Iwasawa modules arising from deformation spaces of p-divisible formal group laws

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Abstract. Let G be a p-divisible formal group law over an algebraically closed field of characteristic p. We show that certain equivariant vector bundles on the universal deformation space of G give rise to pseudocompact modules over the Iwasawa algebra of the automorphism group of G. Passing to global rigid analytic sections, we obtain representations which are topologically dual to locally analytic representations. In studying these, one is led to the consideration of divided power completions of universal enveloping algebras. The latter seem to constitute a novel tool in p-adic representation theory.

Contents

0 Introduction

Let p be a prime number, and let k be an algebraically closed field of characteristic p. Let $W = W(k)$ denote the ring of Witt vectors with coefficients in k, and let K denote the quotient field of W. We fix a p-divisible commutative formal group law G of height h over k and denote by $R := R_G^{\mathrm{def}}$ the universal deformation ring of G representing isomorphism classes of deformations of G to complete noetherian local W-algebras with residue class field k. Denote by $\mathbb{G}$ the universal deformation of G to R and by $\mathrm{Lie}(\mathbb{G})$ the Lie algebra of $\mathbb{G}$. For any integer m, the m-th tensor power $\mathrm{Lie}(\mathbb{G})^{\otimes m}$ of $\mathrm{Lie}(\mathbb{G})$ can be viewed as the space of global sections of a vector bundle on the universal deformation space $\mathrm{Spf}(R)$ which is equivariant for a natural action of the automorphism group $\Gamma := \mathrm{Aut}(G)$ of G.

If G is of dimension one, then the formal scheme $\mathrm{Spf}(R)$ is known as *the moduli space of Lubin-Tate*. It plays a crucial role in Harris' and Taylor's construction of the local Langlands correspondence for $\mathrm{GL}_h(\mathbb{Q}_p)$. Moreover, the Γ-representations $\mathrm{Lie}(\mathbb{G})^{\otimes m}$ and their cohomology figure prominently in stable homotopy theory (cf. the introduction to [7]). Still assuming G to be one dimensional, a detailed study of the Γ-representation R was given in [13]. For $h = 2$ it led to the computation of the continuous Γ-cohomology of R, relying on the foundational work of Devinatz, Gross, Hopkins and Yu. The only prior analysis of p-adic representations stemming from equivariant vector bundles on deformation spaces of p-divisible formal groups concern the p-adic symmetric spaces of Drinfeld. These were studied extensively by Morita, Orlik, Schneider and Teitelbaum (cf. [18], [24] and our remarks at the end of section 2).

The aim of the present article is to generalize and strengthen some of the results of Gross and Hopkins in [9] and of the author in [13]. To this end, section 1 and the first part of section 2 give a survey of the theory of p-divisible commutative formal group laws. This includes the classification results of Dieudonné, Lazard and Manin, as well as the deformation theoretic results of Cartier, Lubin, Tate and Umemura. It follows from the work of Dieudonné and Manin that the group Γ is a compact Lie group over $\mathbb{Q}_p$ (cf. Corollary 1.4).

In the second part of section 2, we prove that the action of Γ on $\mathrm{Lie}(\mathbb{G})^{\otimes m}$ extends to the Iwasawa algebra $\Lambda := W[\![\Gamma]\!]$ of Γ over W. This gives $\mathrm{Lie}(\mathbb{G})^{\otimes m}$ the structure of a pseudocompact module over Λ (cf. Corollary 2.5 and Theorem 2.6). In section 3, we pass to the global rigid analytic sections $(\mathrm{Lie}(\mathbb{G})^{\otimes m})^{\mathrm{rig}}$ of our vector bundles and show that the action of Γ extends to a continuous action of the locally analytic distribution algebra $D(\Gamma)$ of Γ over K. As a consequence, the action of Γ on the strong continuous K-linear dual of $(\mathrm{Lie}(\mathbb{G})^{\otimes m})^{\mathrm{rig}}$ is locally analytic in the sense of Schneider and Teitelbaum (cf. Theorem 3.4 and Theorem 3.5).

We note that the continuity and the differentiability of the action of Γ on R^{rig} were first proven by Gross and Hopkins if G is of dimension one (cf. [9], Proposition 19.2 and Proposition 24.2). Using the structure theory of the algebra $D(\Gamma)$, we arrive at a more precise result for arbitrary m and G, avoiding the use of the period morphism. Our approach essentially relies on a basic lifting lemma for endomorphisms of G which is also at the heart of the strategy followed by Gross and Hopkins (cf. Lemma 2.2 and Proposition 2.3).

A major question that we have to leave open concerns the *coadmissibility* of the $D(\Gamma)$-modules $(\mathrm{Lie}(\mathbb{G})^{\otimes m})^{\mathrm{rig}}$ in the sense of [23], section 6. Taking sections over suitable affinoid subdomains of $\mathrm{Spf}(R)^{\mathrm{rig}}$, it is related to the finiteness properties of the resulting Banach spaces as modules over certain Banach completions of $\Lambda \otimes_W K$. In section 4, we assume G to be of dimension one and consider the restriction of $(\mathrm{Lie}(\mathbb{G})^{\otimes m})^{\mathrm{rig}}$ to an affinoid subdomain of $\mathrm{Spf}(R)^{\mathrm{rig}}$ over which the period morphism of Gross and Hopkins is an open immersion. By spelling out the action of the Lie algebra of Γ, we show that one naturally obtains a continuous module over a complete divided power enveloping algebra $\hat{U}_K^{\mathrm{dp}}(\mathring{\mathfrak{g}})$ constructed by Kostant (cf. Theorem 4.5). Here $\mathring{\mathfrak{g}}$ is a Chevalley order in the split form of the Lie algebra of Γ. If $h = 2$ and $m \geq -1$ then in fact $(\mathrm{Lie}(\mathbb{G})^{\otimes m})^{\mathrm{rig}}$ gives rise to a cyclic module over $\hat{U}_K^{\mathrm{dp}}(\mathring{\mathfrak{g}})$ (cf. Theorem 4.6). This result might indicate that $(\mathrm{Lie}(\mathbb{G})^{\otimes m})^{\mathrm{rig}}$ does not give rise to a coherent sheaf for the Fréchet-Stein structure of $D(\Gamma)$ considered in [23], section 5 (cf. Remark 4.7).

In [9], Gross and Hopkins consider formal modules of dimension one and finite height over the valuation ring $\mathfrak{o}$ of an arbitrary non-archimedean local field. The case of p-divisible formal groups corresponds to the case $\mathfrak{o} = \mathbb{Z}_p$. However, neither the deformation theory nor the theory of the period morphism have been worked out in detail for formal $\mathfrak{o}$-modules of dimension strictly greater than one. This is why we restrict to one dimensional formal groups in section 4 and to p-divisible formal groups throughout.

Conventions and notation. If S is a commutative unital ring, if r is a positive integer, and if $X = (X_1, \ldots, X_r)$ is a family of indeterminates, then we denote by $S[\![X]\!] = S[\![X_1, \ldots, X_r]\!]$ the ring of formal power series in the variables $X_1, \ldots, X_r$ over S. We write $f = f(X) = f(X_1, \ldots, X_r)$ for an element $f \in S[\![X]\!]$. If $n = (n_1, \ldots, n_r) \in \mathbb{N}^r$ is an r-tuple of non-negative integers then we set $|n| := n_1 + \ldots + n_r$ and $X^n := X_1^{n_1} \cdots X_r^{n_r}$. If i and j are elements of a set then we denote by δ_{ij} the Kronecker symbol with value $1 \in S$ if $i = j$ and $0 \in S$ if $i \neq j$. If $\mathfrak{h}$ is a Lie algebra over S then we denote by $U(\mathfrak{h})$ the universal enveloping algebra of $\mathfrak{h}$ over S. Throughout the article, p will denote a fixed prime number.

1 Formal group laws

Let R be a commutative unital ring, and let d be a positive integer. A d-dimensional commutative *formal group law* (subsequently abbreviated to *formal group*) is a d-tuple $G = (G_1, \ldots, G_d)$ of formal power series $G_i \in R[\![X, Y]\!] = R[\![X_1, \ldots, X_d, Y_1, \ldots, Y_d]\!]$, satisfying

(F1) $G_i(X, 0) = X_i$,

(F2) $G_i(X, Y) = G_i(Y, X)$, and

(F3) $G_i(G(X, Y), Z) = G_i(X, G(Y, Z))$

for all $1 \leq i \leq d$. It follows from the formal implicit function theorem (cf. [11], A.4.7) that for a given d-dimensional commutative formal group G there exists a unique d-tuple $\iota_G \in R[\![X]\!]^d$ of formal power series with trivial constant terms such that

$$G_i(X, \iota_G(X)) = 0 \quad \text{for all} \quad 1 \leq i \leq d$$

(cf. also [27], Korollar 1.5). Thus, if S is a commutative R-algebra, and if I is an ideal of S such that S is I-adically complete, then the set I^d becomes a commutative group with unit element $(0, \ldots, 0)$ via

$$x +_G y := G(x, y) \quad \text{and} \quad -x := \iota_G(x).$$

Example 1.1. Let $R = \mathbb{Z}$ and $d = 1$. The formal group $\hat{\mathbb{G}}_a(X, Y) = X + Y$ is called the *one dimensional additive formal group*. We have $\iota_{\hat{\mathbb{G}}_a}(X) = -X$. The formal group $\hat{\mathbb{G}}_m(X, Y) = (1 + X)(1 + Y) - 1$ is called the *one dimensional multiplicative formal group*. We have $\iota_{\hat{\mathbb{G}}_m}(X) = \sum_{n=1}^{\infty} (-X)^n$.

Let G and H be formal groups over R of dimensions d and e, respectively. A *homomorphism from G to H* is an e-tuple $\varphi = (\varphi_1, \ldots, \varphi_e)$ of power series $\varphi_i \in R[\![X]\!] = R[\![X_1, \ldots, X_d]\!]$ in d-variables over R with trivial constant terms, satisfying

$$\varphi(G(X, Y)) = H(\varphi(X), \varphi(Y)).$$

If $\varphi : G \to G'$ and $\psi : G' \to G''$ are homomorphisms of formal groups then we define $\psi \circ \varphi$ through $(\psi \circ \varphi)(X) := \psi(\varphi(X))$. This is a homomorphism from G to G''. We let $\mathrm{End}(G)$ denote the set of *endomorphisms* of a d-dimensional commutative formal group G over R, i.e. of homomorphisms from G to G. It is a ring with unit $1_G = X = (X_1, \ldots, X_d)$, in which addition and multiplication are defined by $(\varphi +_G \psi)(X) := G(\varphi(X), \psi(X))$, $(-\varphi)(X) := \iota_G(\varphi(X))$ and $\psi \cdot \varphi := \psi \circ \varphi$. In particular, $\mathrm{End}(G)$ is a $\mathbb{Z}$-module. Given $m \in \mathbb{Z}$, we denote by $[m]_G \in R[\![X]\!]^d$ the corresponding endomorphism of G. We denote by $\mathrm{Aut}(G)$ the *automorphism group of* G, i.e. the group of units of the ring $\mathrm{End}(G)$.

Denoting by (X) the ideal of $R[\![X]\!]$ generated by $X_1, \ldots, X_d$, the free R-module

$$\mathrm{Lie}(G) := \mathrm{Hom}_R((X)/(X)^2, R)$$

of rank $d = \dim(G)$ is called the *Lie algebra of* G (or its *tangent space at* 1_G). It is an R-Lie algebra for the trivial Lie bracket. Non-commutative Lie algebras occur only for non-commutative formal groups (cf. [27], Kapitel I.7). An R-basis of $\mathrm{Lie}(G)$ is given by the linear forms $(\frac{\partial}{\partial X_i})_{1 \le i \le d}$ sending $f + (X)^2$ to $\frac{\partial f}{\partial X_i}(0)$. Here $\frac{\partial f}{\partial X_i}$ denotes the formal derivative of the power series f with respect to the variable X_i.

Any homomorphism $\varphi : G \to H$ of formal groups as above gives rise to an R-linear ring homomorphism $\varphi^* : R[\![Y_1, \ldots, Y_e]\!] \to R[\![X_1, \ldots, X_d]\!]$, determined by $\varphi^*(Y_i) = \varphi_i$ for all $1 \le i \le e$. It is called the *comorphism of* φ. It maps (Y) to (X), hence $(Y)^2$ to $(X)^2$, and therefore induces an R-linear map

$$\mathrm{Lie}(\varphi) : \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(H)$$

via $\mathrm{Lie}(\varphi)(\delta)(h + (Y)^2) := \delta(\varphi^*(h) + (X)^2)$. In the R-bases $(\frac{\partial}{\partial X_i})_i$ (resp. $(\frac{\partial}{\partial Y_j})_j$) of $\mathrm{Lie}(G)$ (resp. $\mathrm{Lie}(H)$), the map $\mathrm{Lie}(\varphi)$ is given by the *Jacobian matrix* $(\frac{\partial \varphi_i}{\partial X_j}(0))_{i,j} \in R^{e \times d}$ of φ. If $\varphi : G \to G'$ and $\psi : G' \to G''$ are homomorphisms of formal groups, then $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$ and $\mathrm{Lie}(\psi \circ \varphi) = \mathrm{Lie}(\psi) \circ \mathrm{Lie}(\varphi)$. If $H = G$ then one can use **(F1)** to show that the map $(\varphi \mapsto \mathrm{Lie}(\varphi)) : \mathrm{End}(G) \to \mathrm{End}_R(\mathrm{Lie}(G))$ is a homomorphism of rings. In particular, $\mathrm{Lie}(G)$ becomes a module over $\mathrm{End}(G)$ and we have $\mathrm{Lie}([m]_G) = m \cdot \mathrm{id}_{\mathrm{Lie}(G)}$ for any integer m.

If p is a prime number and if R is a complete noetherian local ring of residue characteristic p, then a homomorphism $\varphi : G \to H$ of formal groups is called an *isogeny* if the comorphism φ^* makes $R[\![X]\!]$ a finite free module over $R[\![Y]\!]$ (cf. [25], section 2.2). Of course, this can only happen if $d = e$. A formal group G over a complete noetherian local ring R with residue characteristic p is called *p-divisible*, if the homomorphism $[p]_G : G \to G$ is an isogeny. In this case the rank of $R[\![X]\!]$ over itself via $[p]_G^*$ is a power of p, say p^h (cf. [25], section 2.2; this result can also be deduced from [27], Satz 5.3). The integer $h =: \mathrm{ht}(G)$ is called the *height* of the p-divisible formal group G.

If $R = k$ is a perfect field of characteristic p, the necessary tools to effectively study the category of p-divisible commutative formal groups over k were first developed by Dieudonné (cf. [6], Chapter III). His methods were later generalized by Cartier in order to describe commutative formal groups over arbitrary rings (cf. [16], Chapters III & IV, or [27], Chapters III & IV).

Sticking to the case of a perfect field k of characteristic p, we denote by $W := W(k)$ the ring of Witt vectors over k. Let $\sigma = (x \mapsto x^p)$ denote the Frobenius automorphism of k, as well as its unique lift to a ring automorphism of W. Recall that a *σ^{-1}-crystal over k* is a pair (M, V), consisting of a finitely generated free W-module M and a map $V : M \to M$ which is σ^{-1}-linear, i.e. which is additive and satisfies

$$V(am) = \sigma^{-1}(a)V(m) \quad \text{for all} \quad a \in W,\ m \in M.$$

We shall be interested in those σ^{-1}-crystals (M, V) which satisfy the following two extra conditions (here **D** stands for Dieudonné):

(D1) $pM \subseteq V(M)$

(D2) $V \bmod p$ is a nilpotent endomorphism of M/pM

For the following fundamental result cf. [27], page 109.

Theorem 1.2 (Dieudonné)**.** *If k is a perfect field of characteristic p then the category of p-divisible commutative formal groups over k is equivalent to the category of σ^{-1}-crystals over k, satisfying (D1) and (D2).* □

Let $W[F, V]$ be the non-commutative ring generated by two elements F and V over W subject to the relations

$$VF = FV = p, \quad Va = \sigma^{-1}(a)V \quad \text{and} \quad Fa = \sigma(a)F \quad \text{for all} \quad a \in W.$$

The equivalence of Theorem 1.2 associates with a p-divisible commutative formal group G its *(covariant) Cartier-Dieudonné module* M_G. This is a V-adically separated and complete module over $W[V, F]$ such that the action of V is injective. Since G is p-divisible, also the action of F is injective, and the underlying W-module of M_G is finitely generated and free. In particular, the pair (M_G, V) is a σ^{-1}-crystal over k, satisfying $pM_G = VFM_G \subseteq VM_G$, i.e. condition **(D1)**. Condition **(D2)** follows from the V-adic completeness of M_G. We also note that V and F give rise to a short exact sequence

$$0 \longrightarrow M_G/FM_G \xrightarrow{V} M_G/pM_G \longrightarrow M_G/VM_G \longrightarrow 0,$$

of k-vector spaces in which $\dim_k(M_G/pM_G) = \mathrm{ht}(G)$ and $\dim_k(M_G/VM_G) = \dim(G)$.

Conversely, if (M, V) is a σ^{-1}-crystal over k satisfying **(D1)**, then V is injective. In fact, **(D1)** implies that V becomes surjective (and hence bijective) over the quotient field K of W. Setting $F := V^{-1}p$, the W-module M becomes a module over $W[F, V]$ which is V-adically separated and complete if condition **(D2)** is satisfied.

Recall that a *σ^{-1}-isocrystal over k* is a pair (N, f) consisting of a finite dimensional K-vector space N and a σ^{-1}-linear bijection $f : N \to N$. If (M, V) is a σ^{-1}-crystal over k which satisfies **(D1)** then $(M \otimes_W K, V \otimes \mathrm{id}_K)$ is a σ^{-1}-isocrystal over k. The σ^{-1}-isocrystal which in this way is associated with the

Cartier-Dieudonné module of a p-disivible commutative formal group G over k, classifies G up to isogeny (cf. [27], Satz 5.26 and the remarks on page 110; alternatively, consult [6], Chapter IV.1).

Given integers r and s with $r > 0$, consider the σ^{-1}-isocrystal over k given by $(K[t]/(t^r - p^s), t \circ \sigma)$. Here $K[t]$ denotes the usual commutative polynomial ring in the variable t over K on which σ acts coefficientwise. If k is algebraically closed, we have the following fundamental classification result of Dieudonné and Manin (cf. [27], Satz 6.29, [6], Chapter IV.4, and [16], Proposition VI.7.42).

Theorem 1.3 (Dieudonné-Manin)**.** *If k is an algebraically closed field of characteristic p then the category of σ^{-1}-isocrystals over k is semisimple. The simple objects are given by the σ^{-1}-isocrystals $(K[t]/(t^r - p^s), t \circ \sigma)$, where r and s are relatively prime integers with $r > 0$.* □

To a pair (r, s) of integers as in Theorem 1.3 corresponds a particular p-divisible commutative formal group G_{rs} over k inside the isogeny class determined by the σ^{-1}-isocrystal $(K[t]/(t^r - p^s), t \circ \sigma)$. According to [16], Proposition VI.7.42, the endomorphism ring of G_{rs} is isomorphic to the maximal order of the central division algebra of invariant $\frac{s}{r} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$ and dimension r^2 over $\mathbb{Q}_p$.

Corollary 1.4. *If G is a p-divisible commutative formal group over an algebraically closed field k of characteristic p then the endomorphism ring* $\mathrm{End}(G)$ *of G is an order in a finite dimensional semisimple $\mathbb{Q}_p$-algebra. Endowing* $\mathrm{End}(G)$ *with the p-adic topology and the automorphism group* $\mathrm{Aut}(G)$ *of G with the induced topology,* $\mathrm{Aut}(G)$ *is a compact Lie group over $\mathbb{Q}_p$.*

Proof. That $\mathrm{End}(G)$ is a p-adically separated and torsion free $\mathbb{Z}_p$-module can easily be proved directly, using that G is p-divisible. It also follows from the fact that the Cartier-Dieudonné module of G is free over W. According to Theorem 1.3 and the subsequent remarks there are central division algebras $D_1, \ldots, D_n$ over $\mathbb{Q}_p$ and natural numbers $m_1, \ldots, m_n$ such that

$$\mathrm{End}(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathrm{Mat}(m_1 \times m_1, D_1) \times \ldots \times \mathrm{Mat}(m_n \times m_n, D_n)$$

as $\mathbb{Q}_p$-algebras. Since $\mathrm{End}(G)$ is p-adically separated, it is bounded in $\mathrm{End}(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Thus, it is a lattice in a finite dimensional $\mathbb{Q}_p$-vector space and must be finitely generated over $\mathbb{Z}_p$. This proves the first assertion. Endowing $\mathrm{End}(G)$ with the p-adic topology, it becomes a topological $\mathbb{Z}_p$-algebra and $\mathrm{Aut}(G)$ becomes a compact topological group for the subspace topology. By the above arguments, it is isomorphic to an open subgroup of $\prod_{i=1}^n \mathrm{GL}_{m_i}(D_i)$, hence naturally carries the structure of a Lie group over $\mathbb{Q}_p$. □

2 Deformation problems and Iwasawa modules

We continue to denote by k a fixed algebraically closed field of characteristic p. We also fix a p-divisible commutative formal group G of dimension d over k. Denote by $W = W(k)$ the ring of Witt vectors of k and by $\mathcal{C}_k$ the category of complete noetherian commutative local W-algebras with residue class field k. Let R be an object of $\mathcal{C}_k$ and let $\mathfrak{m}$ be the maximal ideal of R. A *deformation of G to R* is a pair $(G', \rho_{G'})$, where G' is commutative formal group over R

and $\rho_{G'} : G \to G' \bmod \mathfrak{m}$ is an isomorphism of formal groups over k. Two deformations $(G', \rho_{G'})$ and $(G'', \rho_{G''})$ of G to R are said to be isomorphic if there is an isomorphism $f : G' \to G''$ of formal groups over R such that the diagram

$$\begin{tikzcd} & G' \bmod \mathfrak{m} \arrow[dd, "f \bmod \mathfrak{m}"] \\ G \arrow[ur, "\rho_{G'}"] \arrow[dr, "\rho_{G''}"'] & \\ & G'' \bmod \mathfrak{m} \end{tikzcd}$$

is commutative. Let Def_G denote the functor from $\mathcal{C}_k$ to the category *Sets* of sets which associates with an object R of $\mathcal{C}_k$ the set of isomorphism classes of deformations of G to R. If $\dim(G) = 1$, then the following theorem was first proved by Lubin and Tate (cf. [17], Theorem 3.1), building on the work of Lazard. It was later generalized by Cartier und Umemura, independently (cf. [5] and [26]).

Theorem 2.1. *The functor* $\mathrm{Def}_G : \mathcal{C}_k \to$ *Sets is representable, i.e. there is an object* R_G^{def} *of* $\mathcal{C}_k$ *and a deformation* $\mathbb{G}$ *of* G *to* R_G^{def} *with the following universal property. For any object* R *of* $\mathcal{C}_k$ *and any deformation* $(G', \rho_{G'})$ *of* G *to* R *there is a unique* W*-linear local ring homomorphism* $\varphi : R_G^{\mathrm{def}} \to R$ *and a unique isomorphism* $[\varphi] : \varphi_*(\mathbb{G}, \rho_{\mathbb{G}}) \simeq (G', \rho_{G'})$ *of deformations of* G *to* R.* *If* $h = \mathrm{ht}(G)$ *and* $d = \dim(G)$ *denote the height and the dimension of* G*, respectively, then the* W*-algebra* R_G^{def} *is non-canonically isomorphic to the power series ring* $W[\![u_1, \ldots, u_{(h-d)d}]\!]$ *in* $(h-d)d$ *variables over* W. $\square$

It follows from the universal property of the deformation $(\mathbb{G}, \rho_{\mathbb{G}})$ that the automorphism group $\mathrm{Aut}(G)$ of G acts on the universal deformation ring R_G^{def} by W-linear local ring automorphisms. Indeed, given $\gamma \in \mathrm{Aut}(G)$, there is a unique W-linear local ring endomorphism γ of R_G^{def} and a unique isomorphism $[\gamma] : \gamma_*(\mathbb{G}, \rho_{\mathbb{G}}) \simeq (\mathbb{G}, \rho_{\mathbb{G}} \circ \gamma)$ of deformations of G to R_G^{def}. It follows from the uniqueness that the resulting map $\mathrm{Aut}(G) \to \mathrm{End}(R_G^{\mathrm{def}})$ factors through a homomorphism

$$\mathrm{Aut}(G) \longrightarrow \mathrm{Aut}(R_G^{\mathrm{def}})$$

of groups. It is this type of representation that we are concerned with in this article. To ease notation we shall denote by

$$R := R_G^{\mathrm{def}}$$

the universal deformation ring of our fixed p-divisible commutative formal group G over k. Let $\mathfrak{m}$ denote the maximal ideal of R. For any non-negative integer n we denote by $\mathbb{G}_n := \mathbb{G} \bmod \mathfrak{m}^{n+1}$ the reduction of the universal deformation $\mathbb{G}$ modulo the ideal $\mathfrak{m}^{n+1}$ of R. We have $G \simeq \mathbb{G}_0$ via $\rho_{\mathbb{G}}$.

Lemma 2.2. *If* n *is a non-negative integer then the homomorphism of rings* $\mathrm{End}(\mathbb{G}_{n+1}) \to \mathrm{End}(\mathbb{G}_n)$*, induced by reduction modulo* $\mathfrak{m}^{n+1}$*, is injective.*

*Here $\varphi_*(\mathbb{G}, \rho_{\mathbb{G}}) = (\varphi_*(\mathbb{G}), \rho_{\mathbb{G}})$, where $\varphi_*\mathbb{G}$ is obtained by applying φ to the coefficients of $\mathbb{G}$. Since φ induces an isomorphism between the residue class fields of R_G^{def} and R, we may identify $\mathbb{G} \bmod \mathfrak{m}_{R_G^{\mathrm{def}}}$ and $\varphi_*\mathbb{G} \bmod \mathfrak{m}$.

Proof. The formal group $\mathbb{G}_{n+1}$ is p-divisible because the comorphism $[p]^*_{\mathbb{G}_{n+1}}$ is finite and free. Indeed, it is so after reduction modulo $\mathfrak{m}$, and one can use [3], III.2.1 Proposition 14 and III.5.3 Théorème 1, to conclude. Since the ideal $\mathfrak{m}^{n+1}(R/\mathfrak{m}^{n+2})$ of $R/\mathfrak{m}^{n+2}$ is nilpotent, the claim follows from the rigidity theorem in [27], Satz 5.30. $\square$

The preceding lemma allows us to regard all endomorphism rings $\mathrm{End}(\mathbb{G}_n)$ as subrings of $\mathrm{End}(\mathbb{G}_0)$. The main technical result of this section is the following assertion.

Proposition 2.3. *For any non-negative integer n the subring* $\mathrm{End}(\mathbb{G}_n)$ *of* $\mathrm{End}(\mathbb{G}_0)$ *contains* $p^n \,\mathrm{End}(\mathbb{G}_0)$.

Proof. We proceed by induction on n, the case $n = 0$ being trivial. Let $n \geq 1$ and assume the assertion to be true for $n-1$. Set $R_n := R/\mathfrak{m}^{n+1}$. Let $\varphi \in p^{n-1}\,\mathrm{End}(\mathbb{G}_0) \subseteq \mathrm{End}(\mathbb{G}_{n-1})$ and choose a family $\tilde{\varphi} \in R_n[\![X]\!]^d$ of power series with trivial constant terms such that $\tilde{\varphi} \bmod \mathfrak{m}^n R_n = \varphi$. The d-tuple of power series $[p]_{\mathbb{G}_n} \circ \tilde{\varphi}$ is then a lift of $p\varphi$. We claim that it is an endomorphism of $\mathbb{G}_n$.

Note first that $[p]_{\mathbb{G}_n} \circ \tilde{\varphi}$ depends only on φ and not on the choice of a lift $\tilde{\varphi}$. Indeed, if $\tilde{\varphi}'$ is a second lift of φ with trivial constant terms, set $\psi := \tilde{\varphi}' - \tilde{\varphi}$. Setting $\chi := (\tilde{\varphi} + \psi) -_{\mathbb{G}_n} \tilde{\varphi}$, we have $\tilde{\varphi}' = \tilde{\varphi} +_{\mathbb{G}_n} \chi$. Further, the power series χ satisfies $\chi \bmod \mathfrak{m}^n = \varphi -_{\mathbb{G}_{n-1}} \varphi = 0$, hence has coefficients in $\mathfrak{m}^n R_n$. Since $p\mathfrak{m}^n \subseteq \mathfrak{m}^{n+1}$ and $(\mathfrak{m}^n)^m \subseteq \mathfrak{m}^{n+1}$ for any integer $m \geq 2$, we have $[p]_{\mathbb{G}_n} \circ \chi = 0$ and hence

$$[p]_{\mathbb{G}_n} \circ \tilde{\varphi}' = [p]_{\mathbb{G}_n}(\tilde{\varphi} +_{\mathbb{G}_n} \chi) = ([p]_{\mathbb{G}_n} \circ \tilde{\varphi}) +_{\mathbb{G}_n} ([p]_{\mathbb{G}_n} \circ \chi) = [p]_{\mathbb{G}_n} \circ \tilde{\varphi},$$

as desired.

If $\eta \in R_n[\![X]\!]^d$ is a family of power series with trivial constant terms, set $\delta_\eta := \delta_\eta(X, Y) := \eta(X +_{\mathbb{G}_n} Y) -_{\mathbb{G}_n} \eta(X) -_{\mathbb{G}_n} \eta(Y)$. Since $\tilde{\varphi}$ reduces to an endomorphism of $\mathbb{G}_{n-1}$, the power series $\delta_{\tilde{\varphi}}$ has coefficients in $\mathfrak{m}^n$. As above, this implies $[p]_{\mathbb{G}_n} \circ \delta_{\tilde{\varphi}} = 0$ and thus

$$\begin{aligned}
\delta_{[p]_{\mathbb{G}_n} \circ \tilde{\varphi}} &= ([p]_{\mathbb{G}_n} \circ \tilde{\varphi})(X +_{\mathbb{G}_n} Y) -_{\mathbb{G}_n} ([p]_{\mathbb{G}_n} \circ \tilde{\varphi})(X) -_{\mathbb{G}_n} ([p]_{\mathbb{G}_n} \circ \tilde{\varphi})(Y) \\
&= [p]_{\mathbb{G}_n}(\delta_{\tilde{\varphi}}) = 0.
\end{aligned}$$

As a consequence, $[p]_{\mathbb{G}_n} \circ \tilde{\varphi} \in \mathrm{End}(\mathbb{G}_n)$, and thus $p\varphi \in \mathrm{End}(\mathbb{G}_n)$. Since φ was arbitrary, we obtain the desired inclusion $p^n\,\mathrm{End}(\mathbb{G}_0) \subseteq \mathrm{End}(\mathbb{G}_n)$. $\square$

According to Corollary 1.4, the group $\mathrm{Aut}(G)$ is a profinite topological group. A basis of open neighborhoods of its identity is given by the subgroups $1 + p^n\,\mathrm{End}(G)$ with $n \geq 1$. If $\mathfrak{m}$ denotes the maximal ideal of the local ring R, the W-algebra R is a topological ring for the $\mathfrak{m}$-adic topology. We are now ready to prove the following result, a particular case of which was treated in [13], Proposition 3.1. The argument is borrowed from the proof of [9], Lemma 19.3. Let us put

$$\Gamma := \Gamma_0 := \mathrm{Aut}(G) \quad \text{and} \quad \Gamma_n := 1 + p^n\,\mathrm{End}(G) \quad \text{for} \quad n \geq 1.$$

Theorem 2.4. *The action of* Γ *on* $R = R_G^{\mathrm{def}}$ *is continuous in the sense that the map* $((\gamma, f) \mapsto \gamma(f)) : \Gamma \times R \to R$ *is a continuous map of topological spaces. Here* $\Gamma \times R$ *carries the product topology. If* n *is a non-negative integer then the induced action of* Γ_n *on* $R/\mathfrak{m}^{n+1}$ *is trivial.*

Proof. As in the proof of [13], Proposition 3.1, it suffices to prove the second statement. Let $\gamma \in \Gamma_n$ and consider the deformation $(\mathbb{G}_n, \rho_{\mathbb{G}} \circ \gamma)$ of G to $R_n = R/\mathfrak{m}^{n+1}$. Denote by $\mathrm{pr}_n : R \to R_n$ the natural projection and let γ_n denote the unique ring homomorphism $\gamma_n : R \to R_n$ for which there exists an isomorphism of deformations $[\gamma_n] : (\gamma_n)_*(\mathbb{G}, \rho_{\mathbb{G}}) \simeq (\mathbb{G}_n, \rho_{\mathbb{G}} \circ \gamma)$ (cf. Theorem 2.1). Note that also the ring homomorphism $\mathrm{pr}_n \circ \gamma : R \to R_n$ admits an isomorphism of deformations $(\mathrm{pr}_n \circ \gamma)_*(\mathbb{G}, \rho_{\mathbb{G}}) \simeq (\mathbb{G}_n, \rho_{\mathbb{G}} \circ \gamma)$, namely the reduction of $[\gamma]$ modulo $\mathfrak{m}^{n+1}$. By uniqueness, we must have $\gamma_n = \mathrm{pr}_n \circ \gamma$ and $[\gamma_n] = [\gamma]$ mod $\mathfrak{m}^{n+1}$.

Since the map $(\sigma \mapsto \rho_{\mathbb{G}} \circ \sigma \circ \rho_{\mathbb{G}}^{-1})$ is a ring isomorphism $\mathrm{End}(G) \to \mathrm{End}(\mathbb{G}_0)$, Proposition 2.3 shows that $\rho_{\mathbb{G}} \circ \gamma \circ \rho_{\mathbb{G}}^{-1} \in \mathrm{Aut}(\mathbb{G}_n)$ and therefore defines an isomorphism of deformations $(\mathrm{pr}_n)_*(\mathbb{G}, \rho_{\mathbb{G}}) = (\mathbb{G}_n, \rho_{\mathbb{G}}) \simeq (\mathbb{G}_n, \rho_{\mathbb{G}} \circ \gamma)$. By uniqueness again, we must have $\gamma_n = \mathrm{pr}_n \circ \gamma = \mathrm{pr}_n$. This implies that γ acts trivially on R_n and that $[\gamma] \mod \mathfrak{m}^{n+1} = \rho_{\mathbb{G}} \circ \gamma \circ \rho_{\mathbb{G}}^{-1}$. □

If H is a profinite topological group then we denote by

$$\Lambda(H) := W[\![H]\!] := \varprojlim_{n \geq 1,\, N \trianglelefteq_o H} (W/p^n W)[H/N]$$

the *Iwasawa algebra* (or *completed group ring*) of H over W. The above projective limit runs over all positive integers n and over all open normal subgroups N of H. If n and n' are positive integers with $n' \leq n$, and if N and N' are two open normal subgroups of H with $N \subseteq N'$, then the transition map $(W/p^n W)[H/N] \to (W/p^{n'} W)[H/N']$ is the natural homomorphism of group rings induced by the surjective homomorphism $H/N \to H/N'$ of groups and the surjective ring homomorphism $W/p^n W \to W/p^{n'} W$. Endowing each ring $(W/p^n W)[H/N]$ with the discrete topology, $\Lambda(H)$ is a topological ring for the projective limit topology. It is a *pseudocompact ring* in the terminology of [4], page 442, because each of the rings $(W/p^n W)[H/N]$ is Artinian. Recall that a complete Hausdorff topological $\Lambda(H)$-module M is called *pseudocompact*, if it admits a basis $(M_i)_{i \in I}$ of open neighborhoods of zero such that each M_i is a $\Lambda(H)$- submodule of M for which the $\Lambda(H)$-module M/M_i has finite length. For brevity, we will set

$$\Lambda := \Lambda(\mathrm{Aut}(G)).$$

Corollary 2.5. *The action of* $\mathrm{Aut}(G)$ *on* $R = R_G^{\mathrm{def}}$ *extends to an action of* Λ *and gives* R *the structure of a pseudocompact* Λ*-module.*

Proof. Since R is $\mathfrak{m}$-adically separated and complete, we may consider the natural isomorphism

$$R \simeq \varprojlim_{n \geq 0} R/\mathfrak{m}^{n+1}.$$

According to Theorem 2.4, the action of the group ring $W[\mathrm{Aut}(G)]$ on $R/\mathfrak{m}^{n+1}$ factors through $(W/p^{n+1}W)[\mathrm{Aut}(G)/(1+p^n \,\mathrm{End}(G))]$ where $1+p^n \,\mathrm{End}(G)$ is an open normal subgroup of $\mathrm{Aut}(G)$. Thus, $R/\mathfrak{m}^{n+1}$ can be viewed as a Λ-module via the natural ring homomorphism $\Lambda \to (W/p^{n+1}W)[\mathrm{Aut}(G)/(1+p^n \mathrm{End}(G))]$. The transition maps in the above projective limit are then Λ-equivariant. This proves the first assertion.

As for the second assertion, the ideals $\mathfrak{m}^{n+1}$ of R are open and Λ-stable, being the kernels of the Λ-equivariant projections $R \to R/\mathfrak{m}^{n+1}$. They form a basis of open neighborhoods of zero of R, and the quotients $R/\mathfrak{m}^{n+1}$ are even of finite length over $W \subseteq \Lambda$. □

Let $\mathrm{Lie}(\mathbb{G})$ denote the Lie algebra of the universal deformation $\mathbb{G}$ of G. This is a free module of rank $d = \dim(G)$ over R. Given $\gamma \in \mathrm{Aut}(G)$, we extend the ring automorphism $\gamma : R \to R$ to an automorphism $\gamma : R[\![X]\!] \to R[\![X]\!]$ by setting $\gamma(X_i) = X_i$ for all $1 \le i \le d$. It induces a homomorphism $\gamma : \mathrm{Lie}(\mathbb{G}) \to \mathrm{Lie}(\gamma_*\mathbb{G})$ of additive groups. We define $\tilde{\gamma} : \mathrm{Lie}(\mathbb{G}) \to \mathrm{Lie}(\mathbb{G})$ as the composite of the two additive maps

$$\mathrm{Lie}(\mathbb{G}) \xrightarrow{\gamma} \mathrm{Lie}(\gamma_*\mathbb{G}) \xrightarrow{\mathrm{Lie}([\gamma])} \mathrm{Lie}(\mathbb{G}),$$

with $[\gamma] : \gamma_*\mathbb{G} \to \mathbb{G}$ as above. Given a second element $\gamma' \in \mathrm{Aut}(G)$, we define $\gamma' : \mathrm{Lie}(\gamma_*\mathbb{G}) \to \mathrm{Lie}(\gamma'_*(\gamma_*\mathbb{G}))$ as before. Further, $\gamma'_*[\gamma] : \gamma'_*(\gamma_*\mathbb{G}) \to \gamma'_*\mathbb{G}$ denotes the homomorphism obtained by applying $\gamma' \in \mathrm{Aut}(R)$ to the coefficients of $[\gamma] \in R[\![X]\!]^d$. One readily checks that the diagram

$$\begin{tikzcd}
\mathrm{Lie}(\gamma_*\mathbb{G}) \arrow[r, "\mathrm{Lie}([\gamma])"] \arrow[d, "\gamma'"] & \mathrm{Lie}(\mathbb{G}) \arrow[d, "\gamma'"'] \\
\mathrm{Lie}(\gamma'_*(\gamma_*\mathbb{G})) \arrow[r, "\mathrm{Lie}(\gamma'_*[\gamma])"'] & \mathrm{Lie}(\gamma'_*\mathbb{G})
\end{tikzcd}$$

is commutative. Further, the uniqueness assertion in Theorem 2.1 implies that $[\gamma'\gamma] = [\gamma'] \circ \gamma'_*[\gamma]$. Therefore,

$$\begin{aligned}
(\gamma'\gamma)^{\sim} &= \mathrm{Lie}([\gamma'\gamma]) \circ (\gamma'\gamma) = \mathrm{Lie}([\gamma']) \circ \mathrm{Lie}(\gamma'_*[\gamma]) \circ \gamma' \circ \gamma \\
&= \mathrm{Lie}([\gamma']) \circ (\gamma' \circ \mathrm{Lie}([\gamma]) \circ (\gamma')^{-1}) \circ \gamma' \circ \gamma = \tilde{\gamma}' \circ \tilde{\gamma}.
\end{aligned}$$

As a consequence, we obtain an action of $\mathrm{Aut}(G)$ on the additive group $\mathrm{Lie}(\mathbb{G})$ which is semilinear for the action on R in the sense that

$$\tilde{\gamma}(f \cdot \delta) = \gamma(f) \cdot \tilde{\gamma}(\delta) \quad \text{for all} \quad f \in R, \delta \in \mathrm{Lie}(\mathbb{G}).$$

To ease notation, we will again write $\gamma(\delta)$ for $\tilde{\gamma}(\delta)$.

Given a positive integer m we denote by $\mathrm{Lie}(\mathbb{G})^{\otimes m}$ the m-fold tensor product of $\mathrm{Lie}(\mathbb{G})$ over R with itself. This is a free R-module of rank d^m with a semilinear action of $\mathrm{Aut}(G)$ defined by

$$\gamma(\delta_1 \otimes \cdots \otimes \delta_m) := \gamma(\delta_1) \otimes \cdots \otimes \gamma(\delta_m).$$

We also set $\mathrm{Lie}(\mathbb{G})^{\otimes 0} := R$ and $\mathrm{Lie}(\mathbb{G})^{\otimes m} := \mathrm{Hom}_R(\mathrm{Lie}(\mathbb{G})^{\otimes(-m)}, R)$ if m is a negative integer. In the latter case $\mathrm{Lie}(\mathbb{G})^{\otimes m}$ is a free R-module of rank d^{-m} with a semilinear action of $\mathrm{Aut}(G)$ defined through

$$\gamma(\varphi)(\delta_1 \otimes \cdots \otimes \delta_{-m}) := \gamma(\varphi(\gamma^{-1}(\delta_1) \otimes \cdots \otimes \gamma^{-1}(\delta_{-m}))).$$

For any integer m we endow the R-module $\mathrm{Lie}(\mathbb{G})^{\otimes m}$ with the $\mathfrak{m}$-adic topology for which it is Hausdorff and complete. By the semilinearity of the $\mathrm{Aut}(G)$-action, the R-submodules $\mathfrak{m}^n \mathrm{Lie}(\mathbb{G})^{\otimes m}$ are $\mathrm{Aut}(G)$-stable for any non-negative integer n.

As an easy consequence of Proposition 2.3 and Theorem 2.4, we obtain the following result.

Theorem 2.6. *Let m and n be integers with $n \geq 0$. The action of $\mathrm{Aut}(G)$ on $\mathrm{Lie}(\mathbb{G})^{\otimes m}$ is continuous in the sense that the structure map $\mathrm{Aut}(G) \times \mathrm{Lie}(\mathbb{G})^{\otimes m} \to \mathrm{Lie}(\mathbb{G})^{\otimes m}$ of the action is continuous. Here the left hand side carries the product topology. The induced action of $1 + p^{2n+1}\mathrm{End}(G)$ on the quotient $\mathrm{Lie}(\mathbb{G})^{\otimes m}/\mathfrak{m}^{n+1}\mathrm{Lie}(\mathbb{G})^{\otimes m}$ is trivial. In particular, the action of $\mathrm{Aut}(G)$ on $\mathrm{Lie}(\mathbb{G})^{\otimes m}$ extends to an action of Λ and gives $\mathrm{Lie}(\mathbb{G})^{\otimes m}$ the structure of a pseudocompact Λ-module.*

Proof. As in the proof of Theorem 2.4 and Corollary 2.5, it suffices to show that the action of $1 + p^{2n+1}\mathrm{End}(G)$ on $\mathrm{Lie}(\mathbb{G})^{\otimes m}/\mathfrak{m}^{n+1}\mathrm{Lie}(\mathbb{G})^{\otimes m}$ is trivial. By definition of the action and Theorem 2.4 we may assume $m = 1$. Setting $\mathbb{G}_n = \mathbb{G} \mod \mathfrak{m}^{n+1}$, as before, we have $\mathrm{Lie}(\mathbb{G})/\mathfrak{m}^{n+1}\mathrm{Lie}(\mathbb{G}) = \mathrm{Lie}(\mathbb{G}_n)$. Since $2n+1 \geq n$, Theorem 2.4 and its proof show that the map $\gamma \mod \mathfrak{m}^{n+1} : \mathrm{Lie}(\mathbb{G}_n) \to \mathrm{Lie}(\mathbb{G}_n)$ is given by $\mathrm{Lie}(\rho_{\mathbb{G}} \circ \gamma \circ \rho_{\mathbb{G}}^{-1})$ where $\rho_{\mathbb{G}} \circ \gamma \circ \rho_{\mathbb{G}}^{-1}$ is contained in $1 + p^{2n+1}\mathrm{End}(\mathbb{G}_0) \subseteq 1 + p^{n+1}\mathrm{End}(\mathbb{G}_n)$ (cf. Proposition 2.3). Therefore, it suffices to show that the natural action of $1 + p^{n+1}\mathrm{End}(\mathbb{G}_n) \subset \mathrm{End}(\mathbb{G}_n)$ on $\mathrm{Lie}(\mathbb{G}_n)$ is trivial. However, if $\varphi \in \mathrm{End}(\mathbb{G}_n)$ and if $\delta \in \mathrm{Lie}(\mathbb{G}_n)$, then

$$\mathrm{Lie}(1 + p^{n+1}\varphi)(\delta) = \delta + p^{n+1}\mathrm{Lie}(\varphi)(\delta) = \delta,$$

because $p^{n+1} \in \mathfrak{m}^{n+1}$. □

Before we continue, let us point out an important variant of the deformation problem considered above. It concerns the moduli problems considered by Rapoport and Zink (cf. [19]).

Let G be a fixed p-divisible group over the algebraically closed field k of characteristic p, i.e. an *fppf*-group scheme over $\mathrm{Spec}(k)$ for which multiplication by p is an epimorphism. Denoting by Nil_p the category of W-schemes on which p is locally nilpotent, let $\mathcal{M}_G : Nil_p \to Sets$ denote the set valued functor which associates to an object S of Nil_p the set of isomorphism classes of pairs $(G', \rho_{G'})$, where G' is a p-divisible group over S and $\rho_{G'} : G_{\overline{S}} \to G'_{\overline{S}}$ is a *quasi-isogeny* (cf. [19], Definition 2.8). Here $\overline{S}$ denotes the closed subscheme of S defined by the sheaf of ideals $p\mathcal{O}_S$. According to [19], Theorem 2.16, the functor $\mathcal{M}_G$ is represented by a formal scheme which is locally formally of finite type over $\mathrm{Spf}(W)$. If G is a p-divisible one dimensional commutative formal group law as in section 1, then $\mathcal{M}_G$ is the disjoint union of open subschemes $\mathcal{M}_G^n$, $n \in \mathbb{Z}$, which are non-canonically isomorphic to $\mathrm{Spf}(R_G^{\mathrm{def}})$ (cf. [19], Proposition 3.79). The reason is that any quasi-isogeny of height zero between one dimensional p-divisible formal group laws over k is an isomorphism.

One can generalize the moduli problem even further by considering deformations of p-divisible groups with additional structures such as polarizations or actions by maximal orders in finite dimensional semisimple $\mathbb{Q}_p$-algebras (cf. [19], Definition 3.21). The corresponding deformation functors are again representable, as was proven by Rapoport and Zink (cf. [19], Theorem 3.25). An important example was studied by Drinfeld (cf. [19], 3.58). The generic fiber of the representing formal scheme is known as *Drinfeld's upper half space over K*. Instead of continuous representations of $\mathrm{Aut}(G)$ as in Theorem 2.4, it gives rise to an important class of *p-adic locally analytic representations* in the sense of Schneider and Teitelbaum. This particular class of representations was studied extensively by Morita, Orlik, Schneider and Teitelbaum (cf. [18] and [24]). It found arithmetic applications to the de Rham cohomology of varieties which are p-adically uniformized by Drinfeld's upper half space (cf. [12]). In the next section we shall see that the deformation spaces we consider here give rise to locally analytic representations, as well.

3 Rigidification and local analyticity

We keep the notation of the previous section and denote by k an algebraically closed field of characteristic p and by G a fixed commutative p-divisible formal group over k. Let h and d denote the height and the dimension of G, respectively. We denote by W the ring of Witt vectors of k and by K the quotient field of W. We let $R = R_G^{\mathrm{def}}$ denote the universal deformation ring of G (cf. Theorem 2.1).

According to Theorem 2.1, the rigidification $\mathrm{Spf}(R)^{\mathrm{rig}}$ of the formal scheme $\mathrm{Spf}(R)$ in the sense of Berthelot (cf. [10], section 7) is isomorphic to the $(h-d)d$-dimensional rigid analytic open unit polydisc $\mathring{\mathbb{B}}_K^{(h-d)d}$ over K. We let

$$R^{\mathrm{rig}} := \mathcal{O}(\mathrm{Spf}(R)^{\mathrm{rig}})$$

denote the ring of global rigid analytic functions on $\mathrm{Spf}(R)^{\mathrm{rig}}$. Any isomorphism $R \simeq W[\![u]\!]$ of local W-algebras extends to an isomorphism

$$R^{\mathrm{rig}} \simeq \{ \sum_{\alpha \in \mathbb{N}^{(h-d)d}} c_\alpha u^\alpha \mid c_\alpha \in K \text{ and } \lim_{|\alpha| \to \infty} |c_\alpha| r^{|\alpha|} = 0 \text{ for all } 0 < r < 1 \}$$

of K-algebras, where $|\cdot|$ denotes the p-adic absolute value on K. This allows us to view R^{rig} as a topological K-Fréchet algebra whose topology is defined by the family of norms $||\cdot||_\ell$, given by

$$||\sum_\alpha c_\alpha u^\alpha||_\ell := \sup_\alpha \{|c_\alpha| p^{-|\alpha|/\ell}\}$$

for any positive integer ℓ. Letting R_ℓ^{rig} denote the completion of R^{rig} with respect to the norm $||\cdot||_\ell$, the K-algebra R_ℓ^{rig} can be identified with the ring of rigid analytic functions on the affinoid subdomain

$$\mathbb{B}_\ell^{(h-d)d} := \{x \in \mathrm{Spf}(R)^{\mathrm{rig}} \mid |u_i(x)| \le p^{-1/\ell} \text{ for all } 1 \le i \le (h-d)d\}$$

of $\mathrm{Spf}(R)^{\mathrm{rig}}$. Further, $R^{\mathrm{rig}} \simeq \varprojlim_\ell R_\ell^{\mathrm{rig}}$ is the topological projective limit of the K-Banach algebras R_ℓ^{rig}. In fact, by a cofinality argument and [1], 6.1.3 Theorem

1, R^{rig} is the topological projective limit of the system of affinoid K-algebras corresponding to any nested admissible open affinoid covering of $\mathrm{Spf}(R)^{\mathrm{rig}}$.

By functoriality, the automorphism group $\Gamma = \mathrm{Aut}(G)$ of G acts on $\mathrm{Spf}(R)^{\mathrm{rig}}$ by automorphisms of rigid analytic K-varieties. This gives rise to an action of Γ on R^{rig} by K-linear ring automorphisms. By the above cofinality argument, any of these automorphisms is continuous. The goal of this section is to show that the induced action on the strong topological K-linear dual of R^{rig} is *locally analytic* in the sense of Schneider and Teitelbaum (cf. [22], page 451).

Fix an algebraic closure K^{alg} of K. According to [10], Lemma 7.19, the maximal ideals of the ring $R_K := R \otimes_W K$ are in bijection with the points of $\mathrm{Spf}(R)^{\mathrm{rig}}$. It follows from [1], 7.1.1 Proposition 1, that the latter are in bijection with the $\mathrm{Gal}(K^{\mathrm{alg}}|K)$-orbits of

$$\mathring{\mathbb{B}}_K^{(h-d)d}(K^{\mathrm{alg}}) := \{x \in (K^{\mathrm{alg}})^{(h-d)d} \mid |x_i| < 1 \text{ for all } 1 \le i \le (h-d)d\}.$$

A point x representing one of these orbits corresponds to the kernel of the surjective K-linear ring homomorphism $R_K \to K(x) := K(x_1, \ldots, x_{(h-d)d}) \subseteq K^{\mathrm{alg}}$, sending $f(u)$ to $f(x)$.

The following result constitutes the technical heart of this section. It is a straightforward generalization of [9], Lemma 19.3.

Proposition 3.1. *Let n and ℓ be integers with $n \ge 0$ and $\ell \ge 1$. If $\gamma \in \Gamma_n$ and if $f \in R^{\mathrm{rig}}$ then $||\gamma(f) - f||_\ell \le p^{-n/\ell}||f||_\ell$.*

Proof. First assume $f = u_i$ for some $1 \le i \le (h-d)d$. If

$$\mathbb{B}_\ell^{(h-d)d}(K^{\mathrm{alg}}) := \{x \in (K^{\mathrm{alg}})^{(h-d)d} \mid |x_i| \le p^{-1/\ell} \text{ for all } 1 \le i \le (h-d)d\},$$

then $||g||_\ell = \sup\{|g(x)| \mid x \in \mathbb{B}_\ell^{(h-d)d}(K^{\mathrm{alg}})\}$ for any $g \in R^{\mathrm{rig}}$. Thus, we need to see that if $x \in \mathbb{B}_\ell^{(h-d)d}(K^{\mathrm{alg}})$ and if $y := x \cdot \gamma = \gamma(u)(x)$, then $|x_i - y_i| \le p^{-(n+1)/\ell}$.

Denoting by W^{alg} the valuation ring of K^{alg}, consider the commutative diagram

$$\begin{tikzcd} R \arrow[rr, "\gamma"] \arrow[dr] & & R \arrow[dl] \\ & W^{\mathrm{alg}} & \end{tikzcd}$$

of homomorphisms of W-algebras, in which the left and right oblique arrow is given by evaluation at y and x, respectively. Choosing $z \in W^{\mathrm{alg}}$ with $|z| = p^{-1/\ell}$, we have $x_j \in zW^{\mathrm{alg}}$ for any j. Further, $p \in zW^{\mathrm{alg}}$ because $\ell \ge 1$. As a consequence, the right oblique arrow maps $\mathfrak{m}_R$ to zW^{alg}. Note that $\gamma(u_j) \in \mathfrak{m}_R$, so that we obtain $y_j = u_j(x \cdot \gamma) = \gamma(u_j)(x) \in zW^{\mathrm{alg}}$, as well. Therefore, also the left oblique arrow maps $\mathfrak{m}_R$ to zW^{alg}. Now consider the induced diagram

$$\begin{tikzcd} R/\mathfrak{m}_R^{n+1} \arrow[rr, "\gamma"] \arrow[dr] & & R/\mathfrak{m}_R^{n+1} \arrow[dl] \\ & W^{\mathrm{alg}}/(z^{n+1}). & \end{tikzcd}$$

According to Theorem 2.4, the upper horizontal arrow is the identity. It follows that $x_i - y_i \in z^{n+1}W^{\mathrm{alg}}$, i.e. $|x_i - y_i| \leq p^{-(n+1)/\ell}$, as required. In particular, γ stabilizes $\mathbb{B}_\ell^{(h-d)d}(K^{\mathrm{alg}})$ and therefore is an isometry for the norm $||\cdot||_\ell$ on R^{rig}.

To prove the proposition, the continuity of γ allows us to assume $f = u^\alpha$ for some $\alpha \in \mathbb{N}^{(h-d)d}$. The assertion is trivial for $|\alpha| = 0$. If $|\alpha| > 0$ choose an index i with $\alpha_i > 0$. Define β through $\beta_j := \alpha_j$ if $j \neq i$ and $\beta_i := \alpha_i - 1$. If $x \in \mathbb{B}_\ell^{(h-d)d}(K^{\mathrm{alg}})$ and if $y = x \cdot \gamma$, then

$$\begin{aligned} |\gamma(u^\alpha)(x) - u^\alpha(x)| &= |y^\alpha - x^\alpha| = |y_i y^\beta - x_i x^\beta| \\ &\leq \max\{|y_i||y^\beta - x^\beta|, |y_i - x_i||x^\beta|\}. \end{aligned}$$

Here $|y_i||y^\beta - x^\beta| \leq p^{-1/\ell}||\gamma(u^\beta) - u^\beta||_\ell \leq p^{-(n+1)/\ell}||u^\beta||_\ell = p^{-n/\ell}||u^\alpha||_\ell$ by the induction hypothesis. Further, $|y_i - x_i||x^\beta| \leq p^{-(n+1)/\ell}p^{-|\beta|/\ell} = p^{-n/\ell}||u^\alpha||_\ell$, as seen above. Thus, we obtain $|\gamma(u^\alpha)(x) - u^\alpha(x)| \leq p^{-n/\ell}||u^\alpha||_\ell$ for all $x \in \mathbb{B}_\ell^{(h-d)d}(K^{\mathrm{alg}})$. This proves the proposition. $\square$

A topological group is a Lie group over $\mathbb{Q}_p$ if and only if it contains an open subgroup which is a *uniform pro-p group* (cf. [8], Definition 4.1 and Theorem 8.32). For the compact p-adic Lie group $\Gamma = \mathrm{Aut}(G)$ we have the following more precise result. We let

$$\varepsilon := 1 \quad \text{if} \quad p > 2 \quad \text{and} \quad \varepsilon := 2 \quad \text{if} \quad p = 2.$$

Lemma 3.2. *For any non-negative integer n we have $\Gamma_\varepsilon^{p^n} = \Gamma_{\varepsilon+n}$. The open subgroup $\Gamma_{\varepsilon+n}$ of Γ is a uniform pro-p group.*

Proof. As for the first assertion, the proofs of [8], Lemma 5.1 and Theorem 5.2, can be copied word by word on replacing $\mathrm{M}_d(\mathbb{Z}_p)$ by $\mathrm{End}(G)$ and $\mathrm{GL}_d(\mathbb{Z}_p)$ by $\mathrm{Aut}(G)$. Further, $\Gamma_{\varepsilon+n}$ is a powerful pro-p group by [8], Theorem 3.6 (i) and the remark after Definition 3.1. That it is uniform follows from [8], Theorem 3.6 (ii), and the first assertion. $\square$

Fix an integer $n \geq \varepsilon$. By Lemma 3.2 and [8], Theorem 3.6, the group Γ_n/Γ_{n+1} is a finite dimensional $\mathbb{F}_p$-vector space. Choosing elements $\gamma_1, \ldots, \gamma_r \in \Gamma_n$ whose images modulo Γ_{n+1} form an $\mathbb{F}_p$-basis of Γ_n/Γ_{n+1}, [8], Theorem 4.9, shows that $(\gamma_1, \ldots, \gamma_r)$ is an *ordered basis* of Γ_n in the sense that the map $\mathbb{Z}_p^r \to \Gamma_n$, sending λ to $\gamma_1^{\lambda_1} \cdots \gamma_r^{\lambda_r}$, is a homeomorphism.

Set $b_i := \gamma_i - 1 \in \Lambda(\Gamma_n)$ and $b^\alpha := b_1^{\alpha_1} \cdots b_r^{\alpha_r}$ for any $\alpha \in \mathbb{N}^r$. By [8], Theorem 7.20, any element $\delta \in \Lambda(\Gamma_n)$ admits a unique expansion of the form

$$\lambda = \sum_{\alpha \in \mathbb{N}^r} d_\alpha b^\alpha \text{ with } d_\alpha \in W \text{ for all } \alpha \in \mathbb{N}^r.$$

For any $\ell \geq 1$ this allows us to define the K-norm $||\cdot||_\ell$ on the algebra $\Lambda(\Gamma_n)_K := \Lambda(\Gamma_n) \otimes_W K$ through

$$||\sum_\alpha d_\alpha b^\alpha||_\ell := \sup_\alpha\{|d_\alpha| p^{-\varepsilon|\alpha|/\ell}\}.$$

Remark 3.3. A more accurate notation would be the symbol $||\cdot||_\ell^{(n)}$ for the above norm on $\Lambda(\Gamma_n)_K$. It does generally not coincide with the restriction of $||\cdot||_\ell^{(m)}$ to $\Lambda(\Gamma_n)_K \subseteq \Lambda(\Gamma_m)_K$ if $n \geq m$. However, there is an explicit rescaling relation between the families of norms $(||\cdot||_\ell^{(n)})_\ell$ and $(||\cdot||_\ell^{(m)})_\ell$ on $\Lambda(\Gamma_n)_K$ (cf. [20], Proposition 6.2). Since we will never work with two different groups Γ_n and Γ_m at once, we decided to ease notation and use the somewhat ambiguous symbol $||\cdot||_\ell$.

By [20], Proposition 2.1 and [23], Proposition 4.2, the norm $||\cdot||_\ell$ on $\Lambda(\Gamma_n)_K$ is submultiplicative whenever $\ell \geq 1$. As a consequence, the completion

$$\Lambda(\Gamma_n)_{K,\ell} = \{\sum_\alpha d_\alpha b^\alpha \mid d_\alpha \in K,\ \lim_{|\alpha|\to\infty} |d_\alpha| p^{-\varepsilon|\alpha|/\ell} = 0\}$$

of $\Lambda(\Gamma_n)_K$ with respect to $||\cdot||_\ell$ is a K-Banach algebra. The natural inclusions $\Lambda(\Gamma_n)_{K,\ell+1} \to \Lambda(\Gamma_n)_{K,\ell}$ endow the projective limit

$$D(\Gamma_n) := \varprojlim_\ell \Lambda(\Gamma_n)_{K,\ell}$$

with the structure of a K-Fréchet algebra. As is explained in [23], section 4, a theorem of Amice allows us to identify it with the algebra of *K-valued locally analytic distributions on Γ_n*. Similarly, we denote by $D(\Gamma)$ the algebra of K-valued locally analytic distributions on Γ (cf. [22], section 2).

Theorem 3.4. *For any integer $\ell \geq 1$ the action of Γ_ε on R^{rig} extends to R_ℓ^{rig} and makes R_ℓ^{rig} a topological Banach module over the K-Banach algebra $\Lambda(\Gamma_\varepsilon)_{K,\ell}$. The action of Γ on R^{rig} extends to a jointly continuous action of the K-Fréchet algebra $D(\Gamma)$. The action of Γ on the strong continuous K-linear dual $(R^{\mathrm{rig}})'_b$ of R^{rig} is locally analytic in the sense of [22], page 451.*

Proof. First, we prove by induction on $|\alpha|$ that $||b^\alpha f||_\ell \leq ||b^\alpha||_\ell ||f||_\ell$ for any $f \in R^{\mathrm{rig}}$. This is clear if $|\alpha| = 0$. Otherwise, let i be the minimal index with $\alpha_i > 0$ and define β through $\beta_j = \alpha_j$ if $j \neq i$ and $\beta_i := \alpha_i - 1$. In this case, Proposition 3.1 and the induction hypothesis imply

$$\begin{aligned} ||b^\alpha f||_\ell &= ||(\gamma_i - 1)b^\beta f||_\ell \leq p^{-\varepsilon/\ell}||b^\beta f||_\ell \\ &\leq p^{-\varepsilon/\ell}p^{-\varepsilon|\beta|/\ell}||f||_\ell = ||b^\alpha||_\ell ||f||_\ell, \end{aligned}$$

as required. This immediately gives $||\lambda \cdot f||_\ell \leq ||\lambda||_\ell ||f||_\ell$ for all $\lambda \in \Lambda(\Gamma_\varepsilon)_K$ and $f \in R_K$. Thus, the multiplication map $\Lambda(\Gamma_\varepsilon)_K \times R_K \to R_K$ is continuous, if $\Lambda(\Gamma_\varepsilon)_K$ and R_K are endowed with the respective $||\cdot||_\ell$-topologies, and if the left hand side carries the product topology. Since R_K is dense in R_ℓ^{rig}, we obtain a map $\Lambda(\Gamma_\varepsilon)_{K,\ell} \times R_\ell^{\mathrm{rig}} \to R_\ell^{\mathrm{rig}}$ by passing to completions. By continuity, it gives R_ℓ^{rig} the structure of a topological Banach module over $\Lambda(\Gamma_\varepsilon)_{K,\ell}$.

Passing to the projective limit, we obtain a continuous map $D(\Gamma_\varepsilon) \times R^{\mathrm{rig}} \to R^{\mathrm{rig}}$, giving R^{rig} the structure of a jointly continuous module over $D(\Gamma_\varepsilon)$. Since $D(\Gamma)$ is topologically isomorphic to the locally convex direct sum $\oplus_{\gamma\Gamma_\varepsilon \in \Gamma/\Gamma_\varepsilon} \gamma D(\Gamma_\varepsilon)$ (cf. [22], page 447 bottom), R^{rig} is a jointly continuous module over $D(\Gamma)$.

It follows from [21], Proposition 19.9 and the arguments proving the claim on page 98, that the K-Fréchet space R^{rig} is nuclear. Therefore, [22], Corollary 3.4, implies that the locally convex K-vector space $(R^{\mathrm{rig}})'_b$ is of compact type and that the action of Γ obtained by dualizing is locally analytic. $\square$

Using Theorem 2.6, the preceding result can be generalized as follows. Fixing an integer m, the free R-module $\mathrm{Lie}(\mathbb{G})^{\otimes m}$ gives rise to a locally free coherent sheaf on $\mathrm{Spf}(R)$. For any positive integer ℓ we denote by $(\mathrm{Lie}(\mathbb{G})^{\otimes m})_\ell^{\mathrm{rig}}$ the sections of its rigidification over the affinoid subdomain $\mathbb{B}_\ell^{(h-d)d}$ of $\mathrm{Spf}(R)^{\mathrm{rig}}$. This is a free R_ℓ^{rig}-module for which the natural R_ℓ^{rig}-linear map

$$R_\ell^{\mathrm{rig}} \otimes_R \mathrm{Lie}(\mathbb{G})^{\otimes m} \longrightarrow (\mathrm{Lie}(\mathbb{G})^{\otimes m})_\ell^{\mathrm{rig}}$$

is bijective (cf. [10], 7.1.11). We denote by $(\mathrm{Lie}(\mathbb{G})^{\otimes m})^{\mathrm{rig}}$ the space of global sections of the rigidification of $\mathrm{Lie}(\mathbb{G})^{\otimes m}$ over $\mathrm{Spf}(R)^{\mathrm{rig}}$. This is a free R^{rig}-module for which the natural R^{rig}-linear maps

$$(1) \qquad R^{\mathrm{rig}} \otimes_R \mathrm{Lie}(\mathbb{G})^{\otimes m} \longrightarrow (\mathrm{Lie}(\mathbb{G})^{\otimes m})^{\mathrm{rig}} \longrightarrow \varprojlim_\ell (\mathrm{Lie}(\mathbb{G})^{\otimes m})_\ell^{\mathrm{rig}}$$

are bijective. Further, $(\mathrm{Lie}(\mathbb{G})^{\otimes m})^{\mathrm{rig}} \simeq (\mathrm{Lie}(\mathbb{G})^{\mathrm{rig}})^{\otimes m}$, where the latter tensor products and dualities are with respect to R^{rig}.

By functoriality, the group $\Gamma = \mathrm{Aut}(G)$ acts on $(\mathrm{Lie}(\mathbb{G})^{\otimes m})^{\mathrm{rig}}$ in such a way that the left map in (1) becomes Γ-equivariant for the diagonal action on the left. In particular, it is semilinear for the action of Γ on R^{rig}. We endow $(\mathrm{Lie}(\mathbb{G})^{\otimes m})^{\mathrm{rig}}$ and $(\mathrm{Lie}(\mathbb{G})^{\otimes m})_\ell^{\mathrm{rig}}$ with the natural topologies of finitely generated modules over R^{rig} and R_ℓ^{rig}, respectively. This makes them a nuclear K-Fréchet space and a K-Banach space, respectively. The right map in (1) is then a topological isomorphism for the projective limit topology on the right. With the same cofinality argument as for R^{rig} one can show that any element of Γ acts on $(\mathrm{Lie}(\mathbb{G})^{\otimes m})^{\mathrm{rig}}$ through a continuous K-linear automorphism.

Theorem 3.5. *Let m be an integer. For any integer $\ell \geq 1$ the action of $\Gamma_{2\varepsilon-1}$ on $(\mathrm{Lie}(\mathbb{G})^{\otimes m})^{\mathrm{rig}}$ extends to $(\mathrm{Lie}(\mathbb{G})^{\otimes m})_\ell^{\mathrm{rig}}$ and makes $(\mathrm{Lie}(\mathbb{G})^{\otimes m})_\ell^{\mathrm{rig}}$ a topological Banach module over the K-Banach algebra $\Lambda(\Gamma_{2\varepsilon-1})_{K,\ell}$. The action of Γ on $(\mathrm{Lie}(\mathbb{G})^{\otimes m})^{\mathrm{rig}}$ extends to a jointly continuous action of the K-Fréchet algebra $D(\Gamma)$. The action of Γ on the strong continuous K-linear dual $[(\mathrm{Lie}(\mathbb{G})^{\otimes m})^{\mathrm{rig}}]'_b$ of $(\mathrm{Lie}(\mathbb{G})^{\otimes m})^{\mathrm{rig}}$ is locally analytic.*

Proof. Set $M_\ell^m := (\mathrm{Lie}(\mathbb{G})^{\otimes m})_\ell^{\mathrm{rig}}$. Any R-basis $(\delta_1, \ldots, \delta_s)$ of $\mathrm{Lie}(\mathbb{G})^{\otimes m}$ can be viewed as an R_ℓ^{rig}-basis of M_ℓ^m. Writing $M_\ell^m = \oplus_{i=1}^s R_\ell^{\mathrm{rig}} \delta_i$, the topology of M_ℓ^m is defined by the norm

$$||\sum_{i=1}^s f_i \delta_i||_\ell = \sup_i \{||f_i||_\ell\} \quad \text{if} \quad f_1, \ldots, f_s \in R_\ell^{\mathrm{rig}}.$$

We choose an ordered basis $(\gamma_1, \ldots, \gamma_r)$ of $\Gamma_{2\varepsilon-1}$ and let $b_i := \gamma_i - 1$ be as before. By induction on $|\alpha|$ we will first prove the fundamental estimate $||b^\alpha \delta||_\ell \leq ||b^\alpha||_\ell ||\delta||_\ell$ for all $\alpha \in \mathbb{N}^r$ and $\delta \in (\mathrm{Lie}(\mathbb{G})^{\otimes m})^{\mathrm{rig}}$. As in the proof of Theorem 3.4 this is reduced to the case $|\alpha| = 1$, i.e. $b^\alpha = \gamma_i - 1$ for some

$1 \leq i \leq r$. Further, we may assume $\delta = f\delta_j$ for some $f \in R^{\mathrm{rig}}$ and $1 \leq j \leq s$.

There are elements $r_1, \ldots, r_s \in R$ such that $\gamma_i(\delta_j) = \sum_{\nu=1}^{s} r_\nu \delta_\nu$. According to Theorem 2.6 we have $(\gamma_i - 1)(\delta_j) \in \mathfrak{m}^\varepsilon \operatorname{Lie}(\mathbb{G})^{\otimes m}$, i.e. $r_j - 1 \in \mathfrak{m}^\varepsilon$ and $r_\nu \in \mathfrak{m}^\varepsilon$ for $\nu \neq j$. We claim that $||r||_\ell \leq p^{-c/\ell}$ for any integer $c \geq 0$ and any element $r \in \mathfrak{m}^c$. Indeed, this is clear for $c = 0$. For general c, the ideal $\mathfrak{m}^c$ of R is generated by all elements of the form $p^a u^\beta$ with $a \in \mathbb{N}$, $\beta \in \mathbb{N}^{(h-d)d}$ and $a + |\beta| = c$. Since $\ell \geq 1$ we have $|p^a| = p^{-a} \leq p^{-a/\ell}$, and the claim follows from the multiplicativity of the norm $|| \cdot ||_\ell$ on R. Now

$$
\begin{aligned}
||(\gamma_i - 1)(f\delta_j)||_\ell &\leq \max\{||(\gamma_i - 1)(f) \cdot \gamma_i(\delta_j)||_\ell, ||f \cdot (\gamma_i - 1)(\delta_j)||_\ell\} \\
&= \max\{||\sum_\nu (\gamma_i - 1)(f) r_\nu \delta_\nu||_\ell, ||f||_\ell ||\delta_j - \sum_\nu r_\nu \delta_\nu||_\ell\},
\end{aligned}
$$

where $||(\gamma_i - 1)(f) \cdot r_\nu||_\ell \leq ||(\gamma_i - 1)(f)||_\ell \leq p^{(2\varepsilon-1)/\ell}||f||_\ell$ by Proposition 3.1. Here $p^{(2\varepsilon-1)/\ell} \leq p^{-\varepsilon/\ell} = ||\gamma_i - 1||_\ell$. Moreover, $||r_j - 1||_\ell \leq p^{-\varepsilon/\ell}$ and $||r_\nu||_\ell \leq p^{-\varepsilon/\ell}$ if $\nu \neq j$ by the above claim. This finishes the proof of the fundamental estimate.

As an immediate consequence, we obtain that $||\lambda \cdot \delta||_\ell \leq ||\lambda||_\ell ||\delta||_\ell$ for any $\lambda \in \Lambda(\Gamma_{2\varepsilon-1})_K$ and any $\delta \in \operatorname{Lie}(\mathbb{G})^{\otimes m} \otimes_W K$. The proof proceeds now as in Theorem 3.4. □

According to [23], Theorem 4.10, the projective system $(\Lambda(\Gamma_{2\varepsilon-1})_{K,\ell})_\ell$ of K-Banach algebras endow their projective limit $D(\Gamma_{2\varepsilon-1})$ with the structure of a *K-Fréchet-Stein algebra*. In the terminology of [23], section 8, the family $((\operatorname{Lie}(\mathbb{G})^{\otimes m})_\ell^{\mathrm{rig}})_\ell$ is a *sheaf over* $(D(\Gamma_{2\varepsilon-1}), (|| \cdot ||_\ell)_\ell)$ with global sections $(\operatorname{Lie}(\mathbb{G})^{\otimes m})^{\mathrm{rig}}$ for any integer m. One of the main open questions in this setting is whether this sheaf is *coherent*, i.e. whether the $\Lambda(\Gamma_{2\varepsilon-1})_{K,\ell}$-modules $(\operatorname{Lie}(\mathbb{G})^{\otimes m})_\ell^{\mathrm{rig}}$ are finitely generated and whether the natural maps

$$
\Lambda(\Gamma_{2\varepsilon-1})_{K,\ell} \otimes_{\Lambda(\Gamma_{2\varepsilon-1})_{K,\ell+1}} (\operatorname{Lie}(\mathbb{G})^{\otimes m})_{\ell+1}^{\mathrm{rig}} \longrightarrow (\operatorname{Lie}(\mathbb{G})^{\otimes m})_\ell^{\mathrm{rig}}
$$

are always bijective. This would amount to the *admissibility* of the locally analytic Γ-representation $[(\operatorname{Lie}(\mathbb{G})^{\otimes m})^{\mathrm{rig}}]_b'$ in the sense of [23], section 6. Nothing in this direction is known. In the next section, however, we will have a closer look at the case $\dim(G) = 1$ and $\ell = 1$. We will see that in order to obtain finitely generated objects, one might be forced to introduce yet another type of Banach algebras.

4 Non-commutative divided power envelopes

In this final section we assume that our fixed p-divisible formal group G over the algebraically closed field k of characteristic p is of dimension one. If h denotes the height of G then the endomorphism ring of G is isomorphic to the maximal order $\mathfrak{o}_D$ of the central $\mathbb{Q}_p$-division algebra D of invariant $\frac{1}{h} + \mathbb{Z}$ (cf. [9], Proposition 13.10). In the following we will identify $\operatorname{End}(G)$ and $\mathfrak{o}_D$ (resp. $\operatorname{Aut}(G)$ and $\mathfrak{o}_D^*$). We will also exclude the trivial case $h = 1$. We continue to denote by $R = R_G^{\mathrm{def}}$ the universal deformation ring of G (cf. Theorem 2.1).

Consider the *period morphism* $\Phi : \mathrm{Spf}(R)^{\mathrm{rig}} \to \mathbb{P}_K^{h-1}$ of Gross and Hopkins, where $\mathbb{P}_K^{h-1}$ denotes the rigid analytic projective space of dimension $h-1$ over K (cf. [9], section 23). In projective coordinates Φ can be defined by $\Phi(x) = [\varphi_0(x) : \ldots : \varphi_{h-1}(x)]$ where $\varphi_0, \ldots, \varphi_{h-1} \in R^{\mathrm{rig}}$ are certain global rigid analytic functions on $\mathrm{Spf}(R)^{\mathrm{rig}}$ without any common zero. The power series expansions of the functions φ_i in suitable coordinates $u_1, \ldots, u_{h-1}$ can be written down explicitly by means of a closed formula of Yu (cf. [13], Proposition 1.5 and Remark 1.6). According to [9], Lemma 23.14, the function φ_0 does not have any zeroes on $\mathbb{B}_1^{h-1} \subset \mathrm{Spf}(R)^{\mathrm{rig}}$, hence is a unit in R_1^{rig}. We set

$$w_i := \frac{\varphi_i}{\varphi_0} \in R_1^{\mathrm{rig}} \quad \text{for} \quad 1 \le i \le h-1.$$

By [9], Lemma 23.14, any element $f \in R_1^{\mathrm{rig}}$ admits a unique expansion of the form $f = \sum_{\alpha \in \mathbb{N}^{h-1}} d_\alpha w^\alpha$ with $d_\alpha \in K$ and $\lim_{|\alpha| \to \infty} |d_\alpha| p^{-|\alpha|} = 0$. Further, Φ restricts to an isomorphism $\Phi : \mathbb{B}_1^{h-1} \to \Phi(\mathbb{B}_1^{h-1})$ (cf. [9], Corollary 23.15).

Denote by $\mathbb{Q}_{p^h}$ the unramified extension of degree h of $\mathbb{Q}_p$ and by $\mathbb{Z}_{p^h}$ its valuation ring. It was shown by Devinatz, Gross and Hopkins, that there exists an explicit closed embedding $\mathfrak{o}_D^* \hookrightarrow \mathrm{GL}_h(\mathbb{Q}_{p^h})$ of Lie groups over $\mathbb{Q}_p$ such that Φ is $\mathfrak{o}_D^*$-equivariant (cf. [13], Proposition 1.3 and Remark 1.4). Here $\mathfrak{o}_D^*$ acts on $\mathrm{Spf}(R)^{\mathrm{rig}}$ through the identification $\mathfrak{o}_D^* \simeq \mathrm{Aut}(G)$, and it acts by fractional linear transformations on $\mathbb{P}_K^{h-1}$ via the embedding $\mathfrak{o}_D^* \hookrightarrow \mathrm{GL}_h(\mathbb{Q}_{p^h})$.

The morphism Φ is constructed in such a way that $\Phi^* \mathcal{O}_{\mathbb{P}_K^{h-1}}(1) = \mathrm{Lie}(\mathbb{G})^{\mathrm{rig}}$. It follows from general properties of the inverse image functor that $\Phi^* \mathcal{O}_{\mathbb{P}_K^{h-1}}(m) = (\mathrm{Lie}(\mathbb{G})^{\otimes m})^{\mathrm{rig}}$ for any integer m. Restricting to $\mathbb{B}_1^{h-1}$, we obtain an $\mathfrak{o}_D^*$-equivariant and R_1^{rig}-linear isomorphism $(\mathrm{Lie}(\mathbb{G})^{\otimes m})_1^{\mathrm{rig}} \simeq R_1^{\mathrm{rig}} \cdot \varphi_0^m$ of free R_1^{rig}-modules of rank one.

We denote by $\mathfrak{d}$ the Lie algebra of the Lie group $\mathfrak{o}_D^*$ over $\mathbb{Q}_p$. It is isomorphic to the Lie algebra associated with the associative $\mathbb{Q}_p$-algebra D. According to [22], page 450, the universal enveloping algebra $U_K(\mathfrak{d}) := U(\mathfrak{d} \otimes_{\mathbb{Q}_p} K)$ of $\mathfrak{d}$ over K embeds into the locally analytic distribution algebra $D(\Gamma_{2\varepsilon-1})$. Together with the natural map $D(\Gamma_{2\varepsilon-1}) \to \Lambda(\Gamma_{2\varepsilon-1})_{K,1}$, Theorem 3.5 allows us to view

$$M_1^m := (\mathrm{Lie}(\mathbb{G})^{\otimes m})_1^{\mathrm{rig}}$$

as a module over $U_K(\mathfrak{d}) \simeq U(\mathfrak{g} \otimes_{\mathbb{Q}_{p^h}} K) =: U_K(\mathfrak{g})$, where $\mathfrak{g} := \mathfrak{d} \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^h} \simeq \mathfrak{gl}_h$ as Lie algebras over $\mathbb{Q}_{p^h}$. Explicitly, the action of an element $\mathfrak{x} \in \mathfrak{g}$ on M_1^m is given by

$$\mathfrak{x}(\delta) = \frac{d}{dt}(\exp(t\mathfrak{x})(\delta))|_{t=0}.$$

Here $\exp : \mathfrak{g} \dashrightarrow \mathrm{GL}_h(\mathbb{Q}_{p^h})$ is the usual exponential map which is defined locally around zero in $\mathfrak{g}$. Further, a sufficiently small open subgroup of $\mathrm{GL}_h(\mathbb{Q}_{p^h})$ acts on M_1^m through the isomorphism $M_1^m \simeq \mathcal{O}_{\mathbb{P}_K^{h-1}}(m)(\Phi(\mathbb{B}_1^{h-1}))$. Writing an element $\mathfrak{x} \in \mathfrak{g}$ as a matrix $\mathfrak{x} = (a_{rs})_{0 \le r,s \le h-1}$ with coefficients $a_{rs} \in \mathbb{Q}_{p^h}$, fix indices $0 \le i,j \le h-1$ and denote by $\mathfrak{x}_{ij}$ the matrix with entry 1 at the place (i,j) and zero everywhere else. In the following we will formally put $w_0 := 1$.

Lemma 4.1. *Let i,j and m be integers with $0\le i,j\le h-1$. If $f\in R_1^{\mathrm{rig}}$ then*

$$\mathfrak{x}_{ij}(f\varphi_0^m)=\begin{cases} w_i\frac{\partial f}{\partial w_j}\varphi_0^m, & \text{if } j\neq 0,\\ (mf-\sum_{\ell=1}^{h-1}w_\ell\frac{\partial f}{\partial w_\ell})\varphi_0^m, & \text{if } i=j=0,\\ w_i(mf-\sum_{\ell=1}^{h-1}w_\ell\frac{\partial f}{\partial w_\ell})\varphi_0^m, & \text{if } i>j=0.\end{cases}$$

Proof. If $i=j$ and if t is sufficiently close to zero in $\mathbb{Q}_{p^h}$ then $\exp(t\mathfrak{x}_{ii})$ is the diagonal matrix with entry $\exp(t)$ at the place (i,i) and 1 everywhere else on the diagonal. Recall that $\mathrm{GL}_h(\mathbb{Q}_{p^h})$ acts by fractional linear transformations on the projective coordinates $\varphi_0,\ldots,\varphi_{h-1}$ of $\mathbb{P}_K^{h-1}$. Thus, $\exp(t\mathfrak{x}_{ii})(w_\ell)=w_\ell$ if $\ell\neq i\neq 0$, $\exp(t\mathfrak{x}_{ii})(w_i)=\exp(t)w_i$ if $i\neq 0$, and $\exp(t\mathfrak{x}_{00})(w_\ell)=\frac{1}{\exp(t)}w_\ell$ for all $1\le\ell\le h-1$.

If $i\neq j$ then $\exp(t\mathfrak{x}_{ij})=1+t\mathfrak{x}_{ij}$ in $\mathrm{GL}_h(\mathbb{Q}_{p^h})$. Thus, $\exp(t\mathfrak{x}_{ij})(w_\ell)=w_\ell$ if $\ell\neq j\neq 0$, $\exp(t\mathfrak{x}_{ij})(w_j)=w_j+tw_i$ if $j\neq 0$, and $\exp(t\mathfrak{x}_{i0})(w_\ell)=w_\ell/(1+tw_i)$ for all $1\le\ell\le h-1$. Writing $f=f(w_1,\ldots,w_{h-1})$ we have

$$\exp(t\mathfrak{x}_{ij})(f\varphi_0^m)=f(\exp(t\mathfrak{x}_{ij})(w_1),\ldots,\exp(t\mathfrak{x}_{ij})(w_{h-1}))\cdot\exp(t\mathfrak{x}_{ij})(\varphi_0)^m.$$

Here $\exp(t\mathfrak{x}_{ij})(\varphi_0)=\varphi_0$ if $j\neq 0$, $\exp(t\mathfrak{x}_{00})(\varphi_0)=\exp(t)\varphi_0$ and $\exp(t\mathfrak{x}_{i0})(\varphi_0)=\varphi_0+t\varphi_i$ if $1\le i\le h-1$. It is now an exercise in elementary calculus to derive the desired formulae. $\square$

Note that $(\mathrm{Lie}(\mathbb{G})^{\otimes m})^{\mathrm{rig}}$ is a $D(\Gamma_{2\varepsilon-1})$-stable K-subspace of M_1^m and hence is $\mathfrak{g}$-stable. If $m=0$ then Lemma 4.1 shows that in order to describe the $\mathfrak{g}$-action in the coordinates $u_1,\ldots,u_{h-1}$, one essentially has to compute the functional matrix

$$F:=(\frac{\partial u_i}{\partial w_j})_{1\le i,j\le h-1}.$$

Proposition 4.2. *The matrix $A:=(\frac{\partial\varphi_i}{\partial u_j}\varphi_0-\frac{\partial\varphi_0}{\partial u_j}\varphi_i)_{1\le i,j\le h-1}$ over R^{rig} is invertible over the localization $R^{\mathrm{rig}}_{\varphi_0}$. We have $F=\varphi_0^2A^{-1}$, which is a matrix with entries in $\varphi_0R^{\mathrm{rig}}$. Moreover, we have $\sum_{j=1}^{h-1}\varphi_j\frac{\partial u_i}{\partial w_j}\in\varphi_0^2R^{rig}$ for any index $1\le i\le h-1$.*

Proof. Let $B:=(\frac{\partial\varphi_i}{\partial u_j})_{0\le i,j\le h-1}$ with $\frac{\partial\varphi_i}{\partial u_0}:=\varphi_i$. We have $B\in\mathrm{GL}_h(R^{\mathrm{rig}})$ by a result of Gross and Hopkins (cf. [9], Corollary 21.17). Setting

$$N:=\begin{pmatrix}1&0&\cdots&0\\-\varphi_1&\varphi_0&&0\\\vdots&&\ddots&\\-\varphi_{h-1}&0&&\varphi_0\end{pmatrix},\text{ we have } NB=\begin{pmatrix}\varphi_0&\frac{\partial\varphi_0}{\partial u_1}&\cdots&\frac{\partial\varphi_0}{\partial u_{h-1}}\\0&&&\\\vdots&&A&\\0&&&\end{pmatrix}.$$

This already shows that A is invertible over $R^{\mathrm{rig}}_{\varphi_0}$. Denoting by $c_0,\ldots,c_{h-1}$ the columns of $B^{-1}=(c_{ij})_{i,j}\in\mathrm{GL}_h(R^{\mathrm{rig}})$, we obtain

$$(\varphi_0^{-1}\sum_{j=0}^{h-1}\varphi_jc_j,\varphi_0^{-1}c_1,\ldots,\varphi_0^{-1}c_{h-1})=B^{-1}N^{-1}=\begin{pmatrix}\varphi_0^{-1}&*&\cdots&*\\0&&&\\\vdots&&A^{-1}&\\0&&&\end{pmatrix}.$$

By the chain rain rule we have

$$\delta_{ij} = \frac{\partial w_i}{\partial w_j} = \sum_{\ell=1}^{h-1} \frac{\partial w_i}{\partial u_\ell} \cdot \frac{\partial u_\ell}{\partial w_j} = \sum_{\ell=1}^{h-1} \varphi_0^{-2}(\frac{\partial \varphi_i}{\partial u_\ell}\varphi_0 - \frac{\partial \varphi_0}{\partial u_\ell}\varphi_i)\frac{\partial u_\ell}{\partial w_j},$$

so that $F = \varphi_0^2 A^{-1}$. As seen above, the right hand side has entries in $\varphi_0 R^{\mathrm{rig}}$. Further, we have $\sum_{j=1}^{h-1} \varphi_j \frac{\partial u_i}{\partial w_j} = \sum_{j=1}^{h-1} \varphi_j \varphi_0 c_{ij} = -\varphi_0^2 c_{i0} \in \varphi_0^2 R^{\mathrm{rig}}$ for any index $1 \leq i \leq h-1$. □

Together with Lemma 4.1, Proposition 4.2 shows that $\mathfrak{x}(u_i) \in R^{\mathrm{rig}}$ for any $\mathfrak{x} \in \mathfrak{g}$ and any $1 \leq i \leq h-1$, as was clear a priori. For $h = 2$, Lemma 4.1 and Proposition 4.2 reprove [9], formula (25.14).

Coming back to the $\mathfrak{g}$-module M_1^m for general m, consider the subalgebra $\mathfrak{sl}_h$ of $\mathfrak{g}$ over $\mathbb{Q}_{p^h}$. Let $\mathfrak{t}$ denote the Cartan subalgebra of diagonal matrices in $\mathfrak{sl}_h$, and let $\{\varepsilon_1, \ldots, \varepsilon_{h-1}\} \subset \mathfrak{t}^*$ denote the basis of the root system of $(\mathfrak{sl}_h, \mathfrak{t})$ given by $\varepsilon_i(\mathrm{diag}(t_0, \ldots, t_{h-1})) := t_{i-1} - t_i$. We let $\lambda_1 \in \mathfrak{t}^*$ denote the fundamental dominant weight defined by $\lambda_1 := \frac{1}{h}\sum_{i=1}^{h-1}(h-i)\varepsilon_i$. We have

$$\lambda_1(\mathrm{diag}(t_0, \ldots, t_{h-1})) = \frac{1}{h}\sum_{i=1}^{h-1}(h-i)(t_{i-1} - t_i) = \frac{1}{h}((h-1)t_0 - \sum_{i=1}^{h-1} t_i) = t_0$$

for any element $\mathrm{diag}(t_0, \ldots, t_{h-1}) \in \mathfrak{t} \subset \mathfrak{sl}_h$.

Proposition 4.3. *For any integer $m \geq 0$, the subspace $W := \sum_{|\alpha| \leq m} K \cdot w^\alpha \varphi_0^m$ of M_1^m is $\mathfrak{g}$-stable. The action of $\mathfrak{sl}_h$ on W is irreducible. More precisely, W is the irreducible $\mathfrak{sl}_h$-representation of highest weight $m \cdot \lambda_1$.*

Proof. It follows from Lemma 4.1 that W is stable under any element $\mathfrak{x}_{ij}$ with $j \neq 0$ or $i = j = 0$. If $1 \leq i \leq h-1$ and if n is a non-negative integer then

$$\mathfrak{x}_{i0}^n(w^\alpha \varphi_0^m) = [\prod_{\ell=0}^{n-1}(m - |\alpha| - \ell)] \cdot w^\alpha w_i^n \varphi_0^m,$$

as follows from Lemma 4.1 by induction. Therefore, $\mathfrak{x}_{i0}(w^\alpha \varphi_0^m) = 0$ if $|\alpha| = m$. If $|\alpha| < m$ then $\mathfrak{x}_{i0}(w^\alpha)$ has degree $|\alpha| + 1 \leq m$. This proves that W is $\mathfrak{g}$-stable.

The above formula also shows that W is generated by φ_0^m as an $\mathfrak{sl}_h$-representation. If $f\varphi_0^m \in W$ is non-zero, then Lemma 4.1 shows that $(\mathfrak{x}_{01}^{\alpha_1} \cdots \mathfrak{x}_{0(h-1)}^{\alpha_{h-1}})(f\varphi_0^m)$ is a non-zero scalar multiple of φ_0^m for a suitable multi-index α. Therefore, the $\mathfrak{sl}_h$-representation W is irreducible.

Finally, if $\mathfrak{x} = \mathrm{diag}(t_0, \ldots, t_{h-1}) \in \mathfrak{t}$ then $\mathfrak{x}(w^\alpha \varphi_0^m) = (t_0(m - |\alpha|) + \sum_{i=1}^{h-1} \alpha_i t_i) \cdot w^\alpha \varphi_0^m$ by Lemma 4.1. Here,

$$t_0(m - |\alpha|) + \sum_{i=1}^{h-1} \alpha_i t_i = t_0 m + \sum_{i=1}^{h-1} \alpha_i(t_i - t_0) = (m \cdot \lambda_1 - \sum_{i=1}^{h-1} \alpha_i \sum_{\ell=1}^{i} \varepsilon_\ell)(\mathfrak{x}).$$

This shows that $m \cdot \lambda_1$ is the highest weight of the $\mathfrak{sl}_h$-representation W. □

Remark 4.4. The statement of Proposition 4.3 can be deduced from a stronger result of Gross and Hopkins. Namely, if $m = 1$ then $\mathrm{Lie}(\mathbb{G})^{\mathrm{rig}}$ contains an h-dimensional algebraic representation of $\mathfrak{o}_D^*$ (cf. [9], Proposition 23.2). Under the restriction map $\mathrm{Lie}(\mathbb{G})^{\mathrm{rig}} \to \mathrm{Lie}(\mathbb{G})_1^{\mathrm{rig}}$, the derived representation of $\mathfrak{g} = \mathfrak{d} \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^h}$ maps isomorphically to the $\mathfrak{g}$-representation W above.

We will now see that the action of $\mathfrak{g}$ on M_1^m naturally extends to a certain divided power completion of the universal enveloping algebra $U_K(\mathfrak{g})$. Note that if i, j, r and s are indices between 0 and $h-1$, then $\mathfrak{x}_{ij} \cdot \mathfrak{x}_{rs} = \delta_{jr}\mathfrak{x}_{is}$ in $\mathfrak{g} \simeq \mathfrak{gl}_h$. Therefore,

$$[\mathfrak{x}_{ij}, \mathfrak{x}_{rs}] = \delta_{jr}\mathfrak{x}_{is} - \delta_{is}\mathfrak{x}_{rj} = \begin{cases} 0, & \text{if } j \neq r \text{ and } i \neq s, \\ \mathfrak{x}_{is}, & \text{if } j = r \text{ and } i \neq s, \\ -\mathfrak{x}_{rj}, & \text{if } j \neq r \text{ and } i = s, \\ \mathfrak{x}_{ii} - \mathfrak{x}_{jj}, & \text{if } j = r \text{ and } i = s. \end{cases}$$

Setting $\mathfrak{x}'_{ij} := p^{\delta_{0i}-\delta_{0j}}\mathfrak{x}_{ij}$, one readily checks that the same relations hold on replacing $\mathfrak{x}_{ij}$ by $\mathfrak{x}'_{ij}$ and $\mathfrak{x}_{rs}$ by $\mathfrak{x}'_{rs}$ everywhere. It follows that the elements $\mathfrak{x}'_{ij}$ span a free $\mathbb{Z}_{p^h}$-Lie subalgebra of $\mathfrak{g}$ that we denote by $\mathring{\mathfrak{g}}$. Since $\mathrm{ad}(\mathfrak{x}'_{ij})^2 = 0$ if $i \neq j$, and since $(\varepsilon_{i+1} - \varepsilon_j)([\mathfrak{x}'_{ij}, \mathfrak{x}'_{ij}]) = 2$ if $i < j$, it follows from [2], VIII.12.7 Théorème 2 (iii), that the W-lattice $\mathring{\mathfrak{g}}$ of $\mathfrak{g}$ is the base extension from $\mathbb{Z}$ to W of a *Chevalley order* of $\mathfrak{g}$ in the sense of [2], VIII.12.7 Définition 2.

For $0 \leq i \leq h-1$ and $n \geq 0$ we set

$$\binom{\mathfrak{x}'_{ii}}{n} := \frac{\mathfrak{x}'_{ii}(\mathfrak{x}'_{ii} - 1)\cdots(\mathfrak{x}'_{ii} - n + 1)}{n!} \in U_K(\mathfrak{g}).$$

We let $\mathcal{U}$ denote the W-subalgebra of $U_K(\mathfrak{g})$ generated by the elements $(\mathfrak{x}'_{ij})^n/n!$ for $i \neq j$ and $n \geq 0$, as well as by the elements $\binom{\mathfrak{x}'_{ii}}{n}$ for $0 \leq i \leq h-1$ and $n \geq 0$. It follows from [2], VIII.7.12 Théorème 3, that $\mathcal{U}$ is a free W-module and that a W-basis of $\mathcal{U}$ is given by the elements

$$b_{\ell mn} := \Big(\prod_{i<j} \frac{(\mathfrak{x}'_{ij})^{\ell_{ij}}}{\ell_{ij}!}\Big) \cdot \Big(\prod_{i=0}^{h-1} \binom{\mathfrak{x}'_{ii}}{m_i}\Big) \cdot \Big(\prod_{i>j} \frac{(\mathfrak{x}'_{ij})^{n_{ij}}}{n_{ij}!}\Big)$$

with $\ell = (\ell_{ij}), n = (n_{ij}) \in \mathbb{N}^{h(h-1)/2}$ and $m = (m_i) \in \mathbb{N}^h$. Here the products of the $\mathfrak{x}'_{ij}$ for $i < j$ and $i > j$ have to be taken in a fixed but arbitrary ordering of the factors. For split semisimple Lie algebras these constructions and statements are due to Kostant (cf. [14], Theorem 1, where $\mathcal{U}$ is denoted by B).

We denote by $\hat{\mathcal{U}}$ the p-adic completion of the ring $\mathcal{U}$ and set

$$\hat{U}_K^{\mathrm{dp}}(\mathring{\mathfrak{g}}) := \hat{\mathcal{U}} \otimes_W K.$$

According to the above freeness result, any element of $\hat{U}_K^{\mathrm{dp}}(\mathring{\mathfrak{g}})$ can be written uniquely in the form $\sum_{\ell,m,n} d_{\ell mn} b_{\ell mn}$ with coefficients $d_{\ell mn} \in K$ satisfying $d_{\ell mn} \to 0$ as $|\ell| + |m| + |n| \to \infty$. Therefore, $\hat{U}_K^{\mathrm{dp}}(\mathring{\mathfrak{g}})$ is a K-algebra containing $U_K(\mathfrak{g})$. We view it as a K-Banach algebra with unit ball $\hat{\mathcal{U}}$ and call it the *complete divided power enveloping algebra of* $\mathring{\mathfrak{g}}$.

Theorem 4.5. *For any integer m the action of $\mathfrak{g}$ on $(\mathrm{Lie}(\mathbb{G})^{\otimes m})_1^{\mathrm{rig}}$ extends to a continuous action of $\hat{U}_K^{\mathrm{dp}}(\mathring{\mathfrak{g}})$.*

Proof. The ring of continuous K-linear endomorphisms of $M_1^m = (\mathrm{Lie}(\mathbb{G})^{\otimes m})_1^{\mathrm{rig}}$ is a K-Banach algebra for the operator norm. Since the latter is submultiplicative, the set of endomorphisms with operator norm less than or equal to one is a p-adically separated and complete W-algebra. Therefore, it suffices to prove that any element of the form $(\mathfrak{x}'_{ij})^n/n!$, $i \neq j$, or $\binom{\mathfrak{x}'_{ii}}{n}$, $0 \leq i \leq h-1$, has operator norm less than or equal to one on M_1^m whenever $n \geq 0$. If $\alpha \in \mathbb{N}^{h-1}$ and $0 \leq i, j \leq h-1$ then

$$\mathfrak{x}_{ij}^n(w^\alpha \varphi_0^m) = \begin{cases} \alpha_j^n w^\alpha \varphi_0^m, & \text{if } i = j \neq 0, \\ (m - |\alpha|)^n w^\alpha \varphi_0^m, & \text{if } i = j = 0, \\ n!\binom{\alpha_j}{n} w^\alpha w_j^{-n} w_i^n \varphi_0^m, & \text{if } i \neq j \neq 0, \\ n!\binom{m-|\alpha|}{n} w^\alpha w_i^n \varphi_0^m, & \text{if } i \neq j = 0, \end{cases} \tag{2}$$

as follows from Lemma 4.1 by induction. Here the generalized binomial coefficients are defined by

$$\binom{x}{n} := \frac{x(x-1)\cdots(x-n+1)}{n!} \in \mathbb{Z}$$

for any integer x. Now $||(\sum_\alpha d_\alpha w^\alpha)\varphi_0^m||_1 = \sup_\alpha\{|d_\alpha| p^{-|\alpha|}\}$. Bearing in mind our convention $w_0 = 1$, we obtain the claim for $(\mathfrak{x}'_{ij})^n/n!$ if $i \neq j$. If $0 \leq i \leq h-1$ then we obtain

$$\binom{\mathfrak{x}'_{ii}}{n}(w^\alpha \varphi_0^m) = \begin{cases} \binom{\alpha_i}{n} w^\alpha \varphi_0^m, & \text{if } i \neq 0, \\ \binom{m-|\alpha|}{n} w^\alpha \varphi_0^m, & \text{if } i = 0. \end{cases}$$

This completes the proof. □

Theorem 4.6. *Let m be an integer and set $c := w_1^{\max\{-1,m\}+1}\varphi_0^m$. The $U(\mathfrak{g})$-submodule $U(\mathfrak{g}) \cdot c$ of $(\mathrm{Lie}(\mathbb{G})^{\otimes m})_1^{\mathrm{rig}}$ is dense. If $h = 2$ and $m \geq -1$ then $\hat{U}_K^{\mathrm{dp}}(\mathring{\mathfrak{g}}) \cdot c = (\mathrm{Lie}(\mathbb{G})^{\otimes m})_1^{\mathrm{rig}}$.*

Proof. Equation (2) shows that $\mathfrak{x}_{01}^{\max\{-1,m\}+1}\mathfrak{x}_{10}^{\alpha_1}\cdots\mathfrak{x}_{(h-1)0}^{\alpha_{h-1}} \cdot c$ is a non-zero scalar multiple of $w^\alpha \varphi_0^m$. Thus, $K[w] \cdot \varphi_0^m \subset U_K(\mathfrak{g}) \cdot c$, proving the first assertion.

If $h = 2$ and $m \geq -1$ let us be more precise. Setting $m' := \max\{-1, m\} + 1$, $w := w_1$ and $\mathfrak{x} := \mathfrak{x}'_{10}$, we have $\mathfrak{x}^n \cdot c = (-1)^n n! p^{-n} w^{n+m'} \varphi_0^m$ for any $n \geq 0$ because $\binom{-1}{n} = (-1)^n$. If $f = \sum_{n \geq 0} d_n w^n \in R_1^{\mathrm{rig}}$ then $d_n p^n \to 0$ in K. Therefore, $\lambda := \sum_{n \geq 0} d_{n+m'}(-p)^n \frac{\mathfrak{x}^n}{n!}$ converges in $\hat{U}_K^{\mathrm{dp}}(\mathring{\mathfrak{g}})$ and we have $f\varphi_0^m - \lambda \cdot c = \sum_{n=0}^{m'-1} d_n w^n \varphi_0^m$. The latter is contained in $K[w] \cdot \varphi_0^m \subset U_K(\mathfrak{g}) \cdot c$, as seen above. □

Remark 4.7. By a result of Lazard, the image of $U_K(\mathfrak{g}) \simeq U_K(\mathfrak{d})$ in $\Lambda(\Gamma_{2\varepsilon-1})_{K,1}$ is dense (cf. [15], Chapitre IV, Théorème 3.2.5). We state without proof that the completion of $U_K(\mathfrak{g})$ for the norm $||\cdot||_1$ embeds continuously into $\hat{U}_K^{\mathrm{dp}}(\mathring{\mathfrak{g}})$. However, a formal series like $\sum_{n \geq 0} p^n \frac{(\mathfrak{x}'_{10})^n}{n!} = \sum_{n \geq 0} \frac{\mathfrak{x}_{10}^n}{n!}$ does not converge in $\Lambda(\Gamma_{2\varepsilon-1})_{K,1}$. Therefore, one might have doubts whether M_1^m is still finitely generated over $\Lambda(\Gamma_{2\varepsilon-1})_{K,1}$.

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