_{H'} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{uni}}(\mathrm{GL}_{h-1}(\mathfrak{o})) \to \Phi_{\mathcal{E}}^{\mathrm{adm}}$ defined through $\mathbb{D}_{H'}(V) := (\mathcal{E}_{H'} \widehat{\otimes}_{\mathbb{Q}_p} V)^{\mathrm{GL}_{h-1}(\mathfrak{o})}$ is an embedding of categories transforming strict exact sequences into strict exact sequences. A unitary representation V of $\mathrm{GL}_{h-1}(\mathfrak{o})$ over $\mathbb{Q}_p$ is topologically irreducible if and only if the corresponding admissible φ-module over $\mathcal{E}$ admits only the trivial closed φ-invariant $\mathcal{E}$-subspaces.

Proof: With the notation of section 1, $E \simeq \overline{L}_0$, so that the existence of H' follows from Theorem 1.3. The assertions on the functors $\mathbb{D}_{H'}$ follow from Corollary 3.3, Theorem 3.6 and Theorem 3.8. It remains to prove the last assertion of (ii). If V is topologically reducible, then the corresponding φ-module admits a nontrivial closed φ-stable $\mathcal{E}$-subspace because $\mathbb{D}_{H'}$ is fully faithful and exact on strict exact sequences.

Conversely, let M_1 be a closed φ-stable $\mathcal{E}$-subspace of $M_2 := \mathbb{D}_{H'}(V) \simeq \mathbb{D}(V)$, and set $M_3 := M_2/M_1$. If $\mathring{M}_2$ is a φ-stable bounded open $\mathfrak{o}_{\mathcal{E}}$-lattice in M_2 which is an admissible φ-module over $\mathfrak{o}_{\mathcal{E}}$, then we let $\mathring{M}_1$ and $\mathring{M}_3$ be its intersection with M_1 and its image in M_3, respectively. Since M_3 and hence $\mathring{M}_3$ is p-torsion free, the exact sequence $0 \to \mathring{M}_1 \to \mathring{M}_2 \to \mathring{M}_3 \to 0$ is strict. It follows from Lemma 3.5, that the φ-modules M_1 and M_3 are admissible, and Theorem 3.8 implies the sequence

$$0 \longrightarrow \mathbb{V}(M_1) \longrightarrow \mathbb{V}(M_2) \longrightarrow \mathbb{V}(M_3) \longrightarrow 0$$

to be a strict exact sequence in $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{uni}}(H)$. Since $\mathbb{V}(M_2) \simeq V$, we have $\mathbb{V}(M_1) = 0$ or $\mathbb{V}(M_1) = V$. This implies $M_1 = 0$ or $M_1 = M_2$. □

Note that all assertions of Theorem 5.1 remain valid if instead of $E = \overline{L}_0$ we choose the perfect field $E := \overline{L}_0^{\mathrm{rad}} \subseteq \overline{L}_\infty$, because $\overline{L}_\infty | \overline{L}_0^{\mathrm{rad}}$ is Galois with Galois group isomorphic to $\mathrm{GL}_{h-1}(\mathfrak{o})$ (cf. Theorem 1.3 and Corollary 1.5). We then have $\mathfrak{o}_{\mathcal{E}} = W(\overline{L}_0^{\mathrm{rad}})$ and $\mathfrak{o}_{\mathcal{E}_{H'}} = W(\overline{L}_\infty)$.

5.1 char$(K) = p$

Assume the characteristic of K to be p, and let $t_1, \ldots, t_{h-1}$ be indeterminates. With the notation of section 1, let $E := L_0$, which is isomorphic to the quotient field of the complete local ring of $k^{\mathrm{sep}}[[\pi, t_1, \ldots, t_{h-1}]]$ at the prime ideal generated by π, i.e. to $\mathrm{Quot}(k^{\mathrm{sep}}[[t_1, \ldots, t_{h-1}]])((\pi))$. We choose

$$\mathfrak{o}_{\mathcal{E}} := \left(W(k^{\mathrm{sep}})[[t_1, \ldots, t_{h-1}]]\widehat{_{(p)}} \right) [[\pi]]\widehat{_{(p)}}$$

as a Cohen ring of E where, by abuse of notation, π is viewed as a formal variable reducing to π. Let $\mathcal{E}$ be the quotient field of $\mathfrak{o}_{\mathcal{E}}$ and choose an arbitrary lift to $\mathfrak{o}_{\mathcal{E}}$ of the Frobenius endomorphism of E. Setting $H' := \mathrm{Gal}(E^{\mathrm{sep}} | L_\infty)$, the following result is proved as in Theorem 5.1.

Theorem 5.2. *Assume K to be of characteristic p. With the above notation, there is a closed subgroup H' of $H := \mathrm{Gal}(E^{\mathrm{sep}}|E)$ such that $H/H' \simeq P_0$. Let $\mathfrak{o}_{\mathcal{E}_{H'}} := \mathfrak{o}_{\breve{\mathcal{E}}}^{H'}$ and $\mathcal{E}_{H'} := \breve{\mathcal{E}}^{H'}$.*

(i) The functor $\mathbb{D}_{H'} : \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(P_0) \to \Phi^{\mathrm{adm}}_{\mathfrak{o}_{\mathcal{E}}}$ given by $\mathbb{D}_{H'}(V) := (\mathfrak{o}_{\mathcal{E}_{H'}} \widehat{\otimes}_{\mathbb{Z}_p} V)^{P_0}$ is an embedding of categories transforming strict exact sequences into strict exact sequences.

(ii) The functor $\mathbb{D}_{H'} : \mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(P_0) \to \Phi^{\mathrm{adm}}_{\mathcal{E}}$ given by $\mathbb{D}_{H'}(V) := (\mathcal{E}_{H'} \widehat{\otimes}_{\mathbb{Q}_p} V)^{P_0}$ is an embedding of categories transforming strict exact sequences into strict exact sequences. A unitary representation V of P_0 over $\mathbb{Q}_p$ is topologically irreducible if and only if the corresponding admissible φ-module over $\mathcal{E}$ admits only the trivial closed φ-invariant $\mathcal{E}$-subspaces. □

Again, all assertions remain valid if $E = L_0$ is replaced by $E = L_0^{\mathrm{rad}}$. We then have $\mathfrak{o}_{\mathcal{E}} = W(L_0^{\mathrm{rad}})$ and $\mathfrak{o}_{\mathcal{E}_{H'}} = W(L_\infty^{\mathrm{rad}})$. Similarly, we could work with the field $E = Q_\infty^{P_0}$ or with its perfect closure.

With the notation of secton 1, we can also consider $E := Q_0$, which is isomorphic to the quotient field of $k^{\mathrm{sep}}[[\pi, t_1, \ldots, t_{h-1}]]$. As a Cohen ring $\mathfrak{o}_{\mathcal{E}}$ of E we choose the complete local ring of $W(k^{\mathrm{sep}})[[\pi, t_1, \ldots, t_{h-1}]]$ at the prime ideal generated by p. We let $\mathcal{E}$ be the quotient field of $\mathfrak{o}_{\mathcal{E}}$ and choose an arbitrary lift of the Frobenius endomorphism of E to $\mathfrak{o}_{\mathcal{E}}$. Setting $H' := \mathrm{Gal}(E^{\mathrm{sep}}|Q_\infty)$, the following result is proved as in Theorem 5.1.

Theorem 5.3. *Assume K to be of characteristic p. With the above notation, there is a closed subgroup H' of $H := \mathrm{Gal}(E^{\mathrm{sep}}|E)$ such that $H/H' \simeq G_0 := \mathrm{GL}_h(\mathfrak{o})$. Let $\mathfrak{o}_{\mathcal{E}_{H'}} := \mathfrak{o}_{\breve{\mathcal{E}}}^{H'}$ and $\mathcal{E}_{H'} := \breve{\mathcal{E}}^{H'}$.*

(i) The functor $\mathbb{D}_{H'} : \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(G_0) \to \Phi^{\mathrm{adm}}_{\mathfrak{o}_{\mathcal{E}}}$ given by $\mathbb{D}_{H'}(V) := (\mathfrak{o}_{\mathcal{E}_{H'}} \widehat{\otimes}_{\mathbb{Z}_p} V)^{G_0}$ is an embedding of categories transforming strict exact sequences into strict exact sequences.

(ii) The functor $\mathbb{D}_{H'} : \mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(G_0) \to \Phi^{\mathrm{adm}}_{\mathcal{E}}$ given by $\mathbb{D}_{H'}(V) := (\mathcal{E}_{H'}\widehat{\otimes}_{\mathbb{Q}_p} V)^{G_0}$ is an embedding of categories transforming strict exact sequences into strict exact sequences. A unitary representation V of G_0 over $\mathbb{Q}_p$ is topologically irreducible if and only if the corresponding admissible φ-module over $\mathcal{E}$ admits only the trivial closed φ-invariant $\mathcal{E}$-subspaces. □

Once again, all assertions of Theorem 5.3 remain valid on replacing $E = Q_0$ by $E = Q_0^{\mathrm{rad}}$. We then have $\mathfrak{o}_{\mathcal{E}} = W(Q_0^{\mathrm{rad}})$ and $\mathfrak{o}_{\mathcal{E}_{H'}} = W(Q_\infty^{\mathrm{rad}})$.

The group P_0 is a closed subgroup of G_0 so that the corresponding categories of p-adically continuous representations are related to each other. For the sake of brevity, we are only going to consider the respective categories of unitary representations on $\mathbb{Q}_p$-Banach spaces.

First of all, we have the *restriction functor*

$$\mathrm{res} : \mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(G_0) \longrightarrow \mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(P_0).$$

In the other direction, let V be an object of $\mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(P_0)$ and choose a P_0-invariant norm on V, defining its topology. We let $\mathrm{ind}(V) := \mathrm{ind}_{P_0}^{G_0}(V)$ be the $\mathbb{Q}_p$-vector space of all continuous functions $f : G_0 \to V$ such that $f(gp) = p^{-1}f(g)$ for all elements $g \in G_0$ and $p \in P_0$. The space $\mathrm{ind}(V)$ is complete with respect to the supremum norm, and the left regular action makes it a unitary representation of G_0. We obtain the *induction functor*

$$\mathrm{ind} : \mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(P_0) \longrightarrow \mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(G_0).$$

We pass on to admissible φ-modules. In order to avoid the technical difficulties of constructing Cohen rings for different fields, related by ring homomorphisms for which the structures of Galois actions and Frobenius lifts are compatible, we are going to work with perfect fields only (cf. the remarks following Theorem 5.2 and Theorem 5.3).

The embedding $Q_\infty^{\mathrm{rad}} \to L_\infty^{\mathrm{rad}}$ induces an embedding $W(Q_\infty^{\mathrm{rad}}) \to W(L_\infty^{\mathrm{rad}})$ which is P_0-equivariant and commutes with the natural Frobenius endomorphisms. It induces an equivariant embedding $W(Q_\infty^{\mathrm{rad}})[1/p] \to W(L_\infty^{\mathrm{rad}})[1/p]$ of the corresponding quotient fields. These embeddings restrict to embeddings $W(Q_0^{\mathrm{rad}}) \to W(L_0^{\mathrm{rad}})$ and $W(Q_0^{\mathrm{rad}})[1/p] \to W(L_0^{\mathrm{rad}})[1/p]$.

Let (M, φ_M) be an admissible φ-module over $W(F)[1/p]$ for some perfect subfield F of L_0^{rad}. Choose a φ_M-stable bounded open $W(F)$-lattice $\mathring{M}$ of M which is admissible as a φ-module over $W(F)$. The φ-semilinear map $\varphi \otimes \varphi_M$ on $W(L_0^{\mathrm{rad}}) \otimes_{W(F)} \mathring{M}$ extends to the p-adic completion and, after localizing at p, makes $W(L_0^{\mathrm{rad}})[1/p]\widehat{\otimes}_{W(F)[1/p]} M$ an admissible φ-module over $W(L_0^{\mathrm{rad}})$ (note that the algebraic base extension from $W(F)$ to $W(L_0^{\mathrm{rad}})$ preserves étale φ-modules).

If (M, φ_M) is an object of $\Phi^{\mathrm{adm}}_{W(Q')[1/p]}$ for some field Q' with $Q_0^{\mathrm{rad}} \subseteq Q' \subseteq Q_\infty^{\mathrm{rad}}$ then, by restriction of scalars, we obtain a unitary Banach φ-module $\mathrm{res}(M)$ over $W(Q_0^{\mathrm{rad}})$.

Proposition 5.4. *Assume K to be of characteristic p.*

(i) Up to natural equivalence the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(G_0) & \xrightarrow{\ \mathbb{D}\ } & \Phi^{\mathrm{adm}}_{W(Q_0^{\mathrm{rad}})[1/p]} \\ {\scriptstyle \mathrm{res}}\big\downarrow & & \big\downarrow {\scriptstyle W(L_0^{\mathrm{rad}})[1/p]\widehat{\otimes}_{W(Q_0^{\mathrm{rad}})[1/p]}(\,\cdot\,)} \\ \mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(P_0) & \xrightarrow{\ \mathbb{D}\ } & \Phi^{\mathrm{adm}}_{W(L_0^{\mathrm{rad}})[1/p]}. \end{array}$$

(ii) Let $Q' := (Q_\infty^{\mathrm{rad}})^{P_0} \subseteq L_0^{\mathrm{rad}}$. The functor

$$W(L_0^{\mathrm{rad}})[1/p]\widehat{\otimes}_{W(Q')[1/p]}(\,\cdot\,) : \Phi^{\mathrm{adm}}_{W(Q')[1/p]} \longrightarrow \Phi^{\mathrm{adm}}_{W(L_0^{\mathrm{rad}})[1/p]}$$

is an equivalence of categories.

(iii) Let $Q' := (Q_\infty^{\mathrm{rad}})^{P_0}$. If V is an object of $\mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(P_0)$ then the φ-module $\mathrm{res}(\mathbb{D}(V))$ over $W(Q_0^{\mathrm{rad}})[1/p]$ is admissible, and up to natural equivalence the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(G_0) & \xrightarrow{\ \mathbb{D}\ } & \Phi^{\mathrm{adm}}_{W(Q_0^{\mathrm{rad}})[1/p]} \\ {\scriptstyle \mathrm{ind}}\big\uparrow & & \big\uparrow {\scriptstyle \mathrm{res}} \\ \mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(P_0) & \xrightarrow{\ \mathbb{D}\ } & \Phi^{\mathrm{adm}}_{W(Q')[1/p]}. \end{array}$$

Proof: As for (i), let V be an object of $\mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(G_0)$, and let $\mathring{V}$ be a G_0-invariant bounded open $\mathbb{Z}_p$-lattice in V. It suffices to see that the natural map

$$W(L_0^{\mathrm{rad}})\widehat{\otimes}_{W(Q_0^{\mathrm{rad}})}(W(Q_\infty^{\mathrm{rad}})\widehat{\otimes}_{\mathbb{Z}_p}\mathring{V})^{G_0} \longrightarrow (W(L_\infty^{\mathrm{rad}})\widehat{\otimes}_{\mathbb{Z}_p}\mathring{V})^{P_0}$$

is an isomorphism. We may assume that $p^n\mathring{V} = 0$. After the faithfully flat base extension from $W(L_0^{\mathrm{rad}})$ to $W(L_\infty^{\mathrm{rad}})$, the claim follows from Theorem 3.2.

Using that the natural restriction map $P_0 \simeq \mathrm{Gal}(L_\infty^{\mathrm{rad}}|L_0^{\mathrm{rad}}) \to \mathrm{Gal}(Q_\infty^{\mathrm{rad}}|Q')$ is an isomorphism, assertion (ii) is proved similarly.

As for (iii), the map $(f \mapsto f(1)) : \mathrm{ind}(V) \to V$ induces a P_0-equivariant map $W(Q_\infty^{\mathrm{rad}})[1/p]\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{ind}(V) \to W(Q_\infty^{\mathrm{rad}})[1/p]\widehat{\otimes}_{\mathbb{Q}_p}V$, whence a homomorphism $\mathbb{D}(\mathrm{ind}(V)) \to \mathrm{res}(\mathbb{D}(V))$ of unitary Banach φ-modules over $W(Q_0^{\mathrm{rad}})[1/p]$. We need to see that it is an isomorphism. Choose a G_0-invariant bounded open $\mathbb{Z}_p$-lattice $\mathring{V}$ in V. The p-adically complete $\mathbb{Z}_p$-module $\mathrm{ind}(\mathring{V})$ consisting of all continuous maps $f : G_0 \to \mathring{V}$ with the above transformation properties, is a G_0-invariant bounded open $\mathbb{Z}_p$-lattice in $\mathrm{ind}(V)$. It suffices to see that the natural homomorphism $\mathbb{D}(\mathrm{ind}(\mathring{V})) \to \mathrm{res}(\mathbb{D}(\mathring{V}))$ in $\Phi^{\mathrm{cpl}}_{W(Q_0^{\mathrm{rad}})}$ is an isomorphism. Since both sides are p-torsion free, it suffices to see that the map induced modulo p is bijective.

Let W be a P_0-subrepresentation of $\mathring{V}/p\mathring{V}$ which is finite dimensional over $\mathbb{F}_p$. It suffices to see that the natural map

$$(Q_\infty^{\mathrm{rad}} \otimes_{\mathbb{F}_p} \mathrm{ind}(W))^{G_0} \longrightarrow (Q_\infty^{\mathrm{rad}} \otimes_{\mathbb{F}_p} W)^{P_0}$$

is bijective. Since W is finitely generated, the kernel of the P_0-action contains an open subgroup. Therefore, the G_0-representation $\mathrm{ind}(W)$ is the union of the G_0-subrepresentations $\mathrm{ind}(I, W) := \mathrm{ind}_{P_0/P_0 \cap I}^{G_0/I}(W)$ with I running through the open normal subgroups of G_0 such that $P_0 \cap I$ acts trivially on W. It suffices to see that for any such subgroup I the natural map

$$(Q_\infty^{\mathrm{rad}} \otimes_{\mathbb{F}_p} \mathrm{ind}(I, W))^{G_0} \longrightarrow (Q_\infty^{\mathrm{rad}} \otimes_{\mathbb{F}_p} W)^{P_0}$$

is bijective. Set $Q_I := (Q_\infty^{\mathrm{rad}})^I$. We have

$$(Q_\infty^{\mathrm{rad}} \otimes_{\mathbb{F}_p} \mathrm{ind}(I, W))^{G_0} \simeq (Q_I \otimes_{\mathbb{F}_p} \mathrm{ind}(I, W))^{G_0/I},$$

and there is a G_0/I-equivariant isomorphism

$$Q_I \otimes_{\mathbb{F}_p} \mathrm{ind}(I, W) \simeq Q_I \otimes_{Q_0^{\mathrm{rad}}} \mathrm{ind}(I, Q_0^{\mathrm{rad}} \otimes_{\mathbb{F}_p} W).$$

Since $[Q_I : Q_0^{\mathrm{rad}}] < \infty$, Frobenius reciprocity for finite dimensional Q_0^{rad}-linear representations implies

$$\begin{aligned}(Q_I \otimes_{\mathbb{F}_p} \mathrm{ind}(I, W))^{G_0/I} &\simeq \mathrm{Hom}_{G_0/I}(Q_I^*, \mathrm{ind}(I, Q_0^{\mathrm{rad}} \otimes_{\mathbb{F}_p} W)) \\ &\simeq \mathrm{Hom}_{P_0/P_0 \cap I}(Q_I^*, Q_0^{\mathrm{rad}} \otimes_{\mathbb{F}_p} W) \\ &\simeq (Q_I \otimes_{\mathbb{F}_p} W)^{P_0/P_0 \cap I} \simeq (Q_\infty^{\mathrm{rad}} \otimes_{\mathbb{F}_p} W)^{P_0}. \quad \square\end{aligned}$$

With the notation of Theorem 5.2, there is a unique choice of φ on $\mathfrak{o}_{\mathcal{E}} = \Big(W(k^{\mathrm{sep}})[[t_1, \ldots, t_{h-1}]]^{\widehat{}}_{(p)}\Big)[[\pi]]^{\widehat{}}_{(p)}$ whose restriction to $W(k^{\mathrm{sep}})$ is the canonical Frobenius and which satisfies $\varphi(t_i) = t_i^p$ for $1 \leq i \leq h-1$ and $\varphi(\pi) = (1+\pi)^p - 1$. If $\mathfrak{o}_{\mathcal{E}_0} := W(k^{\mathrm{sep}})[[\pi]]^{\widehat{}}_{(p)}$ then $\mathfrak{o}_{\mathcal{E}_0}$ is a p-adically separated and complete φ-stable subring of $\mathfrak{o}_{\mathcal{E}}$, and we have the functor of complete base extension

$$\Phi_{\mathfrak{o}_{\mathcal{E}_0}}^{\mathrm{adm}} \longrightarrow \Phi_{\mathfrak{o}_{\mathcal{E}}}^{\mathrm{adm}}.$$

Note that the restriction homomorphism $\mathrm{Gal}(\breve{K}^{\mathrm{sep}}|\breve{K}) \to I_K := \mathrm{Gal}(K^{\mathrm{sep}}|K^{\mathrm{unr}})$ into the inertia group of K is an isomorphism (cf. [10], Corollaire II.3.4). Since $\mathfrak{o}_{\mathcal{E}_0}$ is a Cohen ring of $\breve{K}$, Theorem 3.6 implies that the category $\Phi_{\mathfrak{o}_{\mathcal{E}_0}}^{\mathrm{adm}}$ of admissible φ-modules over $\mathfrak{o}_{\mathcal{E}_0}$ is equivalent to the category $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(I_K)$ of continuous representations of I_K on p-adically separated and complete $\mathbb{Z}_p$-modules. Although we obtain two functors

$$\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(P_0) \longrightarrow \Phi_{\mathfrak{o}_{\mathcal{E}}}^{\mathrm{adm}} \longleftarrow \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(I_K),$$

this gives only a weak link between the two categories of representations involved.

Proposition 5.5. *Let V be an object of $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(P_0)$. There is an object σ of $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(I_K)$ and an isomorphism $\mathbb{D}(V) \simeq \mathfrak{o}_{\mathcal{E}} \widehat{\otimes}_{\mathfrak{o}_{\mathcal{E}_0}} \mathbb{D}(\sigma)$ in $\Phi_{\mathfrak{o}_{\mathcal{E}}}^{\mathrm{adm}}$ if and only if the action of P_0 on V factors through the determinant. In this case, the action of I_K on σ factors through $\mathrm{Gal}(\breve{K}'|\breve{K}) \simeq \mathfrak{o}^\times$, where $\breve{K}'$ is the extension of $\breve{K}$ obtained by adjoining to $\breve{K}$ all π-power torsion points of a formal Lubin-Tate module of height one over $\mathfrak{o}$.*

Proof: Consider the embedding $\breve{K} \subseteq L_0$. The $\breve{K}$-subalgebra of L_0^{sep} consisting of all elements which are separable algebraic over $\breve{K}$ is a separable closure of $\breve{K}$. Set $H := \mathrm{Gal}(L_0^{\mathrm{sep}}|L_0)$ and consider the restriction homomorphism $H \to I_K$. It was shown in the proof of Theorem 2.2 that $\breve{K}^{\mathrm{sep}} \cap L_0 = \breve{K}$, so that $\breve{K}^{\mathrm{sep}} L_0 \simeq \breve{K}^{\mathrm{sep}} \otimes_{\breve{K}} L_0$. It follows that the above restriction map is surjective, hence gives rise to the inflation functor $\inf : \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(I_K) \to \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(H)$. We claim that up to natural equivalence the diagram

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(I_K) & \xrightarrow{\mathbb{D}} & \Phi_{\mathfrak{o}_{\mathcal{E}_0}}^{\mathrm{adm}} \\ {\scriptstyle \inf}\downarrow & & \downarrow{\scriptstyle \mathfrak{o}_{\mathcal{E}} \widehat{\otimes}_{\mathfrak{o}_{\mathcal{E}_0}} (\,\cdot\,)} \\ \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(H) & \xrightarrow{\mathbb{D}} & \Phi_{\mathfrak{o}_{\mathcal{E}}}^{\mathrm{adm}} \end{array}$$

is commutative. Replacing $\breve{K}$ and L_0 by their perfect closures, it suffices to see that for any object σ of $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(I_K)$ the natural map

$$W(L_0^{\mathrm{rad}}) \widehat{\otimes}_{W(\breve{K}^{\mathrm{rad}})} (W(\breve{K}^{\mathrm{alg}}) \widehat{\otimes}_{\mathbb{Z}_p} \sigma)^{I_K} \longrightarrow (W(L_0^{\mathrm{alg}}) \widehat{\otimes}_{\mathbb{Z}_p} \sigma)^H$$

is an isomorphism. We may assume $p^n \sigma = 0$ for some integer $n \geq 0$. After the faithfully flat base extension from $W(L_0^{\mathrm{rad}})$ to $W(L_0^{\mathrm{alg}})$ the bijectivity of the above map follows from Theorem 3.2.

On the other hand, the functor $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(P_0) \to \Phi_{\mathfrak{o}_{\mathcal{E}}}^{\mathrm{adm}}$ is the composition

$$\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(P_0) \xrightarrow{\inf} \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(H) \xrightarrow{\mathbb{D}} \Phi_{\mathfrak{o}_{\mathcal{E}}}^{\mathrm{adm}}.$$

It follows that if σ exists as in the proposition, then its inflation to H has to be trivial on $H' := \mathrm{Gal}(L_0^{\mathrm{sep}}|L_\infty)$.

Below, we are going to show that $L_\infty \cap \breve{K}^{\mathrm{sep}} = \breve{K}'$. Assuming this, it follows that σ is trivial on $\mathrm{Gal}(\breve{K}^{\mathrm{sep}}|\breve{K}')$, hence factors as desired. For V this means that the action of P_0 is trivial upon restriction to $\mathrm{Gal}(L_\infty|\breve{K}'L_0) \simeq P_0 \cap \mathrm{SL}_h(\mathfrak{o})$ (cf. the proof of [16], Theorem 4.4).

Conversely, if V is an object of $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(P_0)$ which factors through the determinant then the existence of σ is clear from the fact that the restriction homomorphism $P_0/(P_0 \cap \mathrm{SL}_h(\mathfrak{o})) \simeq \mathrm{Gal}(\breve{K}'L_0|L_0) \longrightarrow \mathrm{Gal}(\breve{K}'|\breve{K})$ is an isomorphism.

Thus, it remains to prove $L_\infty \cap \breve{K}^{\mathrm{sep}} = \breve{K}'$. For any integer $m \geq 0$ let $\breve{K}_m$ be the field obtained by adjoining to $\breve{K}$ the π^m-torsion points of a one dimensional Lubin-Tate module of height one over $\mathfrak{o}$. By a result of M. Strauch, we have $\mathfrak{o}_{\breve{K}_m} \subseteq R_m \subseteq \mathfrak{o}_{L_m}$, and if ξ_m denotes a uniformizer of $\breve{K}_m$ then the ring $R_m/\xi_m R_m$ is reduced (cf. [16], Corollary 3.4 and Proposition 4.2). In particular, $\breve{K}' \subseteq L_\infty$. Since $\pi \mathfrak{o}_{L_m} \subseteq \xi_m \mathfrak{o}_{L_m} \subseteq \pi_m \mathfrak{o}_{L_m}$, the ideal $\xi_m \mathfrak{o}_{L_m}$ of $\mathfrak{o}_{L_m}$ is open, and the natural homomorphism of rings

$$(R_m)_{(\pi_m)}/\xi_m \cdot (R_m)_{(\pi_m)} \longrightarrow \mathfrak{o}_{L_m}/\xi_m \mathfrak{o}_{L_m}$$

is an isomorphism. Since the ring on the left is isomorphic to the localization of $R_m/\xi_m R_m$ at the prime ideal generated by π_m, it is a reduced ring. As in the proof of [9], Lemma 1.5, this implies $\breve{K}_m$ to be separably closed in L_m. $\square$

5.2 char$(K) = 0$

Assume the characteristic of K to be zero. Taking up the notation of section 2 and section 4, we let $t_1, \ldots, t_{h-1}$ be indeterminates and choose

$$\mathfrak{o}_{\mathcal{E}(K_0)} := \left(W(k^{\mathrm{sep}})[[t_1, \ldots, t_{h-1}]]\widehat{_{(p)}} \right) [[\pi]]\widehat{_{(p)}}$$

as a Cohen ring of X_{K_∞}. According to [15], section 2.3, the image of the embedding $\mathfrak{o}_{\breve{\mathcal{E}}} \hookrightarrow W(\tilde{X})$ – which depends on the choice of a Frobenius lift – is stable under the action of $\mathrm{Gal}(K_0^{\mathrm{sep}}|K_0)$ if we let φ be the uniquely determined endomorphism of $\mathfrak{o}_{\mathcal{E}(K_0)}$ whose restriction to $W(k^{\mathrm{sep}})$ is the natural Frobenius and which satisfies

$$\varphi(t_i) = t_i^p \text{ for } 1 \leq i \leq h-1 \quad \text{and} \quad \varphi(\pi) = (1+\pi)^p - 1.$$

The $W(k^{\mathrm{sep}})$-linear action of $\mathrm{Gal}(K_\infty|K_0) \simeq \mathbb{Z}_p^{h-1} \rtimes \mathbb{Z}_p^\times$ on $\mathfrak{o}_{\mathcal{E}(K_0)}$ is then given by

$$(b_1, \ldots, b_{h-1}, \alpha) \cdot t_i = (1+\pi)^{b_i} t_i \quad \text{and} \quad (b_1, \ldots, b_{h-1}, \alpha) \cdot \pi = (1+\pi)^\alpha - 1$$

for all $(b_1, \ldots, b_{h-1}) \in \mathbb{Z}_p^{h-1}$ and all $\alpha \in \mathbb{Z}_p^\times$.

Consider the finite totally ramified extension $\breve{K}|\breve{\mathbb{Q}}_p$, and let $G := \mathrm{Gal}(K_0^{\mathrm{sep}}|L_0)$, $H := G \cap \mathrm{Gal}(K_0^{\mathrm{sep}}|K_\infty) \simeq \mathrm{Gal}(K_0^{\mathrm{sep}}|\breve{K}K_\infty)$ and $\Gamma := G/H$. Letting $H' := \mathrm{Gal}(K_0^{\mathrm{sep}}|L_\infty K_\infty)$, any representation of $\mathfrak{P} := G/H' \simeq \mathrm{Gal}(L_\infty K_\infty|L_0)$ is also one of G via inflation. We set $\mathfrak{o}_{\mathcal{E}} := \mathfrak{o}_{\breve{\mathcal{E}}}^H$ and let $\mathcal{E} := \mathrm{Quot}(\mathfrak{o}_{\mathcal{E}})$. The following theorem follows from Corollary 3.3 and Theorem 4.5. The last assertion of (ii) is proved as in Theorem 5.1.

Theorem 5.6. *Assume K to be of characteristic zero and let the notation be as above.*

(i) The functor $\mathbb{D}_{H'} : \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(\mathfrak{P}) \to \Phi\Gamma_{\mathfrak{o}_{\mathcal{E}}}^{\mathrm{adm}}$, with $\mathbb{D}_{H'}(V) := (\mathfrak{o}_{\breve{\mathcal{E}}}^{H'} \widehat{\otimes}_{\mathbb{Z}_p} V)^H$, is an embedding of categories transforming strict exact sequences into strict exact sequences.

(ii) The functor $\mathbb{D}_{H'} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{uni}}(\mathfrak{P}) \to \Phi\Gamma_{\mathcal{E}}^{\mathrm{adm}}$, with $\mathbb{D}_{H'}(V) := (\breve{\mathcal{E}}^{H'} \widehat{\otimes}_{\mathbb{Q}_p} V)^H$, is an embedding of categories transforming strict exact sequences into strict exact sequences. An object V of $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{uni}}(\mathfrak{P})$ is topologically irreducible if and only if the corresponding admissible (φ, Γ)-module over $\mathcal{E}$ admits only the trivial closed (φ, Γ)-invariant $\mathcal{E}$-subspaces. □

Note that $\mathfrak{P}$ admits $P_0 \simeq \mathrm{Gal}(L_\infty|L_0)$ as a quotient. Via inflation, the categories $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(P_0)$ and $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{uni}}(P_0)$ become full subcategories of $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(\mathfrak{P})$ and $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{uni}}(\mathfrak{P})$, respectively. Further, $\Gamma \simeq \Delta \rtimes \Gamma_0$ with $\Gamma_0 \simeq \mathrm{Gal}(\breve{K}(\mu_{p^\infty})|\breve{K})$ (cf. Theorem 2.2). Forgetting the action of Δ gives a functor $f : \Phi\Gamma_{\mathfrak{o}_{\mathcal{E}}}^{\mathrm{adm}} \to (\Phi\Gamma_0)_{\mathfrak{o}_{\mathcal{E}}}^{\mathrm{adm}}$, where $(\Phi\Gamma_0)_{\mathfrak{o}_{\mathcal{E}}}^{\mathrm{adm}}$ is the category of complete φ-modules over $\mathfrak{o}_{\mathcal{E}}$ which carry a semilinear action of Γ_0 such that hypotheses analogous to those of Definition 4.3 are satisfied. Similarly, one defines the category $(\Phi\Gamma_0)_{\mathcal{E}}^{\mathrm{adm}}$. The linear disjointness property in Theorem 2.2 yields the following fact.

Theorem 5.7. *Assume the characteristic of K to be zero. With the above notation, if at least one of q or h is different from 2, then the functors $f \circ \mathbb{D}_{H'}$: $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(P_0) \to (\Phi\Gamma_0)^{\mathrm{adm}}_{\mathfrak{o}_{\mathcal{E}}}$ and $f \circ \mathbb{D}_{H'} : \mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(P_0) \to (\Phi\Gamma_0)^{\mathrm{adm}}_{\mathcal{E}}$ are embeddings of categories.*

Proof: The choice of a section of the projection $\Gamma \to \Gamma/\Delta \simeq \Gamma_0$ gives rise to a section of the projection $\mathfrak{P} \to \mathfrak{P}/\Delta \simeq P_0$. By construction, any object V of $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(P_0)$ is viewed as a continuous representation of $\mathfrak{P} \simeq \Delta \rtimes P_0$ via inflation. Since the action of Δ on

$$V \simeq (\mathfrak{o}^{H'}_{\breve{\mathcal{E}}} \widehat{\otimes}_{\mathfrak{o}_{\mathcal{E}}} \mathbb{D}_{H'}(V))^{\varphi=1}$$

is trivial, the P_0-representation V is determined by the restriction of the diagonal $\mathfrak{P}$-action on the φ-module $\mathfrak{o}^{H'}_{\breve{\mathcal{E}}} \widehat{\otimes}_{\mathfrak{o}_{\mathcal{E}}} \mathbb{D}_{H'}(V)$ to the subgroup P_0. For the latter, however, it suffices to know $f(\mathbb{D}_{H'}(V))$. If V is an object of $\mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(P_0)$, then the proof is similar. □

6 Example: Modular representations of $\mathrm{GL}_{h-1}(\mathfrak{o})$

Let the characteristic of K be arbitrary. Since the functor $\mathbb{D} = \mathbb{D}_{H'}$ of Theorem 5.1 commutes with reduction modulo p, we obtain an embedding of categories

$$\mathbb{D} : \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{F}_p}(GL_{h-1}(\mathfrak{o})) \longrightarrow \Phi^{\mathrm{adm}}_E.$$

The category on the left is that of $\mathbb{F}_p$-linear smooth representations of $\mathrm{GL}_{h-1}(\mathfrak{o})$. The simple objects of this category are finite dimensional and, assuming $p \neq 2$, factor through the quotient map $\mathrm{GL}_{h-1}(\mathfrak{o}) \to \mathrm{GL}_{h-1}(k)$. Viewing the category $\mathrm{Rep}_{\mathbb{F}_p}(\mathrm{GL}_{h-1}(k))$ of finite dimensional $\mathbb{F}_p$-linear representations of $\mathrm{GL}_{h-1}(k)$ as a subcategory of $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{F}_p}(\mathrm{GL}_{h-1}(\mathfrak{o}))$ via inflation, the functor $\mathbb{D}$ restricts to an embedding

$$\mathbb{D} : \mathrm{Rep}_{\mathbb{F}_p}(\mathrm{GL}_{h-1}(k)) \longrightarrow \Phi^{\text{ét}}_E.$$

Since most interesting $\mathbb{F}_p$-linear representations of $\mathrm{GL}_{h-1}(k)$ are defined over k rather than over $\mathbb{F}_p$, we slightly change our point of view and study the embedding of categories $\mathbb{D} : \mathrm{Rep}_k(\mathrm{GL}_{h-1}(k)) \longrightarrow \Phi^{\text{ét}}_E$ using the Frobenius $\varphi := (x \mapsto x^q)$ on E.

Such an embedding can be obtained from any Galois extension $F|E$ of fields of characteristic p with Galois group $\mathrm{GL}_{h-1}(k)$. One simply sets $\mathbb{D}(V) := (F \otimes_k V)^{\mathrm{GL}_{h-1}(k)}$ as before. We are going to specify an extension $F|E$ for which the simple étale φ-modules over E corresponding to many irreducible k-linear representations of $\mathrm{GL}_{h-1}(k)$ can be written down explicitly.

Namely, we let $F := \overline{L}_1 \simeq \mathrm{Quot}(k^{\mathrm{sep}}[[\tau_2, \ldots, \tau_h]])$ with the action of $\mathrm{GL}_{h-1}(k)$ given by $g(\tau_j) := \sum_{i=1}^{h-1} g_{ij}\tau_{i+1}$ for all $2 \leq j \leq h$ and $g = (g_{ij}) \in \mathrm{GL}_{h-1}(k)$. If the characteristic of K is p then one can show this action to coincide with the one considered in section 1, whereas it constitutes a linearized version thereof if K is of characteristic zero. Since the above action of $\mathrm{GL}_{h-1}(k)$ on F is faithful, F is a Galois extension of $E := F^{\mathrm{GL}_{h-1}(k)}$ with Galois group $\mathrm{GL}_{h-1}(k)$.

We let $\mu(X) \in E[X]$ be the minimal polynomial of τ_2 over E. Its roots are the different Galois conjugates of τ_2, i.e.

$$\mu(X) = \prod_{{}^t(\alpha_1,\ldots,\alpha_{h-1}) \in k^{h-1}\setminus\{0\}} (X - \alpha_1\tau_2 - \ldots - \alpha_{h-1}\tau_h).$$

An elementary argument shows that $\mu(X)$ is of the form $\mu(X) = \sum_{j=0}^{h-1} \lambda_j X^{q^j-1}$ with $\lambda_{h-1} = 1$.

Let the natural number f be defined through $q = p^f$. For any integer n with $0 \le n \le f-1$ consider the $(h-1)$-dimensional k-linear representation ρ_n of $\mathrm{GL}_{h-1}(k)$ given by

$$\rho_n : \mathrm{GL}_{h-1}(k) \longrightarrow \mathrm{GL}_{h-1}(k), \quad (g_{ij}) \mapsto (g_{ij}^{p^n}).$$

For example, ρ_0 is the standard representation of $\mathrm{GL}_{h-1}(k)$ on k^{h-1}. Further, we view $\det : \mathrm{GL}_{h-1}(k) \to k^\times$ as a one dimensional k-linear representation.

Proposition 6.1. *There is a basis with respect to which the representing matrix of the étale φ-module $\mathbb{D}(det)$ is given by $\lambda_0^{-1} \in \mathrm{GL}_1(E)$. For any integer n with $0 \le n \le f-1$ there is a basis with respect to which the representing matrix of the étale φ-module $\mathbb{D}(\rho_n)$ is given by*

$$\begin{pmatrix} -(\lambda_1/\lambda_0)^{p^n} & \cdots & -(\lambda_{h-2}/\lambda_0)^{p^n} & -1/\lambda_0^{p^n} \\ 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{pmatrix} \in \mathrm{GL}_{h-1}(E).$$

Proof: Let $e := \prod_{g \in \mathrm{SL}_{h-1}(k)} g(\tau_2)^{-1} \in F^\times$. A direct computation shows that $g(e) = \det(g)^{-1}e$ for all $g \in \mathrm{GL}_{h-1}(k)$ and that $\lambda_0^{-1} = \mathrm{N}_{F|E}(\tau_2^{-1}) = e^{q-1}$. Thus, $\varphi(e) = \lambda_0^{-1}e$, proving the first claim.

Since the functor $\mathbb{D}$ commutes with duals (cf. [6], Proposition 1.2.6), it suffices to see that ρ_n^* is represented by the transpose-inverse of the given matrix. However,

$$\mathbb{D}(\rho_n^*) = (F \otimes_k \rho_n^*)^{\mathrm{GL}_{h-1}(k)} \simeq \mathrm{Hom}_{k[\mathrm{GL}_{h-1}(k)]}(\rho_n, F).$$

Denoting by $(e_1, \ldots, e_{h-1})$ the standard basis of k^{h-1} and by e_i^* the corresponding dual basis vectors, we have $f := \sum_{i=1}^{h-1} \tau_{i+1}^{p^n} \otimes e_i^* \in \mathbb{D}(\rho_n^*)$. The element $\tau_{i+1}^{p^n}$ satisfies the polynomial $\sum_{j=0}^{h-1} \lambda_j^{p^n} X^{q^j}$, independently of i, and we have $f, \varphi(f), \ldots, \varphi^{h-2}(f) \in \mathbb{D}(\rho_n^*)$ with $\varphi^{h-1}(f) = -\sum_{j=0}^{h-2} \lambda_j^{p^n} \varphi^j(f)$. Now $\mathbb{D}(\rho_n^*)$ is a simple φ-module since the representation ρ_n is irreducible. As $f \neq 0$, it follows that $\mathbb{D}(\rho_n^*)$ is represented by the matrix

$$\begin{pmatrix} 0 & \cdots & 0 & -\lambda_0^{p^n} \\ 1 & & 0 & -\lambda_1^{p^n} \\ & \ddots & & \vdots \\ 0 & & 1 & -\lambda_{h-2}^{p^n} \end{pmatrix}. \qquad \square$$

The functor $\mathbb{D}$ commutes with symmetric powers so that we obtain explicit descriptions of $\mathbb{D}(\mathrm{Sym}^{r_0}\rho_0 \otimes \ldots \otimes \mathrm{Sym}^{r_{f-1}}\rho_{f-1} \otimes \det^m)$ for any integer m and all natural numbers $r_0, \ldots, r_{f-1}$. If $h = 3$, for example, the above tensor product representations with $0 \leq m \leq q-1$ and $0 \leq r_i \leq p-1$ are irreducible, pairwise inequivalent and exhaust all irreducible k-linear representations of $\mathrm{GL}_2(k)$. As an example, if $h = 3$ then we see that $\mathbb{D}(\mathrm{Sym}^2\rho_0)$ can be represented by the matrix

$$\begin{pmatrix} \lambda_1^2/\lambda_0^2 & \lambda_1/\lambda_0^2 & 1/\lambda_0^2 \\ -2\lambda_1/\lambda_0 & -1/\lambda_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

7 Unitary representations of noncompact groups

Let the characteristic of K be arbitrary and consider the field $\overline{L}_\infty$ of characteristic p introduced in section 1. According to Corollary 1.5, it is perfect, and by Theorem 1.3 and Corollary 1.6 it carries an action of $\mathrm{GL}_{h-1}(K)$ whose restriction to $\mathrm{GL}_{h-1}(\mathfrak{o})$ makes $\overline{L}_\infty$ a Galois extension of $\overline{L}_0^{\mathrm{rad}}$ with Galois group $\mathrm{GL}_{h-1}(\mathfrak{o})$. By functoriality, there is a $\breve{\mathbb{Z}}_p$-linear continuous action of $\mathrm{GL}_{h-1}(K)$ on the Cohen ring $\mathfrak{o}_{\breve{\mathcal{E}}} := W(\overline{L}_\infty)$ of $\overline{L}_\infty$, commuting with the natural Frobenius.

Let G be the subgroup of $\mathrm{GL}_{h-1}(K)$ consisting of all elements g with $\det(g) \in \mathfrak{o}^\times$. Let $U \subseteq G$ be any compact subgroup and choose $g \in \mathrm{GL}_{h-1}(K)$ such that $gUg^{-1} \subseteq \mathrm{GL}_{h-1}(\mathfrak{o})$. We have $\overline{L}_\infty^U = g^{-1}(\overline{L}_\infty^{gUg^{-1}})$, and the extension $\overline{L}_\infty | \overline{L}_\infty^U$ is Galois with Galois group U.

Varying the notion of a coefficient system, as developed in [12] by P. Schneider and U. Stuhler, we shall see that an object $V \in \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(G)$ is completely determined by the family of admissible φ-modules $(\mathfrak{o}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V)^U$ with U ranging over a particular family of compact open subgroups U of G.

More precisely, let X denote the Bruhat-Tits building of $\mathrm{GL}_{h-1}(K)$, viewed as an abstract simplicial complex. It carries an action of G such that the stabilizer subgroup U_σ of any simplex σ of X is compact and open in G. We let $\mathfrak{o}_\sigma := \mathfrak{o}_{\breve{\mathcal{E}}}^{U_\sigma}$. A *cohomological coefficient system of φ-modules on X* is a family $\underline{M} = (M_\sigma)_\sigma$ of objects $M_\sigma \in \Phi_{\mathfrak{o}_\sigma}^{\mathrm{cpl}}$, parametrized by the simplices σ of X, such that for any two simplices σ' and σ with $\sigma' \subseteq \sigma$ there is a φ-equivariant $\mathfrak{o}_{\sigma'}$-linear map $r_{\sigma'}^\sigma : M_{\sigma'} \to M_\sigma$ satisfying $r_\sigma^\sigma = \mathrm{id}_{M_\sigma}$, as well as $r_{\sigma''}^\sigma = r_{\sigma'}^\sigma \circ r_{\sigma''}^{\sigma'}$ whenever $\sigma'' \subseteq \sigma'$ and $\sigma' \subseteq \sigma$. Note that $U_\sigma \subseteq U_{\sigma'}$ and hence $\mathfrak{o}_{\sigma'} \subseteq \mathfrak{o}_\sigma$ whenever σ' and σ are simplices of X with $\sigma' \subseteq \sigma$.

As in [12], Chapter V, there is a natural notion of G acting on a cohomological coefficient system $\underline{M}$ of φ-modules on X. We call $\underline{M}$ *locally trivial* and *admissible* if all φ-modules M_σ are admissible, if the homomorphisms $r_{\sigma'}^\sigma$ induce isomorphisms $\mathfrak{o}_\sigma \widehat{\otimes}_{\mathfrak{o}_{\sigma'}} M_{\sigma'} \simeq M_\sigma$ whenever $\sigma' \subseteq \sigma$, and if $\underline{M}$ carries an action of G such that for any simplex σ of X the induced $\mathfrak{o}_\sigma$-linear action of U_σ on M_σ is trivial. We denote by $\varphi\text{-}\mathrm{Coeff}_G^{0,\mathrm{adm}}(X)$ the category of locally trivial cohomological coefficient systems of admissible φ-modules on X.

Theorem 7.1. *The assignment* $(V \mapsto \underline{M} = (M_\sigma)_\sigma)$ *with* $M_\sigma := (\mathfrak{o}_{\breve{\mathcal{E}}} \widehat{\otimes}_{\mathbb{Z}_p} V)^{U_\sigma}$ *is an equivalence* $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(G) \to \varphi\text{-}\mathit{Coeff}^{0,\mathrm{adm}}_G(X)$ *of categories.*

Proof: Let $V \in \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(G)$ be given and consider the family $\underline{M}$. For any simplex σ of X the $\mathfrak{o}_\sigma$-module M_σ is an object of $\Phi^{\mathrm{adm}}_{\mathfrak{o}_\sigma}$ by Theorem 3.6. If $\sigma' \subseteq \sigma$ then $U_\sigma \subseteq U_{\sigma'}$, and we have the obvious φ-equivariant and $\mathfrak{o}_{\sigma'}$-linear homomorphism $r^\sigma_{\sigma'} : M_{\sigma'} \to M_\sigma$. It induces an isomorphism $\mathfrak{o}_\sigma \widehat{\otimes}_{\mathfrak{o}_{\sigma'}} M_{\sigma'} \simeq M_\sigma$ as can be seen by reducing modulo p^n, performing the faithfully flat base change $\mathfrak{o}_{\breve{\mathcal{E}}} \otimes_{\mathfrak{o}_\sigma} (\,\cdot\,)$, and by referring to Theorem 3.2. The G-action on $\underline{M}$ comes from the diagonal action of G on $\mathfrak{o}_{\breve{\mathcal{E}}} \widehat{\otimes}_{\mathbb{Z}_p} V$ and makes $\underline{M}$ a locally trivial cohomological coefficient system of admissible φ-modules on X as required.

Conversely, given an object $\underline{M}$ of $\varphi\text{-Coeff}^{0,\mathrm{adm}}_G(X)$ and a simplex σ of X, we let $V_\sigma := (\mathfrak{o}_{\breve{\mathcal{E}}} \widehat{\otimes}_{\mathfrak{o}_\sigma} M_\sigma)^{\varphi_{M_\sigma}=1} \in \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(U_\sigma)$ and set $\tilde{V} := \oplus_\sigma V_\sigma$. Note that if σ' and σ are simplices of X with $\sigma' \subseteq \sigma$ then $r^\sigma_{\sigma'}$ induces a U_σ-equivariant $\mathbb{Z}_p$-linear isomorphism $V_{\sigma'} \to V_\sigma$. We again denote it by $r^\sigma_{\sigma'}$ and let $W_{\sigma'}$ be the $\mathbb{Z}_p$-submodule of $\tilde{V}$ which is the image of the endomorphism $\sum_{\sigma \supseteq \sigma'} (\mathrm{id}_{V_\sigma} - r^\sigma_{\sigma'})$ of $\tilde{V}$. Set $V := \tilde{V} / \sum_{\sigma'} W_{\sigma'}$. The G-action on $\underline{M}$ induces a semilinear G-action on $\oplus_\sigma \mathfrak{o}_{\breve{\mathcal{E}}} \widehat{\otimes}_{\mathfrak{o}_\sigma} M_\sigma$ and $\mathbb{Z}_p$-linear actions on $\tilde{V}$ and V. Note that for any simplex σ of X the natural map $V_\sigma \to V$ is a U_σ-equivariant $\mathbb{Z}_p$-linear isomorphism. To see this one is easily reduced to a maximal simplex and uses the fact that any two such can be joined by a gallery. In particular, the $\mathbb{Z}_p$-module V is p-adically separated and complete. The G-action on V is continuous because its restriction to any of the open subgroups U_σ is and because any $\mathbb{Z}_p$-linear endomorphism of V is automatically continuous for the p-adic topology.

It is a straightforward matter to show that the assignment $(\underline{M} \mapsto V)$ is functorial and quasi inverse to the one above. We leave the details to the reader. □

7.1 char$(K) = p$

Assume the characteristic of K to be p and consider the field L^{rad}_∞ of characteristic p. According to Proposition 1.2 and Theorem 1.3 it carries an action of P whose restriction to P_0 makes L^{rad}_∞ a Galois extension of L^{rad}_0 with Galois group P_0. By functoriality, there is an action of P on the Cohen ring $\mathfrak{o}_{\breve{\mathcal{E}}} := W(L^{\mathrm{rad}}_\infty)$ of L^{rad}_∞. As in section 5 we could use it to embed the category $\mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(P_0)$ into the category $\Phi^{\mathrm{adm}}_{W(L^{\mathrm{rad}}_0)[1/p]}$. However, here we are interested in the functor

$$\mathbb{D} : \mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(P) \longrightarrow \Phi^{\mathrm{uni}}_{W(\breve{K}^{\mathrm{rad}})[1/p]},$$

given by $\mathbb{D}(V) := (\breve{\mathcal{E}} \widehat{\otimes}_{\mathbb{Q}_p} V)^P$. Note that $\breve{\mathcal{E}}^P = W(\breve{K}^{\mathrm{rad}})[1/p]$ by Theorem 1.7. There is also a functor $\mathbb{V}$ in the other direction, given by

$$\mathbb{V}(M) := (\breve{\mathcal{E}} \widehat{\otimes}_{W(\breve{K}^{\mathrm{rad}})[1/p]} M)^{\varphi_M = 1},$$

but we do not expect the functors $\mathbb{D}$ and $\mathbb{V}$ to be as well behaved as for Galois groups. In particular, there is no reason as to why $\mathbb{D}$ should still be exact. Therefore, instead of considering $\mathbb{D}(V) = \mathrm{H}^0_{\mathrm{cont}}(P, \breve{\mathcal{E}} \widehat{\otimes}_{\mathbb{Q}_p} V)$ as the only invariant of the semilinear P-representation $\breve{\mathcal{E}} \widehat{\otimes}_{\mathbb{Q}_p} V$, we suggest to consider also its higher continuous P-cohomology.

Consider the complex $C_b^\bullet := C_b^{\bullet+1}(P, \breve{\mathcal{E}}\widehat{\otimes}_{\mathbb{Q}_p}V)$ of bounded continuous cochains on P with coefficients in $\breve{\mathcal{E}}\widehat{\otimes}_{\mathbb{Q}_p}V$. For each integer $q \geq 0$, the space C_b^q is naturally an $\breve{\mathcal{E}}$-Banach space. We endow it with the φ-semilinear endomorphism φ_q given by $\varphi_q(f)(g_0, \ldots, g_q) := \varphi(f(g_0, \ldots, g_q))$, where $\varphi := \varphi\widehat{\otimes}\mathrm{id}_V$ denotes the Frobenius endomorphism of $\breve{\mathcal{E}}\widehat{\otimes}_{\mathbb{Q}_p}V$. The differentials $d_q : C_b^q \to C_b^{q+1}$, given by

$$(d_q f)(g_0, \ldots, g_{q+1}) := \sum_{i=0}^{q+1} (-1)^i f(g_0, \ldots, \widehat{g_i}, \ldots, g_{q+1}),$$

obviously commute with $\varphi_\bullet$. We let $\mathrm{H}^\bullet_{\mathrm{cont}}(P, \breve{\mathcal{E}}\widehat{\otimes}_{\mathbb{Q}_p}V)$ be the cohomology of the complex $(C_b^\bullet)^P$, which is a complex in $\Phi^{\mathrm{uni}}_{W(\breve{K}^{\mathrm{rad}})[1/p]}$. Usually, the unitary Banach φ-modules over $W(\breve{K}^{\mathrm{rad}})[1/p]$ which appear in this complex will not be admissible, of course, as the example of the trivial one dimensional representation $V = \mathbb{Q}_p$ shows. Indeed, $C_b(P, \breve{\mathcal{E}}\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{Q}_p)^P \simeq \breve{\mathcal{E}}$, viewed as a φ-module over $W(\breve{K}^{\mathrm{rad}})[1/p]$. It is easy to see, however, that L_∞^{rad} is not an admissible φ-module over $\breve{K}^{\mathrm{rad}}$, unless $h = 1$.

If the cohomology of the above complex is Hausdorff, it follows that the cohomology group $\mathrm{H}^q_{\mathrm{cont}}(P, \breve{\mathcal{E}}\widehat{\otimes}_{\mathbb{Q}_p}V)$ is an object of $\Phi^{\mathrm{uni}}_{W(\breve{K}^{\mathrm{rad}})[1/p]}$ for any integer $q \geq 0$. In this case one might even hope the φ-structure to be admissible, in which case $\mathrm{H}^q_{\mathrm{cont}}(P, \breve{\mathcal{E}}\widehat{\otimes}_{\mathbb{Q}_p}V)$ would correspond to an object of $\mathrm{Rep}^{\mathrm{uni}}_{\mathbb{Q}_p}(I_K)$ with $I_K = \mathrm{Gal}(K^{\mathrm{sep}}|K^{\mathrm{unr}}) \simeq \mathrm{Gal}(\breve{K}^{\mathrm{alg}}|\breve{K}^{\mathrm{rad}})$ (cf. Theorem 3.6).

Note, by the way, that if $\mathfrak{o}_{\breve{\mathcal{E}}_o} \subseteq W(\breve{K}^{\mathrm{alg}})$ is an I_K-stable Cohen ring of $\breve{K}^{\mathrm{sep}}$ and if $\mathfrak{o}_{\mathcal{E}_0} := \mathfrak{o}_{\breve{\mathcal{E}}_o}^{I_K}$, then there is a commutative diagram of functors

$$\begin{array}{ccc} & \overset{\mathbb{D}}{\nearrow} & \Phi^{\mathrm{adm}}_{W(\breve{K}^{\mathrm{rad}})} \\ \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(I_K) & & \uparrow {\scriptstyle W(\breve{K}^{\mathrm{rad}})\widehat{\otimes}_{\mathfrak{o}_{\mathcal{E}_0}}(\,\cdot\,)}, \\ & \underset{\mathbb{D}}{\searrow} & \Phi^{\mathrm{adm}}_{\mathfrak{o}_{\mathcal{E}_0}} \end{array}$$

as can be seen as in Proposition 5.4. Now Theorem 3.6 implies the vertical one to be an equivalence of categories, thus generalizing [6], Proposition 1.1.3.

Deeper results relating $\mathbb{D}$ and $\mathbb{V}$ will crucially rely on a better understanding of the P-representations $\breve{\mathcal{E}}$ and L_∞^{rad}, and of their cohomology. We hope to come back to these questions in the future.

References

[1] V. Abrashkin: An analogue of the field-of-norms functor and of the Grothendieck conjecture, *J. Algebraic Geom.* **16**, 2007, pp. 671–730

[2] F. Andreatta: Generalized ring of norms and generalized (φ, Γ)-modules, *Ann. Scient. Éc. Norm. Sup.* **35**, 2006, pp. 599-647

[3] N. Bourbaki: *Algèbre Commutative*, Springer, 2006

[4] P. Colmez: Représentations de $\mathrm{GL}_2(\mathbb{Q}_p)$ et (φ, Γ)-modules, *Astérisque* **330**, 2010, pp. 281–509

[5] V.G. Drinfeld: Elliptic modules, *Math. USSR Sbornik* **23**, No. 4, 1974, pp. 561–592

[6] J-M. Fontaine: Représentations p-adiques des corps locaux I, The Grothendieck Festschrift Vol. II, *Progr. Math.* **87**, Birkhäuser, 1990, pp. 249–309

[7] J-M. Fontaine, Y. Ouyang: Theory of p-adic Galois representations, *preprint*, 2008

[8] B.H. Gross, M.J. Hopkins: Equivariant vector bundles on the Lubin-Tate moduli space, *Contemporary Math.* **158**, 1994, pp. 23–88

[9] J. Kohlhaase: Lubin-Tate and Drinfeld bundles, *preprint*, 2009

[10] J-P. Serre: *Corps locaux*, Troisième Édition, Hermann, Paris, 1968

[11] P. Schneider: p-adic Banach space representations of p-adic groups, Lectures at Jerusalem, 2009, available at `http://wwwmath.uni-muenster.de/u/schneider/publ/lectnotes/`

[12] P. Schneider, U. Stuhler: Representation theory and sheaves on the Bruhat-Tits building, *Publ. Math. IHES* **85**, 1997, pp. 97–191

[13] P. Schneider, J. Teitelbaum: Banach space representations and Iwasawa theory, *Israel J. Mathematics* **127**, 2002, pp. 359–380

[14] T. Schoeneberg: p-adische Galoisdarstellungen und (φ, Γ)-Moduln, Diplomarbeit, Universität Münster, 2009, available at `http://wwwmath.uni-muenster.de/u/schneider/publ/diplom/`

[15] A.J. Scholl: Higher fields of norms and (φ, Γ)-modules, *Doc. Math.*, Extra Volume Coates, 2006, pp. 685–709

[16] M. Strauch: Geometrically connected components of deformation spaces of one-dimensional formal modules, *Pure Appl. Math. Quarterly* **4**, Serre Anniversary Volume, 2008, pp. 1–18

[17] M. Strauch: Deformation spaces of one dimensional formal groups and their cohomology, *Adv. Math.* **217**, 2008, pp. 889–951

[18] M. Strauch: Galois actions on torsion points of universal one-dimensional formal modules, to appear in *J. Number Theory*