On the Iwasawa theory of the Lubin-Tate moduli space

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Abstract. We study the affine formal algebra R of the Lubin-Tate deformation space as a module over two different rings. One is the completed group ring of the automorphism group Γ of the formal module of the deformation problem, the other one is the spherical Hecke algebra of a general linear group. In the most basic case of height two and ground field $\mathbb{Q}_p$, our structure results include a flatness assertion for R over the spherical Hecke algebra and allow us to compute the continuous (co)homology of Γ with coefficients in R.

Contents

0 Introduction

Let K be a non-archimedean local field with valuation ring $\mathfrak{o}$ and residue class field k of characteristic p and cardinality q. Let $\overline{\mathbb{H}}$ denote a fixed one dimensional formal $\mathfrak{o}$-module of height $h \geq 1$ over a separable closure k^{sep} of k, and let $\breve{\mathfrak{o}}$ denote the valuation ring of the completion $\breve{K}$ of the maximal unramified extension of K.

By a famous theorem of Lubin-Tate and Drinfeld, the problem of deforming $\overline{\mathbb{H}}$ to formal $\mathfrak{o}$-modules over complete noetherian local $\breve{\mathfrak{o}}$-algebras with residue class field k^{sep} is represented by a formal scheme $\mathrm{Spf}(R)$ in which $R \simeq \breve{\mathfrak{o}}[[u_1, \ldots, u_{h-1}]]$ is a formal power series ring in $h-1$ variables $u_1, \ldots, u_{h-1}$ over $\breve{\mathfrak{o}}$.

The ring R carries a natural action of the automorphism group Γ of $\overline{\mathbb{H}}$. The latter can be identified with the group of units $\mathfrak{o}_D^*$ of the valuation ring $\mathfrak{o}_D$ of the central K-division algebra D of invariant $1/h$. The action of Γ on R is continuous and extends to an action of the completed group ring

$$\Lambda := \Lambda(\Gamma) = \breve{\mathfrak{o}}[[\Gamma]],$$

the so-called *Iwasawa algebra* of Γ over $\breve{\mathfrak{o}}$. This is explained at the beginning of section 3. By adding level structures to the above deformation problem, one can also show that R is a module over the *spherical Hecke algebra*

$$\mathcal{H} = \breve{\mathfrak{o}}[\mathrm{GL}_h(\mathfrak{o})\backslash \mathrm{GL}_h(K)/ \mathrm{GL}_h(\mathfrak{o})] \simeq \breve{\mathfrak{o}}[T_0, T_0^{-1}][T_1, \ldots, T_{h-1}]$$

of $\mathrm{GL}_h(K)$ over $\breve{\mathfrak{o}}$. The Hecke operators T_i commute with the action of Γ up to twists by outer automorphisms. This is explained at the beginning of section 2.

The formal scheme $\mathrm{Spf}(R)$ and its coverings are of fundamental importance in number theory and notably in understanding the arithmetic of the field K. Moreover, the action of Γ on R is related to important problems in stable homotopy theory (cf. [8], sections 5 and 6). Nonetheless, comparatively little work has been done in understanding the actions of Γ and $\mathcal{H}$ on R itself. We point out that $R_K := R \otimes_{\mathfrak{o}} K$ is topologically dual to a continuous representation of Γ on a $\breve{K}$-Banach space. Continuous and locally analytic representations stemming from equivariant vector bundles on moduli spaces of p-divisible groups have recently found a lot of interest. Notably the case of $\mathrm{GL}_h(K)$ acting on Drinfeld's p-adic upper half space was studied in detail and found applications to the de Rham cohomology of p-adically uniformized varieties (cf. [21]).

With these motivating problems in mind, the present article deals with the equally prominent example of the moduli space of Lubin-Tate. The appearance of the Hecke algebra $\mathcal{H}$ is a novel feature here which is not relevant in Drinfeld's setting (cf. Remark 3.5). Further, the analysis of the Γ-action is significantly complicated by the much more intricate geometry of the period morphism. In the most basic case of height two and ground field $\mathbb{Q}_p$, however, our structural results are rather precise. They allow us to compute the continuous (co)homology of Γ with coefficients in R and prove a flatness assertion for R over the Hecke algebra $\mathcal{H}$.

In order to describe our results more precisely, let π be a uniformizer of $\mathfrak{o}$, let $\overline{R} := R/\pi R$, and denote by $\overline{\mathfrak{m}} := (u_1, \ldots, u_{h-1})\overline{R}$ the maximal ideal of the local ring $\overline{R} \simeq k^{\mathrm{sep}}[[u_1, \ldots, u_{h-1}]]$. The leading term of the action of Γ on $\overline{R}$ with respect to the $\overline{\mathfrak{m}}$-adic filtration was computed by Chai in [6]. We take a different approach here and carry some of Chai's computations further by making use of the rigid analytic *period morphism* $\Phi : \mathrm{Spf}(R)^{\mathrm{rig}} \to \mathbb{P}^{h-1}_{\breve{K}}$ of Gross and Hopkins. There is an explicitly known linear action of Γ on the projective space $\mathbb{P}^{h-1}_{\breve{K}}$ for which Φ is equivariant. The main technical problem we have to overcome is that Φ is not defined over $\breve{\mathfrak{o}}$. In order to obtain information about the action of Γ on $\overline{R}$, we need to carefully analyze the growth behavior of the coordinate functions φ_i of Φ (cf. Lemma 1.7). This analysis is based on a closed formula of Yu (cf. Proposition 1.5). Our algorithm to $\overline{\mathfrak{m}}$-adically approximate the action of Γ on $\overline{R}$ is recorded in Theorem 1.11. Although this is a new approach, the possibility of computing the action of Γ to an arbitrary precision was known before.

Consider the open normal subgroup $\Gamma_1 := 1 + \Pi\mathfrak{o}_D$ of Γ, where Π denotes a uniformizer of D satisfying $\Pi^h = \pi$. For a limited number of elements $\gamma \in \Gamma_1$ we compute the image of the power series $\gamma(u_i)$ in $\overline{R}/\overline{\mathfrak{m}}^{q+2}$ for any $1 \leq i \leq h-1$ (cf. Theorem 1.14). If $h = 2$ we go even further and compute the image of $\gamma(u_1)$ in $\overline{R}/\overline{\mathfrak{m}}^{2q+2}$ (cf. Theorem 1.16). We point out that in contrast to Chai we do not treat elements which are arbitrarily close to 1. Thus, our computations only partially generalize his work. If $\mathfrak{o}_h$ denotes the valuation ring of the unramified extension of degree h of K, then we finally approximate the action of $\mathfrak{o}_h^* \subseteq \Gamma$ on R. This is in fact much easier to treat (cf. Theorem 1.19).

Fix $0 \leq i \leq h-1$ and set $u_0 := \pi$. The main technical result of section 2 is a description of the action of the Hecke operator T_i on the quotient $R/(u_0, \ldots, u_{i-1})R$. This does not seem to have been considered before (cf. Theorem 2.1). It requires an explicit knowledge of how $\mathrm{GL}_h(K)$ acts on the torsion points of the universal formal $\mathfrak{o}$-module $\mathbb{H}$ over R and relies on a subtle analysis of the double cosets of $\mathrm{GL}_h(\mathfrak{o})$ modulo its parahoric subgroups (cf. Lemma 2.3).

Our results completely determine the action of T_1 on $\overline{R}$. If $h = 2$ this allows us to prove our first main theorem, saying that $R/\breve{\mathfrak{o}}$ is a flat module over $\mathcal{H}/(T_0-1)\mathcal{H} \simeq \breve{\mathfrak{o}}[T_1]$ without any restriction on the field K (cf. Theorem 2.6 and Remark 2.7). As in the work [12] of Große-Klönne, this result is supposed to have strong representation theoretic consequences.

If h is arbitrary, we also obtain that the action of T_1 on $R/\breve{\mathfrak{o}}$ is topologically nilpotent (cf. Proposition 2.8). Further, for any non-negative integer n, the endomorphism T_1^n of R is injective, continuous with closed Λ-stable image, and has a torsion free cokernel over $\breve{\mathfrak{o}}$ (cf. Corollary 2.5 and Lemma 3.2). Endowing the $\breve{K}$-vector space $R_K := R \otimes_{\mathfrak{o}} K$ with a suitable locally convex topology, it follows that unless $h = 1$ the Λ_K-module R_K is not topologically of finite length (cf. Proposition 3.3). We compare this with the parallel situation of $\mathrm{GL}_h(K)$ acting on Drinfeld's p-adic symmetric space (cf. Remark 3.5).

The question of whether the Λ-module R (resp. the Λ_K-module R_K) is finitely generated, currently remains open. In the most basic case where $h = 2$ and $K = \mathbb{Q}_p$ we are able to show, however, that any of the Λ-modules $T_1^n(R)/T_1^{n+1}(R)$ is finitely generated (cf. Theorem 3.6 and Corollary 3.7). This is achieved by computing the coinvariants of $T_1^n(\overline{\mathfrak{m}})/T_1^{n+1}(\overline{\mathfrak{m}})$ for the action of Γ_1, using the approximations of section 1. One does obtain a module of finite type, however, by viewing $R/\breve{\mathfrak{o}}$ as a module over a twisted power series ring $\Lambda[[T_1; \sigma_1]]$, taking into account both the action of Λ and that of $\mathcal{H}$ (cf. Remark 3.8).

Let $Z_1 := 1 + \pi\mathfrak{o} \subseteq \Gamma_1$, and let $\overline{\mathfrak{m}}_{\Lambda(\Gamma_1/Z_1)}$ denote the maximal ideal of the local ring $\overline{\Lambda(\Gamma_1/Z_1)} := \Lambda(\Gamma_1/Z_1)/\pi\Lambda(\Gamma_1/Z_1)$. If $K = \mathbb{Q}_p$ with $p > h+1$ then fundamental results of Lazard allow us to determine the structure of the graded ring

$$\mathrm{gr}(\overline{\Lambda(\Gamma_1/Z_1)}) := \bigoplus_{i \geq 0} \overline{\mathfrak{m}}^i_{\Lambda(\Gamma_1/Z_1)}/\overline{\mathfrak{m}}^{i+1}_{\Lambda(\Gamma_1/Z_1)}$$

associated with the $\overline{\mathfrak{m}}_{\Lambda(\Gamma_1/Z_1)}$-adic filtration on $\overline{\Lambda(\Gamma_1/Z_1)}$. It is isomorphic to the universal enveloping algebra $U(\overline{\mathfrak{g}}/\overline{\mathfrak{z}})$ of an (h^2-1)-dimensional nilpotent Lie algebra $\overline{\mathfrak{g}}/\overline{\mathfrak{z}}$ over k^{sep} (cf. Corollary 3.14 and Remark 3.15).

Let $\mathfrak{m}_{\Lambda_1}$ be the maximal ideal of the local ring $\Lambda_1 := \Lambda(\Gamma_1)$. The action of $\mathfrak{o}^* \subseteq \Gamma$ and hence that of Z_1 on R is trivial. Endowing $\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}})$ with the

$\mathfrak{m}_{\Lambda_1}$-adic filtration, the associated graded object

$$\mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}})) := \bigoplus_{i\geq 0}[\mathfrak{m}_{\Lambda_1}^i \cdot (\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))]/[\mathfrak{m}_{\Lambda_1}^{i+1} \cdot (\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))]$$

may therefore be viewed as a module over $\mathrm{gr}(\overline{\Lambda(\Gamma_1/Z_1)}) \simeq U(\overline{\mathfrak{g}}/\overline{\mathfrak{z}})$. If $h = 2$ and $K = \mathbb{Q}_p$ with $p > 3$ then we determine the structure of this module completely (cf. Corollary 3.11 and Theorem 3.16). This in turn allows us to compute the Lie algebra (co)homology of $\mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))$ over $\overline{\mathfrak{g}}/\overline{\mathfrak{z}}$ (cf. Corollary 3.17). By means of a finitely convergent spectral sequence, the latter is related to the continuous (co)homology of $\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}})$ over Γ_1/Z_1. Analyzing the action of Γ on $\mathrm{H}_i(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, \mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}})))$ and $\mathrm{H}^i(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, \mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}})))$ we obtain that

$$\mathrm{H}_i(\Gamma, (R/\breve{\mathfrak{o}})/T_1(R/\breve{\mathfrak{o}})) = \mathrm{H}^i(\Gamma, (R/\breve{\mathfrak{o}})/T_1(R/\breve{\mathfrak{o}})) = 0$$

for all $i \geq 0$, assuming $h = 2$, $K = \mathbb{Q}_p$ and $p > 3$ (cf. Theorem 3.19). By dévissage and passage to the limit we finally obtain our second main theorem, saying that the Γ-equivariant inclusion $\breve{\mathfrak{o}} \to R$ induces isomorphisms

$$\mathrm{H}_i(\Gamma, \breve{\mathfrak{o}}) \simeq \mathrm{H}_i(\Gamma, R) \quad \text{and} \quad \mathrm{H}^i(\Gamma, \breve{\mathfrak{o}}) \simeq \mathrm{H}^i(\Gamma, R)$$

for all $i \geq 0$ under the same hypotheses (cf. Theorem 3.20).

The preceding assertion is related to the behavior of the Adams spectral sequence in the theory of ring spectra and is predicted by Hopkins' chromatic splitting conjecture (cf. [15], Conjecture 4.2). In fact, there are important results from algebraic topology exceeding those in Theorem 3.20. More precisely, let $\mathrm{Lie}(\mathbb{H})$ denote the Lie algebra of the universal formal $\mathfrak{o}$-module $\mathbb{H}$. This is a free R-module of rank one carrying a continuous semilinear action of Γ. Using methods from stable homotopy theory, the cohomology algebra $\mathrm{H}^\bullet(\Gamma, \oplus_{n\in\mathbb{Z}}\mathrm{Lie}(\mathbb{H})^{\otimes n})$ was computed by Shimomura and Yabe (cf. [30]). Their work was later taken up and complemented by Behrens in [1]. However, these results are not easily accessible to the non-topologist, and we hope that our representation-theoretic approach, although spelled out only for $n = 0$, is the more direct one. We also note that the Tate-Farrell cohomology of $\oplus_{n\in\mathbb{Z}}\mathrm{Lie}(\mathbb{H})^{\otimes n}$ was computed by Symonds if $K = \mathbb{Q}_p$ and $h = p - 1 \geq 2$ (cf. [33] and our Remark 3.21).

If $h > 2$ or if $K \neq \mathbb{Q}_p$ then the computations leading to the above results become significantly more complicated. On the other hand, we develop most of the necessary machinery in complete generality. Therefore, we are convinced that our methods will prove important in analyzing the structure of R over Λ and $\mathcal{H}$ in other cases, as well.

Conventions and notation. Let K be a non-archimedean local field. The normalized valuation of K, as well as its extension to an algebraic closure of K, will be denoted by v. We denote by $\mathfrak{o}$ the valuation ring of K and fix a uniformizer π of $\mathfrak{o}$. Let $k := \mathfrak{o}/\pi\mathfrak{o}$ denote the residue class field of $\mathfrak{o}$, and let q and p denote the cardinality and the characteristic of k, respectively. If $K = \mathbb{Q}_p$ we will always choose $\pi = p$.

We denote by $\breve{K}$ the completion of the maximal unramified extension of K, and by $\breve{\mathfrak{o}}$ its valuation ring. The residue class field $\breve{\mathfrak{o}}/\pi\breve{\mathfrak{o}}$ of $\breve{\mathfrak{o}}$ will be identified with a fixed separable closure $k^{\rm sep}$ of the field k. We denote by σ the Frobenius automorphism ($x \mapsto x^q$) of $k^{\rm sep}$, as well as its unique lift to a ring automorphism of $\breve{\mathfrak{o}}$.

We fix a positive integer h and denote by $D = D_h$ the central K-division algebra of invariant $1/h$. The valuation v of K uniquely extends to a valuation v_D of D. We denote by $\mathfrak{o}_D$ the valuation ring of D and fix a uniformizer Π of $\mathfrak{o}_D$ satisfying $\Pi^h = \pi$. Let K_h denote the unramified extension of K of degree h, let $\mathfrak{o}_h$ denote the valuation ring of K_h, and let $k_h := \mathfrak{o}_h/\pi\mathfrak{o}_h$ denote the residue class field of $\mathfrak{o}_h$. We fix an embedding $K_h \hookrightarrow D$ of K-algebras. It restricts to an embedding $\mathfrak{o}_h \hookrightarrow \mathfrak{o}_D$ and induces an isomorphism $k_h \simeq \mathfrak{o}_D/\Pi\mathfrak{o}_D$.

If S is a unital ring then we denote by S^* its group of units. If ℓ is a positive integer and if $u = (u_1, \ldots, u_\ell)$ is a family of indeterminates then we denote by $S[[u]] := S[[u_1, \ldots, u_\ell]]$ the ring of formal power series in the variables $u_1, \ldots, u_\ell$ with coefficients in S. If $n = (n_1, \ldots, n_\ell) \in \mathbb{N}^\ell$ we set $u^n := u_1^{n_1} \cdots u_\ell^{n_\ell}$ and $|n| := n_1 + \ldots + n_\ell$.

1 The action of the automorphism group

We fix a positive integer h and a one dimensional formal $\mathfrak{o}$-module $\overline{\mathbb{H}}$ of height h over $k^{\rm sep}$ which is defined over k. By a fundamental theorem of Lubin-Tate and Drinfeld, $\overline{\mathbb{H}}$ admits a deformation to a formal $\mathfrak{o}$-module $\mathbb{H}$ over the power series ring $R := \breve{\mathfrak{o}}[[u_1, \ldots, u_{h-1}]]$ which is universal in the sense that any deformation of $\overline{\mathbb{H}}$ to a formal $\mathfrak{o}$-module over a complete noetherian local $\breve{\mathfrak{o}}$-algebra with residue class field $k^{\rm sep}$ arises uniquely as a specialization of $\mathbb{H}$ (cf. [9], Proposition 4.2, or [11], Proposition 12.10).

The formal parameters $u_1, \ldots, u_{h-1}$ can be chosen in such a way that $\mathbb{H}(X, Y) = f^{-1}(f(X) + f(Y))$, where the logarithm $f(X) \in X \cdot R[\frac{1}{\pi}][[X]]$ satisfies Hazewinkel's functional equation

$$f(X) = X + \sum_{i=1}^{h} \frac{u_i}{\pi} \varphi^i(f)(X) \tag{1}$$

(cf. [11], Proposition 5.7). Here we set $u_h := 1$, and φ denotes the $\breve{\mathfrak{o}}$-linear ring endomorphism of $R[\frac{1}{\pi}][[X]]$ (and of its subrings $R[\frac{1}{\pi}]$ and R) determined by $\varphi(X) := X^q$ and $\varphi(u_i) := u_i^q$ for all $1 \le i \le h-1$.

It follows from (1) that f is of the form $f(X) = \sum_{n=0}^\infty a_n X^{q^n}$ and that the coefficients $a_n \in R[\frac{1}{\pi}]$ satisfy the recursion formula

$$a_0 = 1 \quad \text{and} \quad \pi a_n = \sum_{i=1}^{\min\{h,n\}} u_i \cdot \varphi^i(a_{n-i}) = \sum_{i=1}^{\min\{h,n\}} u_i^{q^{n-i}} \cdot a_{n-i} \tag{2}$$

(cf. [14], I.3.3, equations (3.3.6) and (3.3.9)).

We let $\Gamma := \mathrm{Aut}_{\mathfrak{o}}(\overline{\mathbb{H}})$ denote the group of automorphisms of the formal $\mathfrak{o}$-module $\overline{\mathbb{H}}$. According to [9], Proposition 1.7, the group Γ is isomorphic to the group of units $\mathfrak{o}_D^*$ of the valuation ring $\mathfrak{o}_D$ of the central K-division algebra D of invariant $1/h$. It acts on R from the left by $\breve{\mathfrak{o}}$-linear local ring automorphisms. More precisely, given an $\mathfrak{o}$-linear automorphism γ of $\overline{\mathbb{H}}$, there is a unique isomorphism $\gamma : R \to R$ of local $\breve{\mathfrak{o}}$-algebras and a unique isomorphism $[\gamma] : \gamma_*\mathbb{H} \to \mathbb{H}$ of formal $\mathfrak{o}$-modules such that the reduction of $[\gamma]$ modulo the maximal ideal $\mathfrak{m}$ of R is γ (cf. [11], Proposition 14.7). Fixing an isomorphism $\Gamma \simeq \mathfrak{o}_D^*$ we shall from now on identify the groups Γ and $\mathfrak{o}_D^*$.

We let $\overline{R} := R/\pi R \simeq k^{\mathrm{sep}}[[u_1, \ldots, u_{h-1}]]$ and denote by $\overline{\mathfrak{m}} := (u_1, \ldots, u_{h-1})\overline{R}$ the maximal ideal of $\overline{R}$. In this section we wish to study the action of Γ on $\overline{R}$, induced by that on R. Using the Cartier-Dieudonné module of $\mathbb{H}$, the leading term of this action with respect to the $\overline{\mathfrak{m}}$-adic filtration of $\overline{R}$ was computed by Chai in [6]. We choose a different method here and compute higher terms of this action for a limited number of elements of Γ.

We denote by K_h the unramified extension of K of degree h, and by $\mathfrak{o}_h$ its valuation ring. We fix an embedding $K_h \hookrightarrow D$ and a uniformizer Π of D, satisfying $\Pi^h = \pi$. Recall that any element $\gamma \in D$ can be written uniquely as

$$\gamma = \sum_{i=0}^{h-1} \Pi^i \cdot \alpha_i \quad \text{with} \quad \alpha_0, \ldots, \alpha_{h-1} \in K_h.$$

We have $\gamma \in \Gamma$ if and only if $\alpha_0 \in \mathfrak{o}_h^*$ and $\alpha_1, \ldots, \alpha_{h-1} \in \mathfrak{o}_h$. Further, $\Pi\alpha = \alpha^\sigma \Pi$ for all $\alpha \in \mathfrak{o}_h$.

The subgroup $\mathfrak{o}_h^*$ of Γ contains the group of roots of unity μ_{q^h-1} of order $q^h - 1$. We first reprove [6], Lemma 2, by using power series methods.

Lemma 1.1. *If $\xi \in \mu_{q^h-1} \subset \Gamma$ then the corresponding $\breve{\mathfrak{o}}$-linear ring automorphism of R satisfies $\xi(u_i) = \xi^{q^i-1} \cdot u_i$ for $1 \leq i \leq h-1$. The unique isomorphism $[\xi] : \xi_*\mathbb{H} \to \mathbb{H}$ which reduces to ξ modulo $\mathfrak{m}$ is given by the power series $[\xi](X) = \xi \cdot X$.*

Proof. Viewing ξ as an element of k_h by reduction modulo $\pi\mathfrak{o}_h$, the automorphism ξ of $\overline{\mathbb{H}}$ is given by the power series $\xi \cdot X \in k^{\mathrm{sep}}[[X]]$. This follows from [11], Proposition 13.6, by reduction. Denoting by ξ the K-linear ring automorphism of $R[\frac{1}{\pi}]$ determined by $\xi(u_i) := \xi^{q^i-1} \cdot u_i$, $1 \leq i \leq h-1$, it suffices to prove that $\xi_* f = \xi^{-1} f(\xi X)$.

We define the $R[\frac{1}{\pi}]$-linear automorphism $(g \mapsto \tilde{g})$ of $R[\frac{1}{\pi}][[X]]$ by $\tilde{g}(X) := \xi^{-1} g(\xi X)$. A direct computation shows that if $g \in R[\frac{1}{\pi}][[X]]$ is of the form $g = \sum_{n \geq 0} a_n X^{q^n}$ with $a_n \in R[\frac{1}{\pi}]$, then $\widetilde{\varphi^i(g)} = \xi^{q^i-1} \cdot \varphi^i(\tilde{g})$ for any integer $i \geq 0$. Applying the transformation $(g \mapsto \tilde{g})$ to (1), we obtain that $\tilde{f}$ satisfies the functional equation

$$\tilde{f}(X) = X + \sum_{i=1}^{h} \frac{\xi^{q^i-1} \cdot u_i}{\pi} \varphi^i(\tilde{f})(X).$$

Another direct computation shows that the endomorphisms φ and ξ_* of $R[\frac{1}{\pi}][[X]]$ commute with each other. Applying ξ_* to (1), we obtain that $\xi_* f$ satisfies the same above functional equation. As in (2), this leads to identical recursive definitions of the coefficients of $\xi_* f$ and $\tilde{f}$, whence $\xi_* f = \tilde{f}$. $\square$

In order to study the action of the higher congruence subgroups

$$\Gamma_i := 1 + \Pi^i \mathfrak{o}_D, \quad i \geq 1,$$

of Γ on $\overline{R}$ we shall make use of the period morphism $\Phi : \mathrm{Spf}(R)^{\mathrm{rig}} \to \mathbb{P}^{h-1}_{\breve{K}}$, constructed in [11], section 23. Here $\mathbb{P}^{h-1}_{\breve{K}}$ denotes the rigid analytic projective space of dimension $h-1$ over $\breve{K}$, and $\mathrm{Spf}(R)^{\mathrm{rig}}$ denotes the rigidification of the formal $\breve{\mathfrak{o}}$-scheme $\mathrm{Spf}(R)$ in the sense of Berthelot (cf. [7], §7). The latter is isomorphic to the rigid analytic open unit polydisc of dimension $h-1$ over $\breve{K}$.

In homogeneous projective coordinates, the morphism Φ is given by $\Phi(x) = [\varphi_0(x) : \ldots : \varphi_{h-1}(x)]$, where $\varphi_0, \ldots, \varphi_{h-1} \in \mathcal{O}(\mathrm{Spf}(R)^{\mathrm{rig}})$ are certain global rigid analytic functions on $\mathrm{Spf}(R)^{\mathrm{rig}}$ without any common zero. They can be constructed from the coefficients a_n of the logarithm $f(X) = \sum_{n\geq 0} a_n X^{q^n}$ of the universal formal $\mathfrak{o}$-module $\mathbb{H}$ of height h over R by the formulae

$$(3) \qquad \begin{aligned} \varphi_0 &:= \lim_{n\to\infty} \pi^n a_{nh} \quad \text{and} \\ \varphi_i &:= \lim_{n\to\infty} \pi^{n+1} a_{nh+i}, \quad \text{if } 1 \leq i \leq h-1. \end{aligned}$$

The convergence holds in the natural $\breve{K}$-Fréchet topology of $\mathcal{O}(\mathrm{Spf}(R)^{\mathrm{rig}})$ (cf. [11], Proposition 21.2).

We denote by D_0 the affinoid subdomain of $\mathrm{Spf}(R)^{\mathrm{rig}}$ defined by

$$D_0 := \{x \in \mathrm{Spf}(R)^{\mathrm{rig}} \mid v(u_i(x)) \geq 1 \text{ for all } 1 \leq i \leq h-1\}.$$

The subsequent results follow from [11], Lemma 23.14 and Proposition 23.15.

Theorem 1.2 (Gross-Hopkins). *We have $\varphi_0 \in \mathcal{O}(D_0)^*$, and $\mathcal{O}(D_0)$ is isomorphic to the free Tate algebra over $\breve{K}$ in the variables $(\frac{\varphi_i}{\pi\varphi_0})_{1\leq i\leq h-1}$. The morphism Φ restricts to an isomorphism $\Phi : D_0 \to \Phi(D_0)$.* $\square$

We set $w_h := 1$ and $w_i := \varphi_i/\varphi_0 \in \mathcal{O}(D_0)$ for $1 \leq i \leq h-1$, so that $\mathcal{O}(D_0) \simeq \breve{K}\langle \pi^{-1}w_1, \ldots, \pi^{-1}w_{h-1}\rangle$ by Theorem 1.2. It is a general fact that the morphism Φ is Γ-equivariant for a certain action of Γ on $\mathbb{P}^{h-1}_{\breve{K}}$. This leads to the following result of Devinatz-Hopkins.

Proposition 1.3 (Devinatz-Hopkins). *Fix an integer i with $1 \leq i \leq h-1$. If $\alpha_0 \in \mathfrak{o}_h^*$, if $\alpha_1, \ldots, \alpha_{h-1} \in \mathfrak{o}_h$, and if $\gamma := \sum_{i=0}^{h} \alpha_i \Pi^i \in \Gamma$, then*

$$(4) \qquad \gamma(w_i) = \frac{\sum_{j=1}^{i} \alpha_{i-j}^{\sigma^j} w_j + \sum_{j=i+1}^{h} \pi\alpha_{h+i-j}^{\sigma^j} w_j}{\alpha_0 + \sum_{j=1}^{h-1} \alpha_{h-j}^{\sigma^j} w_j}.$$

In particular, the subdomain D_0 of $\mathrm{Spf}(R)^{\mathrm{rig}}$ *is Γ-stable.*

Proof. The formula (4) is a straightforward generalization of [8], Remark 2.21, Proposition 3.3 and Lemma 4.9, which treats the case $K = \mathbb{Q}_p$. The Γ-stability of D_0 is then an immediate consequence of (4) and of the definition of D_0. $\square$

Remark 1.4. Formula (4) can also be expressed by saying that the period morphism Φ is Γ-equivariant if any element $\gamma = \sum_{i=0}^{h-1} \alpha_i \Pi^i$ of Γ acts on the homogeneous coordinates $[\varphi_0 : \ldots : \varphi_{h-1}]$ of $\mathbb{P}^{h-1}_{\breve{K}}$ through right multiplication with the matrix

$$C(\gamma) := \begin{pmatrix} \alpha_0 & \pi\alpha_1 & \pi\alpha_2 & \cdots & \cdots & \pi\alpha_{h-1} \\ \alpha_{h-1}^{\sigma} & \alpha_0^{\sigma} & \alpha_1^{\sigma} & \cdots & \cdots & \alpha_{h-2}^{\sigma} \\ \alpha_{h-2}^{\sigma^2} & \pi\alpha_{h-1}^{\sigma^2} & \alpha_0^{\sigma^2} & \cdots & \cdots & \alpha_{h-3}^{\sigma^2} \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ \alpha_1^{\sigma^{h-1}} & \pi\alpha_2^{\sigma^{h-1}} & \cdots & \cdots & \pi\alpha_{h-1}^{\sigma^{h-1}} & \alpha_0^{\sigma^{h-1}} \end{pmatrix}.$$

If $h > 2$ then this formula for $C(\gamma)$ does not coincide with formula (22.9) of [11], page 72. In fact, the latter seems to lead to certain inconsistencies. For example, the fundamental domain

$$\begin{aligned} D &:= \{x \in \mathrm{Spf}(R)^{\mathrm{rig}} \mid v(u_i(x)) \geq (h-i)h^{-1} \text{ for all } 1 \leq i \leq h-1\} \\ &= \{x \in \mathrm{Spf}(R)^{\mathrm{rig}} \mid v(w_i(x)) \geq (h-i)h^{-1} \text{ for all } 1 \leq i \leq h-1\} \end{aligned}$$

of Gross-Hopkins is supposed to be stable under the action of Γ (cf. [10], Remarque I.3.2). Viewing Π as an element of an algebraic closure of $\breve{K}$, we have $x_0 := [1 : \Pi^{h-1} : \Pi^{h-2} : \ldots : \Pi] \in \Phi(D)$ because $v(\Pi) = h^{-1}$. Using formula (22.9) of [11] for the element $\gamma := 1 + \Pi \in \Gamma$, we obtain

$$x_0 \cdot \gamma = [1 + \Pi^{h-1} : \Pi^{h-1} + \Pi^{h-2} : \ldots : \Pi^2 + \Pi : \pi + \Pi],$$

which is not contained in $\Phi(D)$ unless $h \leq 2$. Using (4), we obtain

$$x_0 \cdot \gamma = [1 + \Pi : \pi + \Pi^{h-1} : \Pi^{h-1} + \Pi^{h-2} : \ldots : \Pi^2 + \Pi] \in \Phi(D).$$

By Proposition 1.3, the action of Γ on $\mathcal{O}(\mathrm{Spf}(R)^{\mathrm{rig}})$ extends to a continuous $\breve{K}$-linear action on the $\breve{K}$-Banach algebra $\mathcal{O}(D_0)$. Since both $(\frac{u_1}{\pi}, \ldots, \frac{u_{h-1}}{\pi})$ and $(\frac{w_1}{\pi}, \ldots, \frac{w_{h-1}}{\pi})$ are affinoid generators of $\mathcal{O}(D_0)$, there are power series $g_1, \ldots, g_{h-1} \in \mathcal{O}(D_0)$ such that $u_i = g_i(w_1, \ldots, w_{h-1})$ for any index i with $1 \leq i \leq h-1$. Using the $\breve{K}$-linearity and the continuity of the Γ-action, we obtain the tautological relation

$$\gamma(u_i) = g_i(\gamma(w_1), \ldots, \gamma(w_{h-1})) \in R \subseteq \mathcal{O}(D_0) \tag{5}$$

for any element $\gamma \in \Gamma$.

The power series expansions of the functions φ_i in the variables $u_1, \ldots, u_{h-1}$ can be expressed by a closed formula of Yu (cf. [36], Proposition 8). In the case $h = 2$ this formula already appears in the work of Gross-Hopkins (cf. [11], section 25). Recall that if n is a non-negative integer, then an *ordered partition of* n is a decomposition of the set $\{i \in \mathbb{Z} \mid 0 \leq i < n\}$ into a union of pairwise disjoint non-empty *segments*, i.e. sets of the form $\{i \in \mathbb{Z} \mid m \leq i \leq m'\}$ with $m, m' \in \mathbb{Z}$ and $0 \leq m \leq m' < n$. Given such a decomposition and an integer $j \geq 1$, we denote by S_j the set of minimal elements of all segments of the decomposition with cardinality j. By convention, $S_j = \emptyset$, if no such segment exists. Thus, an ordered partition of n gives rise to a collection of sets $S = (S_j)_{j \geq 1}$, which in turn uniquely determines the ordered partition. If $j \geq 1$ we set $q(S_j) := \sum_{x \in S_j} q^x$ and denote by $|S_j|$ the cardinality of S_j.

Proposition 1.5 (Yu). *Fix an index i with $0 \le i \le h-1$. If ℓ is a non-negative integer, let $P_{i,\ell}$ be the set of ordered partitions $S = (S_j)_{j\ge1}$ of $\ell h + i$ such that $(\ell-1)h + i \notin S_h$ and such that $S_j = \emptyset$ whenever $j > h$. We have*

$$\begin{aligned}\varphi_0(u) &= \sum_{\ell\ge0}\sum_{S\in P_{0,\ell}} \pi^{\ell} \prod_{j=1}^{h} \frac{u_j^{q(S_j)}}{\pi^{|S_j|}} \quad \textit{and} \\ \varphi_i(u) &= \sum_{\ell\ge0}\sum_{S\in P_{i,\ell}} \pi^{\ell+1} \prod_{j=1}^{h} \frac{u_j^{q(S_j)}}{\pi^{|S_j|}} \quad \textit{if} \quad 1 \le i \le h-1.\end{aligned}$$

Proof. This follows from (3) and [36], Proposition 8, together with our convention $u_h = 1$ and the observation that

$$\frac{1}{h}(i + \sum_{j=1}^{h}(h-j)|S_j|) = \frac{1}{h}(i + h\sum_{j=1}^{h}|S_j| - \underbrace{\sum_{j=1}^{h} j|S_j|}_{=i+h\ell}) = \sum_{j=1}^{h}|S_j| - \ell$$

for any $S \in P_{i,\ell}$. □

Remark 1.6. As is implicit in the proof of [36], Proposition 8, if $S = (S_j)_{j\ge1}$ and $T = (T_j)_{j\ge1}$ are two distinct elements of $\bigcup_{i=0}^{h-1}\bigcup_{\ell\ge0} P_{i,\ell}$, then the two monomials $\prod_{j=1}^{h} u_j^{q(S_j)}$ and $\prod_{j=1}^{h} u_j^{q(T_j)}$ are distinct. Indeed, if $\prod_{j=1}^{h} u_j^{q(S_j)} = \prod_{j=1}^{h} u_j^{q(T_j)}$ then $q(S_j) = q(T_j)$ for all $1 \le j \le h-1$. Fix an index j with $1 \le j \le h-1$. We will show that $S_j = T_j$ and may assume that both S_j and T_j are non-empty. Let $x_j := \min(S_j)$ and $y_j := \min(T_j)$. Denoting by v_p the p-adic valuation on $\mathbb{Z}$, the equation $q(S_j) = q(T_j)$ implies that $x_j v_p(q) = v_p(q(S_j)) = v_p(q(T_j)) = y_j v_p(q)$, whence $x_j = y_j$ and $q(S_j \setminus \{x_j\}) = q(T_j \setminus \{y_j\})$. Inductively, we obtain $S_j = T_j$, as desired. Now set $x_S := \max(\bigcup_{j=1}^{h} S_j)$ and $y_T := \max(\bigcup_{j=1}^{h} T_j)$. Since $S \in P_{i,\ell}$ and $T \in P_{i',\ell'}$ for certain indices i, i', ℓ, ℓ', we have $x_S \notin S_h$ and $y_T \notin T_h$. By what we proved above, we obtain $x_S = x_T$, so that S and T are ordered partitions of the same integer $x_S + j_0 + 1$ (where j_0 is chosen so that $x_S = y_T \in S_{j_0} = T_{j_0}$). But then the fact that $S_j = T_j$ for all $j \ne h$ also implies $S_h = T_h$, and we are done. As a consequence, the formulae in Proposition 1.5 give the power series expansions of the functions φ_i in the variables $u_1, \ldots, u_{h-1}$.

Lemma 1.7. *Let F be any of the functions φ_i with $0 \le i \le h-1$. Writing $F(u) = \sum_{n\in\mathbb{N}^{h-1}} b_n u^n$ with $b_n \in \breve{K}$, we have*

$$|n| > -q \cdot v(b_n) \tag{6}$$

for any $n \in \mathbb{N}^{h-1}$, unless $F = \varphi_0$ and $n = 0$, where equality holds.

Proof. First assume $i \ne 0$ and note that the required inequality is trivial if $b_n = 0$. If $\ell \ge 0$, and if $S \in P_{i,\ell}$ is an ordered partition as in Proposition 1.5, then $|S_h| \le \ell$. If $0 \in S_j$ then $q(S_j) \ge q|S_j| - q + 1$ because $x \ge 1$ for all

$x \in S_j \smallsetminus \{0\}$. If $0 \notin S_j$ then $q(S_j) \geq q|S_j|$ by the same argument. Hence,

$$\begin{aligned}\sum_{j=1}^{h-1} q(S_j) &\geq 1 - q + q\sum_{j=1}^{h-1}|S_j| > -q + q\sum_{j=1}^{h-1}|S_j| \\ &= q((\sum_{j=1}^{h}|S_j|) - |S_h| - 1) \geq q((\sum_{j=1}^{h}|S_j|) - \ell - 1).\end{aligned}$$

By Proposition 1.5 and Remark 1.6 this proves the claim for $F = \varphi_i$ with $i \neq 0$.

If $F = \varphi_0$ then $b_0 = 1$ by Remark 1.6 and because the unique element of $P_{0,0}$ is the ordered partition S for which $S_j = \emptyset$ for all $j \geq 1$. If $\ell \geq 1$, and if $S \in P_{0,\ell}$, then the condition $h(\ell-1) \notin S_h$ ensures that $|S_h| \leq \ell - 1$. As above, one obtains

$$\sum_{j=1}^{h-1} q(S_j) > q((\sum_{j=1}^{h}|S_j|) - \ell),$$

proving the claim by Proposition 1.5. □

Corollary 1.8. *Let $1 \leq i \leq h-1$, let $\gamma \in \Gamma$, and let F be any of the functions w_i, g_i or $\gamma(w_i)$ in $\mathcal{O}(D_0)$. Writing $F(u) = \sum_{n \in \mathbb{N}^{h-1}} b_n u^n$ with $b_n \in \breve{K}$, the inequality (6) holds for any $n \in \mathbb{N}^{h-1}$.*

Proof. It is clear that if F_1 and F_2 are two power series satisfying the required condition, then so are $F_1 + F_2$, $F_1 \cdot F_2$ and $\alpha \cdot F_1$ for any $\alpha \in \breve{\mathfrak{o}}$. Therefore, the claim for

$$F = w_i = \frac{\varphi_i}{\varphi_0} = \varphi_i + \sum_{\ell \geq 1} \varphi_i (1 - \varphi_0)^\ell$$

follows from Lemma 1.7. Writing $C(\gamma) = (c_{ij})_{0 \leq i,j \leq h-1} \in \mathrm{GL}_h(\mathfrak{o}_h)$ as in Remark 1.4, Proposition 1.3 implies

$$\gamma(w_i) = \frac{\sum_{j=0}^{h-1} c_{ji}\varphi_j}{\sum_{j=0}^{h-1} c_{j0}\varphi_j}.$$

Since $c_{0i} \in \pi\mathfrak{o}$, any of the summands in the numerator satisfies (6) (cf. Lemma 1.7). Hence, so does the numerator itself. Writing

$$\sum_{j=0}^{h-1} c_{j0}\varphi_j = c_{00} + (c_{00}\varphi_0 - c_{00}) + \sum_{j=1}^{h-1} c_{j0}\varphi_j$$

with $c_{00} \in \mathfrak{o}_h^* \subseteq \breve{\mathfrak{o}}^*$, we can argue as above and obtain the required property for $F = \gamma(w_i)$.

As for the power series g_i, we need to recall the recursive construction of the coefficients of g_i (cf. the formal inverse function theorem in [14], A.4.6). Let (u) be the ideal of $\mathcal{O}(D_0)$ (as well as that of $\breve{K}[[u]]$) generated by the elements $u_1, \ldots, u_{h-1}$. Since $w_i(u) \equiv u_i \mod (u)^2$ (cf. Proposition 1.5 or Proposition 1.13 below), we have $g_i(u) \equiv u_i \mod (u)^2$, as well. Put $g_i^{(1)}(u) := u_i$. Suppose $m \geq 1$ and that we have found a power series $g_i^{(m)}$ satisfying

$g_i^{(m)}(w_1(u),\ldots,w_{h-1}(u)) - u_i \in (u)^{m+1}$. There is then a homogeneous polynomial $h_i^{(m)}$ in u of degree $m+1$ such that $g_i^{(m)}(w_1(u),\ldots,w_{h-1}(u)) - u_i - h_i^{(m)}(u) \in (u)^{m+2}$. Setting $g_i^{(m+1)} := g_i^{(m)} - h_i^{(m)}$, the sequence $(g_i^{(m)})_{m\geq 1}$ converges to g_i both (u)-adically and in the topology of $\mathcal{O}(D_0)$. By induction on m we claim that $g_i^{(m)}$ satisfies the required property on the valuation of its coefficients. This will prove the claim for g_i and follows by applying the subsequent lemma to $g_i^{(m)}(w_1(u),\ldots,w_{h-1}(u))$. □

Lemma 1.9. *If the power series $F(u), F_1(u),\ldots,F_{h-1}(u) \in (u) \subseteq \mathcal{O}(D_0)$ satisfy (6), then so does $F(F_1(u),\ldots,F_{h-1}(u))$.*

Proof. Write $F(u) = \sum_n b_n u^n$ and $F_i(u) = \sum_n c_n^{(i)} u^n$, so that

$$F(F_1(u),\ldots,F_{h-1}(u)) = \sum_n b_n F_1(u)^{n_1}\cdots F_{h-1}(u)^{n_{h-1}} = \sum_m c_m u^m,$$

where $c_m \in \breve{K}$ is a finite sum of terms of the form $c = b_n \prod_{i=1}^{h-1}\prod_{j=1}^{n_i} c^{(i)}_{n(i,j)}$ with $n, n(i,j) \in \mathbb{N}^{h-1} \setminus \{0\}$ and $\sum_{i=1}^{h-1}\sum_{j=1}^{n_i} |n(i,j)| = |m|$. By assumption,

$$\begin{aligned} v(c_m) &\geq \min_c\{v(b_n) + \sum_{i,j} v(c^{(i)}_{n(i,j)})\} > -\frac{|n|}{q} + \sum_{i,j}\frac{-|n(i,j)|+1}{q} \\ &= -\frac{|n|}{q} + \frac{-|m|+|n|}{q} = -\frac{|m|}{q}, \end{aligned}$$

as required. □

Let $\gamma = 1 + \Pi^j\alpha \in \Gamma$ with $\alpha \in \mathfrak{o}_h$ and $1 \leq j \leq h-1$, so that

$$(7) \qquad (1+\alpha w_{h-j})\cdot\gamma(w_i) = \begin{cases} w_i + \pi\alpha^{\sigma^i} w_{h-j+i}, & \text{if } 1 \leq i \leq j-1 \\ w_j + \pi\alpha^{\sigma^j}, & \text{if } i = j \\ w_i + \alpha^{\sigma^i} w_{i-j}, & \text{if } j+1 \leq i \leq h-1, \end{cases}$$

by Proposition 1.3 and since $\gamma = 1 + \alpha^{\sigma^j}\Pi^j$.

Lemma 1.10. *Let $\alpha \in \mathfrak{o}_h$, $1 \leq j \leq h-1$, and set $\gamma := 1 + \Pi^j\alpha \in \Gamma$. Fix an index i with $1 \leq i \leq h-1$ and write $g_i(u) = \sum_{n\in\mathbb{N}^{h-1}} b_n u^n$ with $b_n \in \breve{K}$. If $n, m \in \mathbb{N}^{h-1}$, and if $c_m \in \breve{K}$ denotes the coefficient of u^m in the power series expansion of $b_n\gamma(w_1)^{n_1}\cdot\ldots\cdot\gamma(w_{h-1})^{n_{h-1}}$, then*

$$v(c_m) > (v(\alpha) + 1 - \frac{1}{q})|n| - (v(\alpha)+1)|m|.$$

Proof. Writing $\gamma(w_s) = \sum_r c_r^{(s)} u^r$, $1 \leq s \leq h-1$, the coefficient c_m is a sum of terms of the form $c = b_n \prod_{s=1}^{h-1}\prod_{\ell=1}^{n_s} c^{(s)}_{r(\ell,s)}$ with $n, r(\ell,s) \in \mathbb{N}^{h-1}$ and $\sum_{s,\ell}|r(\ell,s)| = |m|$. Note that for $s \neq h-j$, $\gamma(w_s)$ satisfies (6) (cf. Lemma 1.7) and has trivial constant coefficient. Further, $\gamma(w_{h-j}) = \pi\alpha^{\sigma^j} + g$ where g satisfies (6) and $g(0) = 0$, as well. By omitting all factors $c^{(s)}_{r(\ell,s)}$ from c for which $s = h-j$ and $r(\ell,h-j) = 0$, we obtain

$$v(c) - (v(\alpha)+1)|\{\ell \mid r(\ell,h-j) = 0\}| > -\frac{|m| + |\{\ell \mid r(\ell,h-j)=0\}|}{q},$$

as in the proof of Lemma 1.9. Since

$$|m| = \sum_{s=1}^{h-1} \sum_{\ell=1}^{n_s} |r(\ell, s)| \geq |n| - |\{\ell \mid r(\ell, h-j) = 0\}|,$$

this yields

$$\begin{aligned} v(c_m) \geq \min_c\{v(c)\} &> -\frac{|m|}{q} + (v(\alpha) + 1 - \frac{1}{q})(|n| - |m|) \\ &= (v(\alpha) + 1 - \frac{1}{q})|n| - (v(\alpha) + 1)|m|. \end{aligned}$$

□

Theorem 1.11. *Let $\alpha \in \mathfrak{o}_h$, fix an integer j with $1 \leq j \leq h-1$, and set $\gamma := 1 + \Pi^j\alpha \in \Gamma$. For any integer ℓ with $1 \leq \ell \leq h-1$ and any fixed integer i with $1 \leq i \leq h-1$, let $\gamma(w_\ell)(u) = \sum_n c_n^{(\ell)} u^n$ and $g_i(u) = \sum_n b_n u^n$ be the power series expansions of the functions $\gamma(w_\ell)$ and g_i in the variables $u = (u_1, \ldots, u_{h-1})$. Fix a multi-index $m \in \mathbb{N}^{h-1}$ and set*

$$g_i'(u) := \sum_{|n| < \frac{v(\alpha)+1}{v(\alpha)+1-1/q} \cdot |m|} b_n u^n \quad \textit{and} \quad \gamma(w_\ell)'(u) := \sum_{|n| \leq |m|} c_n^{(\ell)} u^n.$$

If $m' \in \mathbb{N}^{h-1}$ with $|m'| \leq |m|$ then the coefficient of $u^{m'}$ in the power series expansion of $g_i'(\gamma(w_1)', \ldots, \gamma(w_{h-1})') \in \breve{K}[[u]]$ is contained in $\breve{\mathfrak{o}}$ and has the same reduction modulo $\pi\breve{\mathfrak{o}}$ as the corresponding coefficient of $\gamma(u_i)$. Equivalently, $g_i'(\gamma(w)')$ is congruent to $\gamma(u_i)$ in $\mathcal{O}(D_0)$ modulo the additive subgroup $\pi R + (u_1, \ldots, u_{h-1})^{m+1}$.

Proof. Obviously, the m'-th coefficient of $g_i'(\gamma(w_1)', \ldots, \gamma(w_{h-1})')$ coincides with that of $g_i'(\gamma(w_1), \ldots, \gamma(w_{h-1}))$. Therefore, the claim follows directly from Lemma 1.10 and the fact that by (5) we have $g_i(\gamma(w_1), \ldots, \gamma(w_{h-1})) = \gamma(u_i) \in R = \breve{\mathfrak{o}}[[u]]$. □

In other words, if $\gamma = 1 + \Pi^j\alpha \in \Gamma$ with $\alpha \in \mathfrak{o}_h$ and $1 \leq j \leq h-1$, then in order to compute the image $\overline{\gamma(u_i)}$ of $\gamma(u_i)$ in $\overline{R}$ up to degree d, it suffices to compute $\gamma(w_1), \ldots, \gamma(w_{h-1})$ up to degree d and to compute g_i up to degree strictly smaller than $d \cdot \frac{v(\alpha)+1}{v(\alpha)+1-1/q}$.

Example 1.12. Let $1 \leq j \leq h-1$, let $\xi \in \mu_{q^h-1} \subset \mathfrak{o}_h^*$, and let $\gamma := 1+\Pi^j\xi \in \Gamma$. In order to compute $\overline{\gamma(u_i)}$ up to degree $q+1$, we need to compute the power series $w_1, \ldots, w_{h-1}$ up to degree $q+1$, use (7), and compute the power series g_i up to degree strictly smaller than $\frac{q}{q-1}(q+1) = q+2+\frac{2}{q-1} \leq q+4$.

Proposition 1.13. *Let (u) denote the ideal of $\mathcal{O}(D_0)$ generated by $u_1, \ldots, u_{h-1}$, and set $u_0 := 0$. We have*

$$\begin{aligned} w_1(u) = \frac{\varphi_1(u)}{\varphi_0(u)} &\equiv u_1 - \frac{u_1^2 u_{h-1}^q}{\pi} + u_2 u_{h-1}^{q^2} \mod (u)^{q+4} \quad \textit{and} \\ w_i(u) = \frac{\varphi_i(u)}{\varphi_0(u)} &\equiv u_i + \frac{u_1 u_{i-1}^q}{\pi} - \frac{u_1 u_i u_{h-1}^q}{\pi} + \frac{u_2 u_{i-2}^{q^2}}{\pi} \mod (u)^{q+4}, \end{aligned}$$

for all $2 \le i \le h-1$. *Further,*

$$\begin{aligned} g_1(u) &\equiv u_1 + \frac{u_1^2 u_{h-1}^q}{\pi} - u_2 u_{h-1}^{q^2} \mod (u)^{q+4} \quad \textit{and} \\ g_i(u) &\equiv u_i - \frac{u_1 u_{i-1}^q}{\pi} + \frac{u_1 u_i u_{h-1}^q}{\pi} - \frac{u_2 u_{i-2}^{q^2}}{\pi} \mod (u)^{q+4}, \end{aligned}$$

for all $2 \le i \le h-1$. *Unless* $q = 2$, *all monomials of degree* q^2+1 *vanish modulo* $(u)^{q+4}$.

Proof. We first use Proposition 1.5 and Remark 1.6 to compute the first terms of the power series expansions of the functions φ_i, starting with $i = 0$. For later applications we will give better approximations as would be necessary for Example 1.12. We may of course assume $h \ge 2$.

We have $P_{0,0} = \{\{\emptyset\}_{j\ge 1}\}$, giving the term 1.

Because of the condition $0 \notin S_h$, any ordered partition $S = \{S_j\}_{j\ge1} \in P_{0,1}$ satisfies $|S| := \sum_{j=1}^h |S_j| \ge 2$. The partitions $S \in P_{0,1}$ with $|S| = 2$ give rise to the terms $\pi^{-1} u_i u_{h-i}^{q^i}$, $1 \le i \le h-1$. If $h \ge 3$ then there is a unique element $S \in P_{0,1}$ with $|S| = 3$ and $q(S) := \sum_{j=1}^{h-1} q(S_j) = 1 + q + q^2$. It is given by $S_1 = \{0,1\}$, $S_{h-2} = \{2\}$ (resp. the union of these if $h = 3$) and $S_j = \emptyset$, else. It gives rise to the term $\pi^{-2} u_1^{1+q} u_{h-2}^{q^2}$. All other elements $T \in P_{0,1}$ with $|T| \ge 3$ satisfy $q(T) \ge 1 + q + q^3$.

There is a unique element $S \in \bigcup_{\ell\ge2} P_{0,\ell}$ which minimizes $q(S)$. It is given by $S_1 = \{0\}$, $S_{h-1} = \{h+1\}$ (resp. the union of these if $h = 2$), $S_h = \{1\}$ and $S_j = \emptyset$, else. In particular, $S \in P_{0,2}$. The partition S gives rise to the term $\pi^{-1} u_1 u_{h-1}^{q^{h+1}}$. All other partitions $T \in \bigcup_{\ell\ge2} P_{0,\ell}$ satisfy $q(T) \ge 1 + q + q^{h+1} \ge 1 + q + q^3$. Altogether, we obtain

$$\begin{aligned} \varphi_0(u) &\equiv 1 + \frac{u_1 u_{h-1}^q}{\pi} + \frac{u_2 u_{h-2}^{q^2}}{\pi} + \frac{u_3 u_{h-3}^{q^3}}{\pi} \\ &\quad + \frac{u_1^{1+q} u_{h-2}^{q^2}}{\pi^2} + \frac{u_1 u_{h-1}^{q^{h+1}}}{\pi} \mod (u)^{1+q+q^3}, \end{aligned} \tag{8}$$

with the convention that $u_i = 0$ for $i > h$ or $i < 1$.

Next assume $i = 1$. The unique element of $P_{1,0}$ gives rise to the term u_1. The elements S of $P_{1,1}$ with $|S| = 2$ and $q(S) < 1 + q + q^3$ give rise to the terms $u_2 u_{h-1}^{q^2}$ and $u_3 u_{h-2}^{q^3}$. The unique element $S \in P_{1,1}$ with $|S| \ge 3$ and $q(S) < 1+q+q^3$ is given by $S_1 = \{0,1\}$, $S_{h-1} = \{2\}$ (resp. the union of these if $h = 2$) and $S_j = \emptyset$, else. It gives rise to the term $\pi^{-1} u_1^{1+q} u_{h-1}^{q^2}$. If $S \in \bigcup_{\ell\ge2} P_{1,\ell}$ then $(\ell-1)h + 1 \notin S_h$ implies $q(S) \ge q^{(\ell-1)h+2} \ge q^{h+2} \ge q^{1+q+q^3}$, so that

$$\varphi_1(u) \equiv u_1 + u_2 u_{h-1}^{q^2} + u_3 u_{h-2}^{q^3} + \frac{u_1^{1+q} u_{h-1}^{q^2}}{\pi} \mod (u)^{1+q+q^3}. \tag{9}$$

Assuming $h > 2$, a similar reasoning for $i \geq 2$ yields

$$(10) \qquad \begin{aligned} \varphi_i(u) \quad \equiv \quad & u_i + \frac{u_1 u_{i-1}^q}{\pi} + \frac{u_2 u_{i-2}^{q^2}}{\pi} + \frac{u_3 u_{i-3}^{q^3}}{\pi} \\ & + \frac{u_1^{1+q} u_{i-2}^{q^2}}{\pi} + u_{i+1} u_{h-1}^{q^{i+1}} \mod (u)^{1+q+q^3}. \end{aligned}$$

As a consequence,

$$\begin{aligned} w_1(u) = \frac{\varphi_1(u)}{\varphi_0(u)} \quad & \equiv \quad u_1 - \frac{u_1^2 u_{h-1}^q}{\pi} + u_2 u_{h-1}^{q^2} \mod (u)^{q+4} \quad \text{and} \\ w_i(u) = \frac{\varphi_i(u)}{\varphi_0(u)} \quad & \equiv \quad u_i + \frac{u_1 u_{i-1}^q}{\pi} - \frac{u_1 u_i u_{h-1}^q}{\pi} + \frac{u_2 u_{i-2}^{q^2}}{\pi} \mod (u)^{q+4}, \end{aligned}$$

if $2 \leq i \leq h-1$. We note that the monomials of degree q^2+1 vanish modulo $(u)^{q+4}$ if $q \neq 2$. Even for $q = 2$ they vanish modulo $(u)^{q+3}$. We thus obtain the uniform approximation

$$w_i(u) \equiv u_i + \frac{u_1 u_{i-1}^q}{\pi} - \frac{u_1 u_i u_{h-1}^q}{\pi} \mod (u)^{q+3}$$

for all $1 \leq i \leq h-1$.

If the polynomials $g_i'(u) \in \breve{K}[u]$ are defined by

$$\begin{aligned} g_1'(u) \quad & := \quad u_1 + \frac{u_1^2 u_{h-1}^q}{\pi} - u_2 u_{h-1}^{q^2} \quad \text{and} \\ g_i'(u) \quad & := \quad u_i - \frac{u_1 u_{i-1}^q}{\pi} + \frac{u_1 u_i u_{h-1}^q}{\pi} - \frac{u_2 u_{i-2}^{q^2}}{\pi} \text{ for } 2 \leq i \leq h-1, \end{aligned}$$

then a direct calculation using the above approximations of w_1 and w_i shows that $g_i'(w_1, \ldots, w_{h-1}) \equiv u_i \mod (u)^{q+4}$ for all $1 \leq i \leq h-1$. This proves $g_i' \equiv g_i \mod (u)^{q+4}$. $\square$

We will now approximate the action of Γ_1 on $\overline{R}$ in the $\overline{\mathfrak{m}}$-adic topology, restricting ourselves to representatives of Γ_1/Γ_h. If $\gamma \in \Gamma$, and if $1 \leq i \leq h-1$, then we denote by $\overline{\gamma(u_i)}$ the image of $\gamma(u_i)$ in $\overline{R}$. For simplicity, we will also assume $q \neq 2$, allowing $p = 2$, however.

Theorem 1.14. *Assume $q \neq 2$. Let $\xi \in \mu_{q^h-1} \subset \mathfrak{o}_h^*$, fix an integer j with $1 \leq j \leq h-1$, and let $\gamma_j := 1 + \Pi^j \xi \in \Gamma$. If $j = 1$ we have*

$$\begin{aligned} \overline{\gamma_1(u_1)} \cdot (1 + \xi u_{h-1}) \quad & \equiv \quad u_1 + \xi^{1+q} u_1 u_{h-2}^q + 2\xi^q u_1 u_{h-1}^q \mod \overline{\mathfrak{m}}^{q+2} \quad \textit{and} \\ \overline{\gamma_1(u_i)} \cdot (1 + \xi u_{h-1}) \quad & \equiv \\ & u_i + \xi^{q^i} u_{i-1} - \xi^q (u_{i-1}^q + \xi^{q^i} u_{i-2}^q) + \xi^q (u_i + \xi^{q^i} u_{i-1})(u_{h-1}^q + \xi u_{h-2}^q) \\ & - \sum_{\ell=1}^{q-1} \frac{1}{\pi} \binom{q}{\ell} u_1 u_{i-1}^{q-\ell} \xi^{\ell q^{i-1}} u_{i-2}^\ell \mod \overline{\mathfrak{m}}^{q+2}, \end{aligned}$$

for $2 \leq i \leq h-1$*. If* $1 < j < i \leq h-1$ *we have*

$$\begin{aligned}\overline{\gamma_j(u_i)} \cdot (1+\xi u_{h-j}) \quad &\equiv \quad u_i + \xi^{q^i} u_{i-j} - \xi^q u_{h-j+1}(u_{i-1}^q + \xi^{q^i} u_{i-1-j}^q) \\ &\qquad - \sum_{\ell=1}^{q-1} \frac{1}{\pi}\binom{q}{\ell} u_1 u_{i-1}^{q-\ell} \xi^{\ell q^{i-1}} u_{i-1-j}^{\ell} \mod \overline{\mathfrak{m}}^{q+2}.\end{aligned}$$

If $1 \leq i < j \leq h-1$ *or* $1 < i = j \leq h-1$ *we have*

$$\overline{\gamma_j(u_i)} \cdot (1+\xi u_{h-j}) \equiv u_i - \xi^q u_{h-j+1} u_{i-1}^q + \xi^{q^i} u_1 u_{h-j+i-1}^q \mod \overline{\mathfrak{m}}^{q+2}.$$

Proof. According to Theorem 1.11 and Example 1.12, the assertions follow from the following computations. As before, we denote by (u) the ideal of $\mathcal{O}(D_0)$ generated by $u_1, \ldots, u_{h-1}$. If $1 \leq i, j \leq h-1$ then

$$(1+\xi u_{h-j}+\xi\frac{u_1 u_{h-j-1}^q}{\pi})\gamma_j(w_i) \equiv \begin{cases} u_i + \frac{u_1 u_{i-1}^q}{\pi} + \pi\xi^{q^i}(u_{h-j+i} + \frac{u_1 u_{h-j+i-1}^q}{\pi}), & i < j, \\ \pi\xi^{q^i} + u_i + \frac{u_1 u_{i-1}^q}{\pi} & , i = j, \\ u_i + \frac{u_1 u_{i-1}^q}{\pi} + \xi^{q^i}(u_{i-j} + \frac{u_1 u_{i-j-1}^q}{\pi}) & , j < i, \end{cases}$$

modulo $(u)^{q+2}$ by (7) and Proposition 1.13. Further,

$$g_1(u) \equiv u_1 + \frac{u_1^2 u_{h-1}^q}{\pi} \mod (u)^{q+4}$$

by Proposition 1.13 and since $q \neq 2$. Let us first assume $h > 2$. Then

$$\begin{aligned}\gamma_1(w_1) \quad &\equiv \quad (1+\xi u_{h-1} + \xi\frac{u_1 u_{h-2}^q}{\pi})^{-1}(\pi\xi^q + u_1) \quad \text{and} \\ \gamma_1(w_{h-1}) \quad &\equiv \\ &(1+\xi u_{h-1} + \xi\frac{u_1 u_{h-2}^q}{\pi})^{-1}(u_{h-1} + \xi^{q^{h-1}} u_{h-2} + \frac{u_1 u_{h-2}^q}{\pi} + \xi^{q^{h-1}}\frac{u_1 u_{h-3}^q}{\pi})\end{aligned}$$

modulo $(u)^{q+2}$. We plug these approximations into the above approximation of $g_1(u)$. Modulo the additive subgroup $\pi R + (u)^{q+2}$ of $\mathcal{O}(D_0)$, the resulting power series then coincides with $\gamma_1(u_1)$ (cf. Theorem 1.11). Write $\overline{\gamma_1(u_1)} = \sum_{n \geq 0} c_n$ where $c_n \in k^{\mathrm{sep}}[[u]]$ is homogeneous of degree n.

Note that $\gamma_1(w_{h-1}) \in (u)$, so that $\gamma_1(u_1) \equiv \gamma_1(w_1) \mod \pi R + (u)^q$. Using the above approximation of $\gamma_1(w_1)$, one computes

$$\overline{\gamma_1(u_1)} \equiv \sum_{\ell=1}^{q-1} (-\xi)^{\ell-1} u_1 u_{h-1}^{\ell-1} \mod \overline{\mathfrak{m}}^q.$$

The q-th homogeneous component of $\gamma_1(w_1)$ is $\pi\xi^q(-\xi u_{h-1})^q + u_1(-\xi u_{h-1})^{q-1}$, whereas that of $\pi^{-1}\gamma_1(w_1)^2\gamma_1(w_{h-1})^q$ is $\pi^{-1}(\pi\xi^q)^2(u_{h-1} + \xi^{q^{h-1}} u_{h-2})^q$. It follows that $c_q = u_1(-\xi u_{h-1})^{q-1}$.

The $(q+1)$-th homogeneous component of $\gamma_1(w_1)$ is

$$\pi\xi^q(-\xi \cdot \frac{u_1 u_{h-2}^q}{\pi} + (-\xi u_{h-1})^{q+1}) + u_1(-\xi u_{h-1})^q,$$

whereas that of $\pi^{-1}\gamma_1(w_1)^2\gamma_1(w_{h-1})^q$ is

$$\frac{1}{\pi}(u_{h-1}+\xi^{q^{h-1}}u_{h-2})^q((\pi\xi^q)^2(-q-2)\xi u_{h-1}+2\pi\xi^q u_1).$$

Thus,

$$\begin{aligned} c_{q+1} &\equiv -\xi^{q+1}u_1u_{h-2}^q-\xi^q u_1u_{h-1}^q+2\xi^q u_1(u_{h-1}^q+\xi u_{h-2}^q) \\ &\equiv \xi^{q+1}u_1u_{h-2}^q+\xi^q u_1u_{h-1}^q. \end{aligned}$$

This proves the theorem for $j=i=1$ in the case $h>2$. The case $h=2$ is similar and will be treated more generally below (cf. Theorem 1.16).

For the rest of the proof assume $h>2$. If $i=2$ then

$$g_i(u)=g_2(u)\equiv u_2-\frac{u_1^{1+q}}{\pi}+\frac{u_1u_2u_{h-1}^q}{\pi} \quad \bmod (u)^{q+4}.$$

Using the above approximations for $\gamma_1(w_1)$ and $\gamma_1(w_{h-1})$, as well as

$$\gamma_1(w_2)\equiv(u_2+\xi^{q^2}u_1+\frac{u_1^{1+q}}{\pi})(1+\xi u_{h-1}+\xi\frac{u_1u_{h-2}^q}{\pi})^{-1} \quad \bmod (u)^{q+2},$$

we proceed as before. The first $q-1$ homogeneous components of $\pi^{-1}\gamma_1(w_1)^{q+1}$ are contained in πR. Further, $\gamma_1(w_2)$ and $\gamma_1(w_{h-1})$ are contained in (u). Therefore, the first $q-1$ homogeneous components of $\overline{\gamma_1(u_2)}$ are the reductions of those of $\gamma_1(w_2)$, i.e.

$$c_0=0 \qquad \text{and} \qquad c_n=(u_2+\xi^{q^2}u_1)(-\xi u_{h-1})^{n-1} \quad \text{for} \quad 1\le n\le q-1.$$

The homogeneous component of degree q of $\gamma_1(w_2)$ is $(u_2+\xi^{q^2}u_1)(-\xi u_{h-1})^{q-1}$. The q-th homogeneous component of $\pi^{-1}\gamma_1(w_1)^{q+1}$ is contained in R, as well, and is congruent to $\frac{1}{\pi}(q+1)\pi\xi^q u_1^q$ modulo πR. Hence,

$$c_q=(u_2+\xi^{q^2}u_1)(-\xi u_{h-1})^{q-1}-\xi^q u_1^q.$$

The homogeneous component of degree $q+1$ of $\gamma_1(w_2)$ is given by

$$(u_2+\xi^{q^2}u_1)(-\xi u_{h-1})^q+\pi^{-1}u_1^{1+q}.$$

Up to additive terms in πR, that of $\pi^{-1}\gamma_1(w_1)^{1+q}$ is $\pi^{-1}u_1^{1+q}-(q+1)\xi^{q+1}u_1^q u_{h-1}$, whereas that of $\pi^{-1}\gamma_1(w_1)\gamma_1(w_2)\gamma_1(w_{h-1})^q$ is $\xi^q(u_2+\xi^{q^2}u_1)(u_{h-1}+\xi^{q^{h-1}}u_{h-2})^q$. Thus,

$$c_{q+1}\equiv(u_2+\xi^{q^2}u_1)(-\xi u_{h-1})^q+\xi^q(u_2+\xi^{q^2}u_1)(u_{h-1}^q+\xi u_{h-2}^q)+\xi^{q+1}u_1^q u_{h-1}.$$

This proves the claim for $j=1$ and $i=2$.

If finally $j=1$ and $2<i\le h-1$ then

$$\begin{aligned} g_i(u) &\equiv u_i-\frac{u_1u_{i-1}^q}{\pi}+\frac{u_1u_iu_{h-1}^q}{\pi} \quad \bmod (u)^{q+4} \text{ and} \\ \gamma_1(w_{i-1}) &\equiv (u_{i-1}+\xi^{q^{i-1}}u_{i-2}+\frac{u_1u_{i-2}^q}{\pi}+\xi^{q^{i-1}}\frac{u_1u_{i-3}^q}{\pi})(1+\xi u_{h-1}+\xi\frac{u_1u_{h-2}^q}{\pi})^{-1}, \\ \gamma_1(w_i) &\equiv (u_i+\xi^{q^i}u_{i-1}+\frac{u_1u_{i-1}^q}{\pi}+\xi^{q^i}\frac{u_1u_{i-2}^q}{\pi})(1+\xi u_{h-1}+\xi\frac{u_1u_{h-2}^q}{\pi})^{-1}, \end{aligned}$$

modulo $(u)^{q+2}$ because $1 < i-1$. Together with the above approximations of $\gamma_1(w_1)$ and $\gamma_1(w_{h-1})$, we plug these approximations of $\gamma_1(w_{i-1})$ and $\gamma_1(w_i)$ into the given approximation of $g_i(u)$. Computing modulo the additive subgroup $\pi R + (u)^{q+2}$ and referring to Theorem 1.11, we find $c_0 = 0$, $c_n = (u_i + \xi^{q^i} u_{i-1})(-\xi u_{h-1})^{n-1}$ for $1 \le n \le q-1$,

$$\begin{aligned} c_q &\equiv (u_i + \xi^{q^i} u_{i-1})(-\xi u_{h-1})^{q-1} - \frac{1}{\pi}\pi\xi^q(u_{i-1} + \xi^{q^{i-1}} u_{i-2})^q \\ &\equiv (u_i + \xi^{q^i} u_{i-1})(-\xi u_{h-1})^{q-1} - \xi^q(u_{i-1}^q + \xi^{q^i} u_{i-2}^q), \end{aligned}$$

and

$$\begin{aligned} c_{q+1} &\equiv (u_i + \xi^{q^i} u_{i-1})(-\xi u_{h-1})^q + \frac{u_1 u_{i-1}^q}{\pi} + \xi^{q^i} \frac{u_1 u_{i-2}^q}{\pi} \\ &\quad - \frac{1}{\pi}(u_{i-1} + \xi^{q^{i-1}} u_{i-2})^q(\pi\xi^q(q+1)(-\xi u_{h-1}) + u_1) \\ &\quad + \frac{1}{\pi}\pi\xi^q(u_i + \xi^{q^i} u_{i-1})(u_{h-1} + \xi^{q^{h-1}} u_{h-2})^q \\ &\equiv (u_i + \xi^{q^i} u_{i-1})(-\xi u_{h-1})^q + \xi^{q+1} u_{h-1}(u_{i-1}^q + \xi^{q^i} u_{i-2}^q) \\ &\quad + \xi^q(u_i + \xi^{q^i} u_{i-1})(u_{h-1}^q + \xi u_{h-2}^q) - \sum_{\ell=1}^{q-1} \frac{1}{\pi}\binom{q}{\ell} u_1 u_{i-1}^{q-\ell} \xi^{\ell q^{i-1}} u_{i-2}^{\ell}. \end{aligned}$$

This proves the claim for $j = 1$ and $2 < i \le h-1$.

For $j > 1$, the proof proceeds by distinguishing a long list of cases. First, one treats the case $h = 3$ and then assumes $h > 3$. If $i = 1$, one has to distinguish the two cases $j = h-1$ and $1 < j < h-1$. If $i = 2$ one has to distinguish the three cases $j \in \{2, h-1\}$ and $2 < j < h-1$. If $i = h-1$ one has to distinguish the three cases $j \in \{h-2, h-1\}$ and $1 < j < h-2$. If $2 < i < h-1$ one has to distinguish the five cases $j \in \{i-1, i, h-1\}$, $1 < j < i-1$ and $i < j < h-1$. We will only present the last two cases. The formulae that we obtain specialize to the correct formulae in all other cases. We leave the verification of this assertion to the reader.

Assume $h > 3$, $2 < i < h-1$ and $i < j < h-1$. We have the same approximation of $g_i(u)$ as before. Further,

$$\begin{aligned} \gamma_j(w_1) &\equiv u_1 + \pi\xi^q u_{h-j+1} \mod (u)^2, \\ \gamma_j(w_{i-1}) &\equiv u_{i-1} + \pi\xi^{q^{i-1}} u_{h-j+i-1} \mod (u)^2, \\ \gamma_j(w_{h-1}) &\equiv u_{h-1} + \xi^{q^{h-1}} u_{h-1-j} \mod (u)^2, \end{aligned}$$

as well as

$$\gamma_j(w_i) \equiv (u_i + \pi\xi^{q^i} u_{h-j+i} + \frac{u_1 u_{i-1}^q}{\pi} + \xi^{q^i} u_1 u_{h-j+i-1}^q)(1 + \xi u_{h-j} + \xi\frac{u_1 u_{h-j-1}^q}{\pi})^{-1}$$

modulo $(u)^{q+2}$. Thus, $\gamma_j(w_1)\gamma_j(w_{i-1})^q \in (u)^{q+1}$ and $\gamma_j(w_1)\gamma_j(w_i)\gamma_j(w_{h-1})^q \in (u)^{q+2}$. As above, we obtain $c_0 = 0$, $c_n = u_i(-\xi u_{h-j})^{n-1}$, for $1 \le n \le q$, and

$$\begin{aligned} c_{q+1} &\equiv u_i(-\xi u_{h-j})^q + \frac{u_1 u_{i-1}^q}{\pi} + \xi^{q^i} u_1 u_{h-j+i-1}^q \\ &\quad - \frac{(u_1 + \pi\xi^q u_{h-j+1})(u_{i-1} + \pi\xi^{q^{i-1}} u_{h-j+i-1})^q}{\pi} \\ &\equiv u_i(-\xi u_{h-j})^q + \xi^{q^i} u_1 u_{h-j+i-1}^q - \xi^q u_{h-j+1} u_{i-1}^q. \end{aligned}$$

If $2 < i < h-1$ and $1 < j < i-1$, the approximations for $\gamma_j(w_{i-1})$ and $\gamma_j(w_i)$ have to be replaced by $\gamma_j(w_{i-1}) \equiv u_{i-1} + \xi^{q^{i-1}} u_{i-1-j} \mod (u)^2$ and

$$\gamma_j(w_i) \equiv (u_i + \xi^{q^i} u_{i-j} + \frac{u_1 u_{i-1}^q}{\pi} + \xi^{q^i} \frac{u_1 u_{i-j-1}^q}{\pi})(1 + \xi u_{h-j} + \xi \frac{u_1 u_{h-j-1}^q}{\pi})^{-1}$$

modulo $(u)^{q+2}$. Following the same procedure as above, we obtain $c_0 = 0$, $c_n = (u_i + \xi^{q^i} u_{i-j})(-\xi u_{h-j})^{n-1}$, for $1 \le n \le q$, and

$$\begin{aligned} c_{q+1} &\equiv (u_i + \xi^{q^i} u_{i-j})(-\xi u_{h-j})^q + \frac{u_1 u_{i-1}^q}{\pi} + \xi^{q^i} \frac{u_1 u_{i-j-1}^q}{\pi} \\ &\quad - \frac{(u_1 + \pi\xi^q u_{h-j+1})(u_{i-1} + \xi^{q^{i-1}} u_{i-1-j})^q}{\pi} \\ &\equiv (u_i + \xi^{q^i} u_{i-j})(-\xi u_{h-j})^q - \xi^q u_{h-j+1}(u_{i-1}^q + \xi^{q^i} u_{i-1-j}^q) \\ &\quad - \sum_{\ell=1}^{q-1} \frac{1}{\pi} \binom{q}{\ell} u_1 u_{i-1}^{q-\ell} \xi^{\ell q^{i-1}} u_{i-1-j}^{\ell}. \end{aligned}$$

This completes the proof. $\square$

Example 1.15. Let $\xi \in \mu_{q^h-1} \subset \mathfrak{o}_h^*$, and let $\gamma := 1 + \Pi^{h-1}\xi \in \Gamma$. If $h = 2$, we will improve the approximations of Theorem 1.14 by computing the action of γ on $\overline{R}$ modulo $\overline{\mathfrak{m}}^{2q+2}$. By Theorem 1.11, this requires to compute the power series $w_1(u_1)$ modulo $u_1^{2q+2}\mathcal{O}(D_0)$. Further, the power series $g_1(u_1)$ has to be computed modulo $(u_1)^{d+1}$ with $d < \frac{q}{q-1}(2q+1) = 2q + 3 + \frac{3}{q-1}$.

Theorem 1.16. *Assume $h = 2$. Let $\xi \in \mu_{q^h-1} \subset \mathfrak{o}_h^*$, and let $\gamma := 1 + \Pi\xi \in \Gamma$. For $-1 \le n \le q$ let $a_n := \sum_{i=1}^{n} \frac{q}{i}(n-i+1) \in \breve{\mathfrak{o}}$, so that $a_{-1} = a_0 = 0$ and $a_n \in p\breve{\mathfrak{o}} \subseteq \pi\breve{\mathfrak{o}}$, if $n < q$. Writing the image of $\gamma(u_1)$ in $\overline{R} \simeq k^{sep}[[u_1]]$ as $\sum_{n=0}^{\infty} c_n u_1^n$ with $c_n \in k^{sep}$ for all $n \ge 0$, we have*

$$\begin{aligned} c_0 &= 0, \\ c_n &\equiv (-\xi)^{n-1} \textit{ for } 1 \le n \le q, \\ c_n &\equiv -n(-\xi)^{n-1} + \frac{a_{n-q-2}}{\pi}(-\xi)^{n-q-2} \textit{ for } q+1 \le n \le 2q \textit{ and} \\ c_{2q+1} &\equiv \frac{a_{q-1}}{\pi}(-\xi)^{q-1}. \end{aligned}$$

Proof. Set $u := u_1$, $w := w_1$ and $g := g_1$. According to (8) and (9) we have

$$\begin{aligned} \varphi_0(u) &\equiv 1 + \frac{u^{1+q}}{\pi} + \frac{u^{1+q^3}}{\pi} \mod (u)^{1+q+q^3} \quad \text{and} \\ \varphi_1(u) &\equiv u + u^{q^2} + \frac{u^{1+q+q^2}}{\pi} \mod (u)^{1+q+q^3}. \end{aligned}$$

Let us first assume that $q > 3$, so that we need to compute $g(u)$ up to degree $2q + 3$ (cf. Example 1.15). If $q > 3$ we have $q^2 > 2q + 3$ and obtain

$$\begin{aligned} w(u) &\equiv u - \frac{u^{2+q}}{\pi} + \frac{u^{2q+3}}{\pi^2} \mod (u)^{2q+4} \quad \text{and} \\ g(u) &\equiv u + \frac{u^{2+q}}{\pi} + (q+1)\frac{u^{2q+3}}{\pi^2} \mod (u)^{2q+4}. \end{aligned}$$

We need to compute $\gamma(w(u))$ modulo $(u)^{2q+2}$, plug the corresponding truncation into the above approximation of $g(u)$ and compute the image of the resulting power series modulo the additive subgroup $\pi R + (u)^{2q+2}$ of $\mathcal{O}(D_0)$. According to Theorem 1.11 this image coincides with that of $\gamma(u)$.

By (7) we have $\gamma(w) = (\pi\xi^q + w)(1 + \xi w)^{-1}$, where

$$\begin{aligned} \pi\xi^q + w &\equiv \pi\xi^q + u - \frac{u^{2+q}}{\pi} \mod (u)^{2q+2} \text{ and} \\ (1+\xi w)^{-1} &\equiv (1 + \xi u - \xi\frac{u^{2+q}}{\pi})^{-1} \\ &\equiv \sum_{n=0}^{2q+1}(-\xi u)^n - \sum_{n=0}^{q} n(-\xi)^n\frac{u^{1+n+q}}{\pi} \mod (u)^{2q+2}. \end{aligned}$$

Therefore, modulo πR, the first $2q+1$ homogeneous components of $\gamma(w(u))$ are given by $\pi\xi^q u^0$, $(\pi\xi^q(-\xi)^n + (-\xi)^{n-1})u^n$ for $1 \le n \le q+1$, and

$$\begin{aligned} & u^n \cdot \pi\xi^q[(-\xi)^n - (n-q-1)\frac{(-\xi)^{n-q-1}}{\pi}] \\ & +u[(-\xi)^{n-1} - (n-q-2)\frac{(-\xi)^{n-q-2}}{\pi}]u^{n-1} - \frac{1}{\pi}(-\xi)^{n-q-2}u^n \\ \equiv\ & [(n-q)(-\xi)^{n-1} - (n-q-1)\frac{(-\xi)^{n-q-2}}{\pi}]u^n, \end{aligned}$$

for $q + 2 \le n \le 2q + 1$.

Next, we will compute $\frac{1}{\pi}\gamma(w)^{q+2}$ in $\mathcal{O}(D_0)$ modulo the additive subgroup $\pi R + (u)^{q+2}$. Note first that

$$(1+\xi u - \frac{\xi u^{2+q}}{\pi})^{-(2+q)} \equiv (1+\xi u)^{-(2+q)}(1+(2+q)\xi\frac{u^{2+q}}{\pi}(1+\xi u)^{-1}) \mod (u)^{2q+2}$$

with $(1 + \xi u)^{-(2+q)} \in R$, and that

$$\begin{aligned} & \frac{1}{\pi}(\pi\xi^q + u - \frac{u^{2+q}}{\pi})^{2+q}(1 + (2+q)\xi\frac{u^{2+q}}{\pi}(1+\xi u)^{-1}) \\ \equiv\ & \frac{1}{\pi}(\pi\xi^q + u)^{2+q} - (2+q)(\pi\xi^q + u)^{1+q}\frac{u^{2+q}}{\pi^2} \\ & +(q+2)\xi\frac{u^{2+q}}{\pi^2}(1+\xi u)^{-1}(\pi\xi^q + u)^{2+q} \mod \pi R + (u)^{2q+2} \\ \equiv\ & \frac{u^{2+q}}{\pi} + (2+q)\xi^q u^{1+q} \mod \pi R + (u)^{2q+2}. \end{aligned}$$

Since this power series has u-order $q+1$ and since the valuations of its coefficients are all at least -1, it now suffices to compute $(1+\xi u)^{-(2+q)}$ modulo $\pi^2 R+(u)^{1+q}$. Note that $(1+\xi u)^{-2} = \sum_{n=0}^{\infty}(n+1)(-\xi u)^n$, so that we obtain

$$(1+\xi u)^{-(2+q)} \equiv (1-\sum_{i=1}^{q}\binom{q}{i}\xi^i u^i)\cdot\sum_{n=0}^{q}(n+1)(-\xi u)^n \mod \pi^2 R+(u)^{1+q}.$$

If $0\le n\le q$ then the n-th coefficient of this power series is

$$\begin{aligned} & (n+1)(-\xi)^n - \sum_{i=1}^{n}\binom{q}{i}\xi^i(n-i+1)(-\xi)^{n-i} \\ \equiv\ & (n+1+a_n)(-\xi)^n \mod \pi^2\breve{\mathfrak{o}}, \end{aligned}$$

because $\binom{q}{i}\equiv\frac{q}{i}(-1)^{i-1} \mod \pi^2\breve{\mathfrak{o}}$ if $1\le i\le q$. As a consequence, modulo πR, the first $2q+1$ homogeneous components of $\pi^{-1}\gamma(w)^{2+q}$ are given by $0\cdot u^n$ for $0\le n\le q$, $2\xi^q u^n$ for $n=1+q$, and

$$[(n-q-1+a_{n-q-2})\frac{(-\xi)^{n-q-2}}{\pi} - 2(n-q+a_{n-q-1})(-\xi)^{n-1}]u^n$$

for $2+q\le n\le 2q+1$.

As for the term $(q+1)\pi^{-2}\gamma(w)^{2q+3}$, note that

$$\begin{aligned} & (1+\xi u-\xi\frac{u^{2+q}}{\pi})^{-(2q+3)} \\ \equiv\ & (1+\xi u)^{-(2q+3)}(1+(2q+3)\frac{\xi u^{2+q}}{\pi}(1+\xi u)^{-1}) \mod (u)^{2q+2} \\ \equiv\ & 1-(2q+3)\xi u \mod (u)^2. \end{aligned}$$

The first congruence shows that in order to determine $(q+1)\pi^{-2}\gamma(w)^{2q+3} \mod \pi R+(u)^{2q+2}$, it suffices to determine $\frac{q+1}{\pi^2}(\pi\xi^q+u-\frac{u^{2+q}}{\pi})^{2q+3} \mod \pi^2 R+(u)^{2q+2}$. We have

$$\begin{aligned} & \frac{q+1}{\pi^2}(\pi\xi^q+u-\frac{\xi u^{2+q}}{\pi})^{2q+3} \\ \equiv\ & \frac{1+q}{\pi^2}(\pi\xi^q+u)^{2q+3} - \frac{(q+1)(2q+3)}{\pi^3}(\pi\xi^q+u)^{2q+2}\xi u^{2+q} \mod (u)^{2q+2} \\ \equiv\ & (q+1)\binom{2q+3}{2}\xi^{2q}u^{2q+1} + \pi\xi^{3q}(q+1)\binom{2q+3}{3}u^{2q} \mod \pi^2 R+(u)^{2q+2}. \end{aligned}$$

As a consequence,

$$\begin{aligned} (q+1)\frac{\gamma(w)^{2q+3}}{\pi^2} &\equiv (q+1)\xi^{2q}\binom{2q+3}{2}u^{2q+1} \\ &\equiv 3\xi^{2q}u^{2q+1} \mod \pi R+(u)^{2q+2}. \end{aligned}$$

Combining our results, we obtain $c_0=0$, $c_n\equiv(-\xi)^{n-1}$ for $1\le n\le q$, $c_{q+1}\equiv(-\xi)^q+2\xi^q=\xi^q$, and

$$\begin{aligned} c_n \equiv\ & (n-q)(-\xi)^{n-1} - (n-q-1)\frac{(-\xi)^{n-q-2}}{\pi} \\ & +(n-q-1+a_{n-q-2})\frac{(-\xi)^{n-q-2}}{\pi} - 2(n-q+a_{n-q-1})(-\xi)^{n-1} \\ \equiv\ & -n(-\xi)^{n-1} + \frac{a_{n-q-2}}{\pi}(-\xi)^{n-q-2},\ \text{for } q+2\le n\le 2q, \end{aligned}$$

because $a_n \in \pi\breve{\mathfrak{o}}$ for $n < q$. Finally,

$$\begin{aligned} c_{2q+1} &\equiv (q+1)(-\xi)^{2q} - q\frac{(-\xi)^{q-1}}{\pi} \\ &\quad +(q+a_{q-1})\frac{(-\xi)^{q-1}}{\pi} - 2(q+1+a_q)(-\xi)^{2q} + 3\xi^{2q} \\ &\equiv \frac{a_{q-1}}{\pi}(-\xi)^{q-1}, \end{aligned}$$

because $a_q \equiv 1 \mod \pi\breve{\mathfrak{o}}$. Note that the above formula for c_n, $q+2 \leq n \leq 2q$, specializes to ξ^q if n is formally put equal to $q+1$. This finishes the proof if $q > 3$.

For $q = 3$ we need to approximate $g(u)$ up to degree 10 (cf. Example 1.15). Using the approximation of w given at the beginning of the proof, we find

$$\begin{aligned} w(u) &\equiv u - \frac{u^5}{\pi} + (1+\frac{1}{\pi^2})u^9 \mod (u)^{13} \text{ and} \\ g(u) &\equiv u + \frac{u^5}{\pi} - (1-\frac{4}{\pi^2})u^9 \mod (u)^{11}, \end{aligned}$$

if $q = 3$. If $q = 2$ we need to approximate $g(u)$ up to degree 9 and find

$$\begin{aligned} w(u) &\equiv u + (1-\frac{1}{\pi})u^4 + (\frac{1}{\pi^2}+\frac{1}{\pi})u^7 \mod (u)^{10} \text{ and} \\ g(u) &\equiv u - (1-\frac{1}{\pi})u^4 + (4-\frac{7}{\pi}+\frac{3}{\pi^2})u^7 \mod (u)^{10}. \end{aligned}$$

A straightforward computation shows that the asserted formulae for the first $2q+1$ coefficients of $\overline{\gamma(u)}$ are also valid in these exceptional cases. $\square$

Remark 1.17. If $\mathrm{char}(K) = p > 0$ then $q = 0$ in $\breve{\mathfrak{o}}$ and $a_n = 0$ for all $n < q$. If $\mathrm{char}(K) = 0$, and if $K|\mathbb{Q}_p$ is ramified, then $p \in \pi^2\breve{\mathfrak{o}}$ and we have $\frac{a_n}{\pi} \in \pi\breve{\mathfrak{o}}$ for all $n < q$. For $K = \mathbb{Q}_p$ and $\pi = p$ we have $q = p$ and hence $a_1 = p$. Further, the coefficient a_{p-1} can be computed to

$$a_{p-1} = \sum_{i=1}^{p-1} \frac{p}{i}(p-i) \equiv -p(p-1) \equiv p \mod p^2\breve{\mathfrak{o}}.$$

This implies $c_{2q+1} = 1$ if $K = \mathbb{Q}_p$.

Remark 1.18. If $h = 2$ and $K = \mathbb{Q}_2$ then the computations leading to the result in Theorem 1.16 are simple enough to be carried even further. If $\gamma = 1 + \Pi\xi$ is as before and $u := u_1$, we find

$$\overline{\gamma(u)} \equiv u + \xi u^2 + \xi^2 u^3 + \xi u^5 + u^7 + \xi^2 u^9 \mod \overline{\mathfrak{m}}^{10}.$$

To give another concrete example, if $K = \mathbb{Q}_5$ then the approximation of $\overline{\gamma(u)}$ in Theorem 1.16 reads

$$\begin{aligned} \overline{\gamma(u)} &\equiv u - \xi u^2 + \xi^2 u^3 - \xi^3 u^4 + \xi^4 u^5 + \xi^5 u^6 \\ &\quad -2\xi^6 u^7 + (3\xi^7 - \xi)u^8 - 4\xi^8 u^9 - \xi^3 u^{10} + \xi^4 u^{11} \mod \overline{\mathfrak{m}}^{12}. \end{aligned}$$

We end this section by considering the action of the subgroup $\mathfrak{o}_h^* \subseteq \Gamma$ on R, which can be approximated in the (u)-adic topology. Here (u) denotes the ideal of R generated by $u_1, \ldots, u_{h-1}$.

Theorem 1.19. *Assume $q \neq 2$, let $\alpha \in \mathfrak{o}_h^* \subseteq \Gamma$, and consider α also as an element of $\breve{K}$. For $0 \leq j \leq h-1$ set $\alpha_j := \alpha^{\sigma^j - 1}$. If $1 \leq i \leq h-1$ we have*

$$\alpha(u_i) \equiv \alpha_i u_i + \frac{\alpha_i - \alpha_1 \alpha_{i-1}^q}{\pi} u_1 u_{i-1}^q + \frac{\alpha_i(\alpha_1 \alpha_{h-1}^q - 1)}{\pi} u_1 u_i u_{h-1}^q \mod (u)^{q+3}.$$

In particular, if $\alpha = 1 + \xi\pi$ for some element $\xi \in \mu_{q^h-1} \subset \mathfrak{o}_h^*$, then

$$\overline{\alpha(u)} \equiv u_i + (\xi^{q^i} - \xi^q) u_1 u_{i-1}^q + (\xi^q - \xi) u_1 u_i u_{h-1}^q \mod \overline{\mathfrak{m}}^{q+3}.$$

If $h = 2$ and $q > 3$ we have

$$\begin{aligned} \alpha(u_1) &\equiv \alpha_1 u_1 + \frac{\alpha_1^{q+2} - \alpha_1}{\pi} u_1^{q+2} \\ &\quad + \frac{\alpha_1 - (q+2)\alpha_1^{q+2} + (q+1)\alpha_1^{2q+3}}{\pi^2} u_1^{2q+3} \mod (u)^{2q+4}. \end{aligned}$$

In particular, if $\alpha = 1 + \xi\pi$ for some element $\xi \in \mu_{q^2-1} \subset \mathfrak{o}_2^*$, then

$$\overline{\alpha(u_1)} \equiv u_1 + (\xi^q - \xi) u_1^{q+2} + \left(\frac{q}{\pi}(\xi^q - \xi) + (\xi^q - \xi)^2\right) u_1^{2q+3} \mod \overline{\mathfrak{m}}^{2q+4}.$$

Proof. The reason that the action is so much simpler to compute in this situation is that $\alpha(w_j) = \alpha_j w_j$ has trivial constant coefficient for all j (cf. Proposition 1.3 and Proposition 1.13). By (5) and Proposition 1.13 this implies

$$\begin{aligned} \alpha(u_i) &= g_i(\alpha(w_1), \ldots, \alpha(w_{h-1})) \\ &\equiv \alpha_i w_i - \frac{\alpha_1 \alpha_{i-1}^q}{\pi} w_1 w_{i-1}^q + \frac{\alpha_1 \alpha_i \alpha_{h-1}^q}{\pi} w_1 w_i w_{h-1}^q \mod (u)^{q+3} \\ &\equiv \alpha_i \left(u_i + \frac{u_1 u_{i-1}^q}{\pi} - \frac{u_1 u_i u_{h-1}^q}{\pi}\right) - \frac{\alpha_1 \alpha_{i-1}^q}{\pi} u_1 u_{i-1}^q \\ &\quad + \frac{\alpha_1 \alpha_i \alpha_{h-1}^q}{\pi} u_1 u_i u_{h-1}^q \mod (u)^{q+3}, \end{aligned}$$

proving the first claim.

If $h = 2$ then we use the better approximation of $g(u)$ appearing in the proof of Theorem 1.16 and proceed as above.

If h is arbitrary, and if $\alpha = 1 + \xi\pi$, then

$$\alpha_j = \frac{1 + \xi^{q^j}\pi}{1 + \xi\pi} \equiv 1 + (\xi^{q^j} - \xi)\pi + (\xi^2 - \xi^{q^j+1})\pi^2 \mod \pi^3 \breve{\mathfrak{o}}.$$

This leads to the required formulae for $\overline{\alpha(u)}$. □

2 The action of the spherical Hecke algebra

Recall that we fixed a one dimensional formal $\mathfrak{o}$-module $\overline{\mathbb{H}}$ over k^{sep} which is defined over k. The deformation problem of section 1 can be generalized as follows (cf. [9], §4, and [31]). If $(S, \mathfrak{m}_S)$ is a complete noetherian commutative unital $\breve{\mathfrak{o}}$-algebra with residue class field $S/\mathfrak{m}_S \simeq k^{\text{sep}}$, and if m is a non-negative integer, then one considers isomorphism classes $[(H, \iota, \varphi)]$ of triples (H, ι, φ) where H is a one dimensional formal $\mathfrak{o}$-module of height h over S, $\iota : H \mod \mathfrak{m}_S \to \overline{\mathbb{H}}$ is a quasi-isogeny of arbitrary height and φ is a so-called *level-m structure*. The latter means that $\varphi : \pi^{-m}\mathfrak{o}^h/\mathfrak{o}^h \to (\mathfrak{m}_S, +_H)$ is a homomorphism of abstract $\mathfrak{o}$-modules such that the power series $\prod_{\alpha \in \pi^{-m}\mathfrak{o}^h/\mathfrak{o}^h}(X - \varphi(\alpha))$ divides the power series $[\pi^m]_H$ in $S[[X]]$. Here $(\alpha \mapsto [\alpha]_H)$ denotes the map $\mathfrak{o} \to \mathrm{End}(H)$ defining the $\mathfrak{o}$-linear structure of H.

The generalized deformation problem just described is represented by a formal scheme $\mathfrak{X}_m$ which decomposes as a disjoint union

$$\mathfrak{X} = \coprod_{n \in \mathbb{Z}} \mathfrak{X}_{m,n}$$

of open affine formal subschemes $\mathfrak{X}_{m,n} = \mathrm{Spf}(R_{m,n})$ (cf. [9], Proposition 4.3 and [26], Proposition 3.79). There are non-canonical isomorphisms $R_{m,n} \simeq R_{m,n'}$ for any two integers n, n', and $\mathfrak{X}_{0,0} \simeq \mathrm{Spf}(R)$ with R as in section 1. If m and m' are two non-negative integers with $m' \geq m$ then there is a finite flat morphism $\mathfrak{X}_{m'} \to \mathfrak{X}_m$ which is compatible with the above decompositions. The collection of these morphisms gives the family $(\mathfrak{X}_m)_{m \geq 0}$ the structure of a projective system of formal $\mathfrak{o}$-schemes.

The local ring $R_m := R_{m,0}$ is regular, and if $(e_1, \ldots, e_h)$ is an $\mathfrak{o}/\pi^m\mathfrak{o}$-basis of $\pi^{-m}\mathfrak{o}^h/\mathfrak{o}^h$, then its image $(\varphi_m(e_1), \ldots, \varphi_m(e_h))$ under the universal level-m-structure $\varphi_m : \pi^{-m}\mathfrak{o}^h/\mathfrak{o}^h \to (\mathfrak{m}_{R_m}, +_{\mathbb{H}})$ is a system of regular coordinates of R_m (cf. [9], Proposition 4.3).

The group of quasi-isogenies of $\overline{\mathbb{H}}$ is isomorphic to D^* and functorially acts on each of the spaces $\mathfrak{X}_m$ over $\mathrm{Spf}(\breve{\mathfrak{o}})$. In fact, if $\delta \in D^*$, then the morphism $\delta : \mathfrak{X}_m \to \mathfrak{X}_m$ restricts to an isomorphism $\mathfrak{X}_{m,n} \to \mathfrak{X}_{m,n-hv_D(\delta)}$ for any $n \in \mathbb{Z}$. Note that we assume v_D to extend v so that $v_D(\Pi) = 1/h$. The induced action of $\mathfrak{o}_D^*$ on $R \simeq \mathcal{O}(\mathfrak{X}_{0,0})$ coincides with the one considered in section 1. The morphisms $\mathfrak{X}_{m'} \to \mathfrak{X}_m$ are D^*-equivariant. Further, there is a functorial action of $G := \mathrm{GL}_h(K)$ on the projective system $(\mathfrak{X}_m)_{m \geq 0}$ which induces an $\breve{\mathfrak{o}}$-linear action on $\varinjlim_m \mathcal{O}(\mathfrak{X}_m)$ (cf. [31], section 2.2.2). For any integer n, the ring $R_{\infty,n} := \varinjlim_m R_{m,n}$ is stable under the subgroup $G^0 := \{g \in \mathrm{GL}_h(K) \mid \det(g) \in \mathfrak{o}^*\}$, and if $G_m := \ker(\mathrm{GL}_h(\mathfrak{o}) \to \mathrm{GL}_h(\mathfrak{o}/\pi^m\mathfrak{o}))$ then $R_{\infty,n}^{G_m} = R_{m,n}$ for any $m \geq 0$. In fact, the ring extension $R_{\infty,n}[\frac{1}{\pi}]|R_{0,n}[\frac{1}{\pi}]$ is étale and Galois with Galois group $G_0 = \mathrm{GL}_h(\mathfrak{o})$ (cf. [31], Theorem 2.1.2). The actions of G and D^* on $\varinjlim_m \mathcal{O}(\mathfrak{X}_m)$ commute with each other.

We let $\text{c-Ind}_{G_0}^G(\breve{\mathfrak{o}})$ be the $\breve{\mathfrak{o}}$-module of all functions $f : G \to \breve{\mathfrak{o}}$ with compact support, satisfying $f(gg_0) = f(g)$ for all $g \in G$ and $g_0 \in G_0$. It is an $\breve{\mathfrak{o}}$-linear representation of G via $g(f)(g') := f(g^{-1}g')$ for $f \in \text{c-Ind}_{G_0}^G(\breve{\mathfrak{o}})$ and $g, g' \in G$. We let $\mathcal{H} := \mathrm{End}_G(\text{c-Ind}_{G_0}^G(\breve{\mathfrak{o}}))$ be the ring of G-equivariant and $\breve{\mathfrak{o}}$-linear endomorphisms of $\text{c-Ind}_{G_0}^G(\breve{\mathfrak{o}})$. The $\breve{\mathfrak{o}}$-algebra $\mathcal{H}$ is called the *spherical Hecke algebra of G over* $\breve{\mathfrak{o}}$ and is $\breve{\mathfrak{o}}$-linearly isomorphic to the module of all compactly supported G_0-biinvariant functions from G to $\breve{\mathfrak{o}}$. Its structure can be made explicit as follows. For any integer i with $0 \leq i \leq h-1$ let $t_i := \mathrm{diag}(1, \ldots, 1, \pi, \ldots, \pi) \in G$ be the diagonal matrix whose first i diagonal entries (counted from top to bottom) are equal to 1 and the remaining ones equal to π. Because of the Cartan decomposition

$$G = \coprod_{n_0 \in \mathbb{Z},\ n_1, \ldots n_{h-1} \in \mathbb{Z}_{\geq 0}} G_0 t_0^{n_0} t_1^{n_1} \cdots t_{h-1}^{n_{h-1}} G_0,$$

$\mathcal{H}$ is a free $\breve{\mathfrak{o}}$-module, a basis of which is given by the characteristic functions of the double cosets $G_0 t_0^{n_0} t_1^{n_1} \cdots t_{h-1}^{n_{h-1}} G_0$ with $n_0, \ldots, n_{h-1}$ as above. In fact, if T_i denotes the characteristic function of $G_0 t_i G_0$, $0 \leq i \leq h-1$, then

$$\mathcal{H} \simeq \breve{\mathfrak{o}}[T_0, T_0^{-1}, T_1, \ldots, T_{h-1}], \tag{11}$$

i.e. $\mathcal{H}$ is commutative and, as an $\breve{\mathfrak{o}}$-algebra, is generated by $T_0, T_0^{-1}, T_1, \ldots, T_{h-1}$ subject to the only relation $T_0 T_0^{-1} = 1$. This is an integral version of the classical Satake isomorphism which is due to Herzig, Henniart, Schneider, Teitelbaum and Vignéras (cf. [12], Proposition 2.1). Note that by Frobenius reciprocity

$$\mathcal{O}(\mathfrak{X}_0) = (\varinjlim_m \mathcal{O}(\mathfrak{X}_m))^{G_0} \simeq \mathrm{Hom}_G(\text{c-Ind}_{G_0}^G(\breve{\mathfrak{o}}), \varinjlim_m \mathcal{O}(\mathfrak{X}_m))$$

is naturally a module over $\mathcal{H}$. Explicitly, if $f \in \mathcal{O}(\mathfrak{X}_0)$ then

$$T_i(f) = \sum_{g \in G_0 t_i G_0 / G_0} g(f) = \sum_{g \in G_0/(G_0 \cap t_i G_0 t_i^{-1})} (g t_i)(f). \tag{12}$$

Since the actions of $\mathcal{H}$ and D^* on $\mathcal{O}(\mathfrak{X}_0)$ commute with each other, we obtain an induced structure of $\mathcal{H}$-module on $\mathcal{O}(\mathfrak{X}_0)^{\Pi^{\mathbb{Z}}} \simeq \mathcal{O}(\mathfrak{X}_{0,0}) \simeq R$. We note that this action of $\mathcal{H}$ on R depends on the choice of a uniformizer Π of D. Therefore, it is non-canonical and does not quite commute with that of Γ. In fact, the above isomorphisms are given by mapping $f \in R$ to the family $(\Pi^n(f))_{n \in \mathbb{Z}}$. If $0 \leq i \leq h-1$ then the endomorphism T_i of the direct product $\mathcal{O}(\mathfrak{X}_0)$ is of degree $i-h$, as follows from (12) and the definition of the $\mathrm{GL}_h(K)$-action. Therefore, the family $T_i(f)$ has entry $T_i(\Pi^{h-i}(f))$ in degree zero. If we denote by σ_i the outer automorphism of Γ defined by conjugation with the element Π^{h-i} of D^*, then this implies

$$T_i(\gamma(f)) = \sigma_i(\gamma)(T_i(f)) \quad \text{for all} \quad \gamma \in \Gamma,\ f \in R. \tag{13}$$

The reduction of the action of $\mathcal{H}$ on R modulo certain prime ideals of R can be described as follows.

Theorem 2.1. *If i is an integer with $0 \leq i \leq h-1$, and if $f \in R = \breve{\mathfrak{o}}[[u_1, \ldots, u_{h-1}]]$, then*

$$T_i(f)(u) \equiv f(u_1^{q^i}, \ldots, u_{h-1}^{q^i}) \mod (u_0, u_1, \ldots, u_{i-1})R,$$

where we change our previous convention and set $u_0 := \pi$.

In order to prove this theorem, we need some preparation. Let $\mathfrak{m}_i$ be the ideal of R generated by $u_0, \ldots, u_{i-1}$. For any positive integer m let $\mathfrak{m}_{i,m}$ be the ideal of R_m generated by $\varphi_m(e_m^1), \ldots, \varphi_m(e_m^i)$, where $\varphi_m : \pi^{-m}\mathfrak{o}^h/\mathfrak{o}^h \to (\mathfrak{m}_{R_m}, +_{\mathbb{H}})$ denotes the universal level-m structure and $(e_m^1, \ldots, e_m^h)$ denotes the standard basis of the free $(\mathfrak{o}/\pi^m\mathfrak{o})$-module $\pi^{-m}\mathfrak{o}^h/\mathfrak{o}^h$. According to [19], Proposition

1.2 – which builds on a result of Strauch – we have $\mathfrak{m}_{i,m'} \cap R_m = \mathfrak{m}_{i,m}$ for any two integers $m' \geq m \geq 0$. Further, $\mathfrak{m}_{i,\infty} := \varinjlim_m \mathfrak{m}_{i,m}$ is a prime ideal of $R_\infty := \varinjlim_m R_m$, lying above $\mathfrak{m}_i$.

Let $B_i := G_0 \cap t_i G_0 t_i^{-1}$, which is the parahoric subgroup of $\mathrm{GL}_h(K)$ consisting of all matrices $g \in G_0 = \mathrm{GL}_h(\mathfrak{o})$ of the form

$$g = \begin{pmatrix} A & B \\ \pi C & D \end{pmatrix}$$

with $A \in \mathrm{GL}_i(\mathfrak{o})$, $D \in \mathrm{GL}_{h-i}(\mathfrak{o})$, $B \in \mathfrak{o}^{i \times (h-i)}$ and $C \in \mathfrak{o}^{(h-i) \times i}$. Further, let W be the subgroup of permutation matrices in G_0, identified with the Weyl group of $\mathrm{GL}_h(K)$ with respect to the diagonal torus. Let ℓ denote the length function on W corresponding to the generating set of simple transpositions $(j\ j+1)$, $1 \leq j \leq h-1$. Let $W_i \subseteq W$ be the Weyl group of the Levi subgroup of $\mathrm{GL}_h(K)$ consisting of all matrices of the form

$$g = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

with $A \in \mathrm{GL}_i(K)$, $D \in \mathrm{GL}_{h-i}(K)$. Finally, let $N_0 \subseteq G_0$ be the subgroup of all upper triangular unipotent matrices in G_0, and let $B := \cap_{i=1}^{h-1} B_i$.

Lemma 2.2. *The group G_0 is the disjoint union of the double cosets BwB_i with $w \in W/W_i$. If $w \in W/W_i$ then $BwB_i = N_0 w B_i = N_0 \tilde{w} B_i$, where $\tilde{w}$ denotes the unique element of minimal length in wW_i. We have $|BwB_i/B_i| = q^{\ell(\tilde{w})}$.*

Proof. The group B (resp. B_i) is the inverse image under the reduction map $G_0 = \mathrm{GL}_h(\mathfrak{o}) \to \mathrm{GL}_h(k)$, of the group of upper triangular matrices (resp. of upper triangular $i \times (h-i)$-block matrices) with coefficients in k. Therefore, the first two assertions follow from the generalized Bruhat decomposition of $\mathrm{GL}_h(k)$ (cf. [2], 21.16). The remaining assertions follow from [2], 21.21 and Proposition 21.29. □

We need to give a more precise version of the standard result in Lemma 2.2. If $1 \leq r < s \leq h$ we let $N_{rs} \subseteq N_0$ denote the root subgroup of G_0 consisting of all unipotent matrices possessing nonzero off-diagonal entries at most in place (r,s). The set $\Phi := \{(r,s) \,|\, 1 \leq r,s \leq h,\ r \neq s\}$ can be identified with the root system of GL_n, and $\Phi^+ := \{(r,s) \in \Phi \,|\, r < s\}$ corresponds to the set of positive roots for the basis determined by the Borel subgroup of upper triangular matrices. We let $\Phi^- := \Phi \smallsetminus \Phi^+$, and note that the symmetric group W acts on Φ in such a way that $wN_{rs}w^{-1} = N_{w(r)w(s)}$ for all $(r,s) \in \Phi$ and all $w \in W$. Letting $w \in W$ and $\Psi_w := \{\alpha \in \Phi^+ \,|\, w^{-1}(\alpha) \in \Phi^-\}$, the references given in the above proof show that the map $\prod_{\alpha \in \Psi_{\tilde{w}}} N_\alpha \to BwB_i/B_i$, defined by $(n_\alpha)_{\alpha \in \Psi_{\tilde{w}}} \mapsto (\prod_{\alpha \in \Psi_{\tilde{w}}} n_\alpha) w B_i$, is surjective for any fixed but arbitrary ordering of the factors. It induces a bijection

$$\prod_{\alpha \in \Psi_{\tilde{w}}} N_\alpha(k) \longrightarrow BwB_i/B_i. \tag{14}$$

Here we abuse notation and also write N_α for the group scheme over $\mathfrak{o}$ whose group of $\mathfrak{o}$-rational points is N_α.

Lemma 2.3. *If $w \in W \setminus W_i$, and if $\tilde{w}$ denotes the unique element of minimal length in wW_i, then $\Psi_{\tilde{w}}$ contains a root (r,s) with $r \le i$.*

Proof. The assertion is trivial if $i = 0$ so that we may assume $1 \le i \le h-1$.

Our proof relies on the fact that the minimal coset representatives of W/W_i are explicitly known. Given integers j and ℓ_j with $0 \le \ell_j \le j < h$, we let $w_j(\ell_j)$ be the (ℓ_j+1)-cycle $(j+1-\ell_j, \ldots, j+1)$ in W, sending $j+1$ to $j+1-\ell_j$. According to [32], Theorem 2, there is a unique sequence of integers $(\ell_i, \ell_{i+1}, \ldots, \ell_{h-1})$, satisfying $0 \le \ell_{h-1} \le \ldots \le \ell_i \le i$, such that $\tilde{w} = w_{h-1}(\ell_{h-1}) \cdots w_i(\ell_i)$.

Since $w \notin W_i$, we have $\ell_i \ne 0$ and consider the root $(i+1, i+1-\ell_i) \in \Phi^-$. If $j > i$ then $j+1-\ell_j > i+1-\ell_j \ge i+1-\ell_i$, so that the cycle $w_j(\ell_j)$ fixes $i+1-\ell_i$. Since $w_i(\ell_i)(i+1) = i+1-\ell_i$, we obtain $\tilde{w}(i+1) = i+1-\ell_i$. Letting $i_0 := \max\{j \,|\, j \ge i,\, \ell_j = \ell_i\}$, a similar reasoning shows that $\tilde{w}(i+1-\ell_i) = i_0+2-\ell_i > i+1-\ell_i$. Thus,

$$(i+1-\ell_i, i_0+2-\ell_i) = \tilde{w}(i+1, i+1-\ell_i) \in \Phi^+.$$

When writing $(i+1, i+1-\ell_i)$ as an integral linear combination of the basis $\Delta := \{(r, r+1) \,|\, 1 \le r < h\}$ of Φ, it has a negative contribution from the positive simple root $(i, i+1)$. In the terminology of [2], 21.23, it is contained in $\Psi(\Delta \setminus \{(i,i+1)\})^-$. Thus,

$$(i+1-\ell_i, i_0+2-\ell_i) \in \Phi^+ \cap \tilde{w}(\Psi(\Delta \setminus \{(i,i+1)\})^-) = \Psi_{\tilde{w}},$$

the last equality being [2], 21.23 (4), where $\Psi_{\tilde{w}}$ is denoted by $\Phi'_{\tilde{w}}$. $\square$

Proof of Theorem 2.1. We first recall some of the constructions which are used in the proofs of [19], Proposition 1.2 and Theorem 1.4. Fix an element $g \in G$ and an integer $r \ge 0$ such that the matrix $\pi^r g^{-1}$ has coefficients in $\mathfrak{o}$. Fixing an integer $m \ge 1$ with $\pi^{-r} g \mathfrak{o}^h \subseteq \pi^{-m+1}\mathfrak{o}^h$, we may view $\pi^{-r} g\mathfrak{o}^h/\mathfrak{o}^h$ as an $\mathfrak{o}$-submodule of $\pi^{-m}\mathfrak{o}^h/\mathfrak{o}^h$. Consider the power series

$$\mathrm{pr}_g(X) := \prod_{\alpha \in \pi^{-r} g \mathfrak{o}^h/\mathfrak{o}^h} \mathbb{H}(X, \varphi_m(\alpha)) \in R_m[[X]].$$

Letting $1 \le j \le h$, and choosing an element $\beta_j \in \pi^{-m}\mathfrak{o}^h/\mathfrak{o}^h$ whose image in $\pi^{-m}\mathfrak{o}^h/\pi^{-r}g\mathfrak{o}^h$ is the image of e_1^j under the injection $\pi^{-r}g : \pi^{-1}\mathfrak{o}^h/\mathfrak{o}^h \hookrightarrow \pi^{-m}\mathfrak{o}^h/\pi^{-r}g\mathfrak{o}^h$, we have

$$g(\varphi_1(e_1^j)) = \mathrm{pr}_g(\varphi_m(\beta_j)) = \prod_{\alpha \in \pi^{-r} g \mathfrak{o}^h/\mathfrak{o}^h} \mathbb{H}(\varphi_m(\beta_j), \varphi_m(\alpha)). \tag{15}$$

If $g = t_0$ then we may take $r = 1$, $m = 2$ and obtain $\pi^{-r}g = 1$, as well as $\mathrm{pr}_{t_0}(X) = \mathbb{H}(X, 0) = X$. Since the diagram

$$\begin{array}{ccc}
\pi^{-2}\mathfrak{o}^h/\mathfrak{o}^h & \xrightarrow{\varphi_2} & (\mathfrak{m}_{R_2}, +_{\mathbb{H}}) \\
\uparrow & & \uparrow \\
\pi^{-1}\mathfrak{o}^h/\mathfrak{o}^h & \xrightarrow{\varphi_1} & (\mathfrak{m}_{R_1}, +_{\mathbb{H}})
\end{array}$$

is commutative, we obtain $t_0(\varphi_1(e_1^j)) = \varphi_1(e_1^j)$ for all j. Since $(\varphi_1(e_1^j))_{1\leq j\leq h}$ is a family of regular parameters for R_1, this implies that t_0 stabilizes R_1 and acts as the identity. By (12) this implies $T_0(f) = t_0(f) = f$ for all $f \in R$, proving the theorem for $i = 0$.

Now assume $1 \leq i \leq h-1$. For $g = t_i$ we may again take $r = 1$, $m = 2$, and obtain

$$\mathrm{pr}_{t_i}(X) = \prod_{\alpha_1,\ldots,\alpha_i\in\mathfrak{o}/\pi\mathfrak{o}} \mathbb{H}(X, \varphi_2(\alpha_1\pi e_2^1 + \ldots + \alpha_i\pi e_2^i)).$$

Now

$$\varphi_2(\alpha_1\pi e_2^1 + \ldots + \alpha_i\pi e_2^i) = [\alpha_1]_{\mathbb{H}}([\pi]_{\mathbb{H}}(\varphi_2(e_2^1))) +_{\mathbb{H}} \ldots +_{\mathbb{H}} [\alpha_i]_{\mathbb{H}}([\pi]_{\mathbb{H}}(\varphi_2(e_2^i)))$$

is contained in $\mathfrak{m}_{i,2}$, because $[\alpha_j]_{\mathbb{H}}(X), [\pi]_{\mathbb{H}}(X) \in XR[[X]]$ and since $\mathbb{H}(X,Y)$ has trivial constant coefficient. Thus,

$$\mathrm{pr}_{t_i}(X) \equiv X^{q^i} \mod \mathfrak{m}_{i,2}R_2[[X]],$$

which, as above, implies $t_i(\varphi_1(e_1^j)) \equiv \varphi_1(e_1^j)^{q^i} \mod \mathfrak{m}_{i,2}$ for all $1 \leq j \leq h$.

Before we consider the action of more general elements of the form gt_i with $g \in G_0$, let us show that $t_i(f)(u) \equiv f(u_1^{q^i}, \ldots, u_{h-1}^{q^i}) \mod \mathfrak{m}_i$ for all $f \in R$. As in the proof of [19], Proposition 1.2, one can use the above form of pr_{t_i} to see that t_i stabilizes $\mathfrak{m}_{i,\infty}$. In particular, it defines a ring automorphism of $R_\infty/\mathfrak{m}_{i,\infty}$. However, $R_1/\mathfrak{m}_{i,1}$ is a regular local k^{sep}-algebra with residue class field k^{sep}. Therefore, $R_1/\mathfrak{m}_{i,1} \simeq k^{\mathrm{sep}}[[\varphi_1(e_1^{i+1}), \ldots, \varphi_1(e_1^h)]]$ (cf. [4], VIII.5.2 Corollaire 3). It follows from the above that the subring $R_1/\mathfrak{m}_{i,1}$ of $R_\infty/\mathfrak{m}_{i,\infty}$ is stabilized by t_i, and that t_i acts by raising the variables $\varphi_1(e_1^j)$ to the power of q^i. Thus, it suffices to see that the elements u_j are contained in $k[[\varphi_1(e_1^1), \ldots, \varphi_1(e_1^h)]]$. This can be deduced from the explicit construction of R_1 from R_0, as presented in the proof of [9], Proposition 4.3. Namely, extend the automorphism σ of $\breve{\mathfrak{o}}$ to a ring automorphism of R by letting it fix the variables u. Since the logarithm f of $\mathbb{H}$ has coefficients in $\mathfrak{o}[[u]][\frac{1}{\pi}]$, the power series $[\pi]_{\mathbb{H}}(X) = f^{-1}(\pi f(X))$ has coefficients in $\mathfrak{o}[[u]] = R^{\sigma=1}$. Hence, so does $f_0(X) := [\pi]_{\mathbb{H}}(X)/X$. As a consequence, σ can be extended to a ring automorphism of $L_1 := R[[X]]/(f_0(X))$ such that the image ϑ_1 of X in L_1 is invariant under σ. Since $[\alpha]_{\mathbb{H}}(X) \in \mathfrak{o}[[u]][[X]]$, any element of the form $[\alpha]_{\mathbb{H}}(\vartheta_1)$ with $\alpha \in \mathfrak{o}/\pi\mathfrak{o}$ is σ-invariant, as well. Considering $f_1(X) := [\pi]_{\mathbb{H}}(X)/\prod_{\alpha\in\mathfrak{o}/\pi\mathfrak{o}}(X - [\alpha]_{\mathbb{H}}(\vartheta_1))$, we see similarly that σ extends to $L_2 := L_1[[X]]/(f_1(X))$ in such a way that the image ϑ_2 of X in L_2 is σ-invariant. Proceeding inductively, σ extends to a ring automorphism of $R_1 = L_h$ in such a way that the zeros $\varphi_1(e_1^1), \ldots, \varphi_1(e_1^h)$ of $[\pi]_{\mathbb{H}}(X)$ are all σ-invariant. Its reduction modulo $\mathfrak{m}_{i,1}$ acts as the q-power map on the coefficients of any power series in $R_1/\mathfrak{m}_{i,1} \simeq k^{\mathrm{sep}}[[\varphi_1(e_1^{i+1}), \ldots, \varphi_1(e_1^h)]]$. Since the variables u_j are all σ-invariant, the claim is proved.

As above, let $B_i := G_0 \cap t_iG_0t_i^{-1}$, fix $w \in W \smallsetminus W_i$, and let $\tilde{w}$ be the unique element of minimal length in the coset wW_i. Let $n \in N_0$ and consider the element $g := nwt_i \in \mathrm{GL}_h(K)$. For the computation of $\mathrm{pr}_g(X)$ we may again take $r = 1$ and $m = 2$. Let $\chi_i : \pi^{-m}\mathfrak{o}^h/\mathfrak{o}^h \to \oplus_{j=i+1}^h(\mathfrak{o}/\pi^m\mathfrak{o})e_m^j$ be the natural projection. If $\alpha \in \pi^{-1}g\mathfrak{o}^h/\mathfrak{o}^h \subseteq \pi^{-m}\mathfrak{o}^h/\mathfrak{o}^h$ then $\alpha = \chi_i(\alpha) + (\alpha - \chi_i(\alpha))$ with $\alpha - \chi_i(\alpha) \in \oplus_{j=1}^{i}(\mathfrak{o}/\pi^m\mathfrak{o})e_m^j$. As in the computation of $\mathrm{pr}_{t_i}(X)$ we see that $\varphi_2(\alpha - \chi_i(\alpha)) \in \mathfrak{m}_{i,2}$ and that $\varphi_2(\alpha) \equiv \varphi_2(\chi_i(\alpha)) \mod \mathfrak{m}_{i,2}$. This in turn implies that $\mathbb{H}(X, \varphi_2(\alpha)) \equiv \mathbb{H}(X, \varphi_2(\chi_i(\alpha))) \mod \mathfrak{m}_{i,2}R_2[[X]]$. Thus,

$$\mathrm{pr}_g(X) \equiv \prod_{\alpha \in \pi^{-1}g\mathfrak{o}^h/\mathfrak{o}^h} \mathbb{H}(X, \varphi_2(\chi_i(\alpha))) \mod \mathfrak{m}_{i,2}R_2[[X]].$$

Now if n' is contained in a root subgroup $N_{rs} \subseteq N_0$ with $r \leq i$, and if $\alpha \in \pi^{-1}g\mathfrak{o}^h/\mathfrak{o}^h$, then $\chi_i(n'\alpha) = \chi_i(\alpha)$. Therefore,

$$\begin{aligned}
\mathrm{pr}_{n'g}(X) &\equiv \prod_{\alpha \in \pi^{-1}n'g\mathfrak{o}^h/\mathfrak{o}^h} \mathbb{H}(X, \varphi_2(\chi_i(\alpha))) = \prod_{\alpha \in \pi^{-1}g\mathfrak{o}^h/\mathfrak{o}^h} \mathbb{H}(X, \varphi_2(\chi_i(n'\alpha))) \\
&= \prod_{\alpha \in \pi^{-1}g\mathfrak{o}^h/\mathfrak{o}^h} \mathbb{H}(X, \varphi_2(\chi_i(\alpha))) \equiv \mathrm{pr}_g(X) \mod \mathfrak{m}_{i,2}R_2[[X]].
\end{aligned}$$

As a consequence of (15), $g(\varphi_1(e_1^j)) \equiv (n'g)(\varphi_1(e_1^j)) \mod \mathfrak{m}_{i,\infty}$ for all $1 \leq j \leq h$. Since the elements $\varphi_1(e_1^{i+1}), \ldots, \varphi_1(e_1^h)$ topologically generated $R_1/\mathfrak{m}_{i,1}$ over k^{sep}, we obtain $g(f) \equiv (n'g)(f) \mod \mathfrak{m}_{i,\infty}$ for all $f \in R_1$. If $f \in R$ then $(n'g)(f)$ depends only on the image of n' in $N_{rs}(k)$ (cf. (14)). Since $w \in W \setminus W_i$, we may choose (r, s) as in Lemma 2.3 and obtain

$$\begin{aligned}
\sum_{g \in BwB_i/B_i} (gt_i)(f) &= \sum_{n \in \prod_{\alpha \in \Psi_{\tilde{w}}} N_\alpha(k)} (n\tilde{w}t_i)(f) \\
&= \sum_{n' \in N_{rs}(k)} \Big(\sum_{n \in \prod_{\alpha \in \Psi_{\tilde{w}} \setminus \{(r,s)\}} N_\alpha(k)} (n'n\tilde{w}t_i)(f) \Big) \\
&\equiv \sum_{n' \in N_{rs}(k)} \Big(\sum_{n \in \prod_{\alpha \in \Psi_{\tilde{w}} \setminus \{(r,s)\}} N_\alpha(k)} (n\tilde{w}t_i)(f) \Big) \mod \mathfrak{m}_{i,\infty} \\
&\equiv 0 \mod \mathfrak{m}_{i,\infty},
\end{aligned}$$

because $|N_{rs}(k)| = q$ and since $R_\infty/\mathfrak{m}_{i,\infty}$ is of characteristic p. According to Lemma 2.2 and (12) we obtain

$$T_i(f) \equiv t_i(f) \equiv f(u_1^{q^i}, \ldots, u_{h-1}^{q^i}) \mod \mathfrak{m}_i,$$

where the second congruence was proved above. $\square$

As a direct consequence of Theorem 2.1 we obtain the following result.

Corollary 2.4. *If $1 \leq i \leq h$ then the prime ideal $\mathfrak{m}_i := (\pi, u_1, \ldots, u_{i-1})R$ of R is stable under the action of the $\breve{\mathfrak{o}}$-subalgebra of $\mathcal{H}$ generated by $T_0, \ldots, T_i$.* $\square$

In the following, we shall always view R as a topological ring with respect to its $\mathfrak{m}$-adic topology. Note that the $\mathfrak{m}$-adic topology gives R the structure of a *pseudo-compact $\breve{\mathfrak{o}}$-module* in the sense that R is a complete Hausdorff topological $\breve{\mathfrak{o}}$-module which is the projective limit of discrete $\breve{\mathfrak{o}}$-modules of finite length.

Corollary 2.5. *For any integer i with $0 \leq i \leq h-1$, and for any non-negative integer n, the $\breve{\mathfrak{o}}$-linear endomorphism T_i^n of R is continuous with closed image. The $\breve{\mathfrak{o}}$-linear endomorphism T_1^n of R is injective. If $m > n$ then the $\breve{\mathfrak{o}}$-module $T_1^n(R)/T_1^m(R)$ is torsion free. It is non-zero unless $h = 1$.*

Proof. The ring homomorphism $t_i : R \to R_\infty^{t_iG_0t_i^{-1}} \hookrightarrow R_\infty^{G_0\cap t_iG_0t_i^{-1}}$ is local, hence is continuous for the topologies defined by the maximal ideals. Moreover, the trace map $R_\infty^{G_0\cap t_iG_0t_i^{-1}} \to R$ is R-linear, hence is continuous for the $\mathfrak{m}$-adic topologies. By Krull's intersection theorem, the $\mathfrak{m}$-adic topology on $R_\infty^{G_0\cap t_iG_0t_i^{-1}}$ coincides with the topology defined by the maximal ideal of $R_\infty^{G_0\cap t_iG_0t_i^{-1}}$. Therefore, it follows from (12) that T_i is continuous. It is a general fact that continuous $\breve{\mathfrak{o}}$-linear maps between pseudo-compact $\breve{\mathfrak{o}}$-modules have closed image (cf. [27], Theorem 22.3).

If $T_1^n(f) = 0$ then the injectivity of T_1^n modulo πR (cf. Theorem 2.1) implies that $f \in \pi R$, i.e. $f = \pi f'$ for some element $f' \in R$. Since T_1^n is $\breve{\mathfrak{o}}$-linear and since R is torsion free over $\breve{\mathfrak{o}}$, we obtain $T_1^n(f') = 0$, as well. Proceeding inductively, we find $f \in \cap_{m\geq 0}\pi^m R = \{0\}$.

Finally, let $g, f \in R$ be such that $\pi T_1^n(g) = T_1^m(f)$. According to Theorem 2.1 we have

$$f(u_1^{q^m}, \dots, u_{h-1}^{q^m}) \equiv T_1^m(f) \equiv \pi T_1^n(g) \equiv 0 \mod \pi R.$$

Obviously, this implies $f \in \pi R$. Writing $f = \pi f'$ for some element $f' \in R$, we obtain $\pi T_1^n(g) = \pi T_1^m(f')$, whence $T_1^n(g) = T_1^m(f')$ because R is torsion free over $\breve{\mathfrak{o}}$. This proves that $T_1^n(R)/T_1^m(R)$ is torsion free over $\breve{\mathfrak{o}}$. That it is non-zero, provided $h \neq 1$, follows from Theorem 2.1. □

Note that the image of $\breve{\mathfrak{o}}$ in $\varinjlim_m \mathcal{O}(\mathfrak{X}_m)$ is pointwise fixed by $\mathrm{GL}_h(K)$, so that $\breve{\mathfrak{o}}$ is naturally an $\mathcal{H}$-submodule of R with T_i acting by multiplication with the index $(G_0 : G_0 \cap t_iG_0t_i^{-1})$. In the case of height two, we can now prove the following result.

Theorem 2.6. *If $h = 2$ then $R/\breve{\mathfrak{o}}$ is a flat module over $\mathcal{H}/(T_0 - 1)\mathcal{H} \simeq \breve{\mathfrak{o}}[T_1]$.*

Proof. It follows from Theorem 2.1 that the action of $\mathcal{H}$ on R (and hence that on $R/\breve{\mathfrak{o}}$) factors through $\mathcal{H}/(T_0 - 1)\mathcal{H}$. The identification of this quotient with $\breve{\mathfrak{o}}[T_1]$ follows from the integral Satake isomorphism (11). Letting $R_K := R \otimes_{\mathfrak{o}} K \simeq R \otimes_{\breve{\mathfrak{o}}} \breve{K}$, it suffices to see that $R_K/\breve{K}$ and $\overline{R}/k^{\mathrm{sep}}$ are flat over $\breve{K}[T_1]$ and $k^{\mathrm{sep}}[T_1]$, respectively (cf. [3], 2.6 Lemma 1).

Since $k^{\mathrm{sep}}[T_1]$ is a principal ideal domain, it suffices to see that $\overline{R}/k^{\mathrm{sep}}$ is torsion free over $k^{\mathrm{sep}}[T_1]$. Note that the k^{sep}-subspace $\overline{\mathfrak{m}}$ of $\overline{R}$ is in fact a $k^{\mathrm{sep}}[T_1]$-submodule which is isomorphic to $\overline{R}/k^{\mathrm{sep}}$. If $F \in k^{\mathrm{sep}}[T_1]$ and $f \in \overline{\mathfrak{m}} = u_1 k^{\mathrm{sep}}[[u_1]]$ then

$$\mathrm{ord}_{u_1}(F(f)) = \mathrm{ord}_{u_1}(f) \cdot q^{\mathrm{ord}_{T_1}(F)},$$

by Theorem 2.1, whence $\overline{\mathfrak{m}}$ is torsion free over $k^{\mathrm{sep}}[T_1]$.

To complete the proof, we will show that $R_K/\breve{K}$ is torsion free over $\breve{K}[T_1]$. Let $f \in R_K$ and $F \in \breve{K}[T_1] \smallsetminus \{0\}$ be such that $F(f) \in \breve{K}$. We need to see that $f \in \breve{K}$. If $f - f(0) \neq 0$ choose integers r and s such that $\pi^r F \in \breve{\mathfrak{o}}[T_1]$ and $\pi^s(f - f(0)) \in R$ with non-trivial images in $k^{\mathrm{sep}}[T_1]$ and $\overline{R}$, respectively. Since $\pi^r F(\pi^s(f - f(0))) \in R \cap \breve{K} = \breve{\mathfrak{o}}$, the case we treated above shows that $\pi^s(f - f(0)) \in \pi R$. This contradicts the choice of s and shows that indeed $f = f(0) \in \breve{K}$. □

Remark 2.7. Without any restriction on h, the above proof can be adjusted to show that $R/\breve{\mathfrak{o}}$ is flat over the subalgebra $\breve{\mathfrak{o}}[T_1]$ of $\mathcal{H}/(T_0-1)\mathcal{H}$.

Let h be arbitrary again. The proof of Theorem 2.6 could have been slightly simplified by referring to the following result.

Proposition 2.8. *The endomorphism T_1 of $R/\breve{\mathfrak{o}}$ is topologically nilpotent in the sense that $\cap_{n\geq 0}T_1^n(R/\breve{\mathfrak{o}})=\{0\}$. Equivalently, $\cap_{n\geq 0}T_1^n(R)=\breve{\mathfrak{o}}$. The action of the ring $\breve{\mathfrak{o}}[T_1]$ on $R/\breve{\mathfrak{o}}$ extends to an action of $\breve{\mathfrak{o}}[[T_1]]$.*

Proof. Note first that the action of T_1 on $\breve{\mathfrak{o}}$ is bijective after reduction modulo π (cf. Theorem 2.1), hence is bijective itself. This shows that $\breve{\mathfrak{o}}$ is contained in any submodule $T_1^n(R)$, i.e. $\breve{\mathfrak{o}}\subseteq \cap_{n\geq 0}T_1^n(R)$.

Conversely, assume $f\in\cap_{n\geq 0}T_1^n(R)$. The image of f in $\overline{R}$ is contained in $\cap_{n\geq 0}T_1^n(\overline{R})=k^{\mathrm{sep}}$, the last equality following from Theorem 2.1. Therefore, we can write $f=\alpha+\pi f'$ with $\alpha\in\breve{\mathfrak{o}}$ and $f'\in R$. By what we already know, we must have $\pi f'=f-\alpha\in\cap_{n\geq 0}T_1^n(R)$, as well. As a consequence of Corollary 2.5, this implies $f'\in\cap_{n\geq 0}T_1^n(R)$. Inductively, this yields $f\in\cap_{n\geq 0}(\breve{\mathfrak{o}}+\pi^nR)=\breve{\mathfrak{o}}$, the last equality coming from the fact that $\breve{\mathfrak{o}}$ is closed in R (cf. [27], Lemma 22.2). This proves the first assertion of the proposition. Together with Corollary 2.5 and [27], Lemma 22.1, it implies that the natural $\breve{\mathfrak{o}}[T_1]$-linear homomorphism of pseudo-compact $\breve{\mathfrak{o}}$-modules

$$R/\breve{\mathfrak{o}}\longrightarrow \varprojlim_{n\geq 0}(R/\breve{\mathfrak{o}})/T_1^n(R/\breve{\mathfrak{o}})$$

is bijective. Obviously, the action of $\breve{\mathfrak{o}}[T_1]$ on the right hand side extends to $\varprojlim_{n\geq 0}\breve{\mathfrak{o}}[T_1]/T_1^n\breve{\mathfrak{o}}[T_1]\simeq\breve{\mathfrak{o}}[[T_1]]$. $\square$

3 Iwasawa theoretic structure theorems

The group Γ is a profinite topological group with a basis of open neighborhoods of the identity given by the subgroups $\Gamma_i=1+\Pi^i\mathfrak{o}_D$ for $i\geq 1$. The following assertion is a direct consequence of a result of Gross and Hopkins (cf. [11], Lemma 19.3).

Proposition 3.1. *Endowing the ring R with its $\mathfrak{m}$-adic topology and the direct product $\Gamma\times R$ with the product topology, the action of Γ on R is continuous in the sense that the map $((\gamma,f)\mapsto\gamma(f)):\Gamma\times R\to R$ is a continuous map of topological spaces.*

Proof. The group Γ acts on R by local ring automorphisms which are continuous for the $\mathfrak{m}$-adic topology. For the same reason we have $\Gamma(\mathfrak{m}^n)=\mathfrak{m}^n$ for any integer $n\geq 0$, so that the map $\Gamma\times R\to R$ is continuous at $(1,0)$. Therefore, it suffices to show that if $f\in R$ is an arbitrary element, then the map $(\gamma\mapsto\gamma(f)):\Gamma\to R$ is continuous.

Fix an integer $n\geq 1$. It suffices to prove that $\gamma(f)-f\in\mathfrak{m}^n$ for any element $\gamma\in\Gamma_{h(n-1)}$. Since $\mathfrak{m}^n$ is an ideal of R, one can further reduce to the case

$f = u_i$ for some index i with $1 \leq i \leq h-1$. Consider the affinoid subdomain $D_0 \subset \mathrm{Spf}(R)^{\mathrm{rig}}$ of section 1. The spectral norm of $\mathcal{O}(D_0)$ is given by

$$|g|_{D_0} := \sup_{x \in D_0} |g(x)| = \sup_{\alpha \in \mathbb{N}^{h-1}} |d_\alpha| q^{-|\alpha|},$$

if $g = \sum_{\alpha \in \mathbb{N}^{h-1}} d_\alpha u^\alpha \in \mathcal{O}(D_0)$. In particular, an element $g \in R \subseteq \mathcal{O}(D_0)$ is contained in $\mathfrak{m}^n$ if and only if $|g|_{D_0} \leq q^{-n}$. Applying [11], Lemma 19.3 with $e = 1$, we have

$$|\gamma(u_i) - u_i|_{D_0} = \sup_{x \in D_0} |u_i(x \cdot \gamma) - u_i(x)| \leq q^{-n}$$

for any $\gamma \in \Gamma_{h(n-1)}$. By our previous remark this implies $\gamma(u_i) - u_i \in \mathfrak{m}^n$, as required. □

For any profinite group H we define the completed group ring $\Lambda(H) = \breve{\mathfrak{o}}[[H]]$ of H over $\breve{\mathfrak{o}}$ by

$$\Lambda(H) = \breve{\mathfrak{o}}[[H]] := \varprojlim_{N \trianglelefteq_o H} \breve{\mathfrak{o}}[H/N],$$

where the projective limit runs over all *open* normal subgroups N of H. If N and N' are two open normal subgroups of H with $N' \subseteq N$, then the transition map $\breve{\mathfrak{o}}[H/N'] \to \breve{\mathfrak{o}}[H/N]$ of this projective limit is the natural homomorphism of group rings induced by the surjective homomorphism $H/N' \to H/N$. We note that if N is an open normal subgroup of H, then the group ring $\breve{\mathfrak{o}}[H/N]$ is the projective limit of the Artinian rings $(\breve{\mathfrak{o}}/\pi^m)[H/N]$ with $m \geq 0$. Therefore, $\Lambda(H)$ is a pseudo-compact topological ring in the sense of [5], page 442.

We shall abbreviate $\Lambda := \Lambda(\Gamma)$ and $\Lambda_1 := \Lambda(\Gamma_1)$. It follows from Proposition 3.1 that if n is any positive integer then there is an open normal subgroup N of Γ such that N acts trivially on $R/\mathfrak{m}^n$. In fact, the proof of Proposition 3.1 shows that we may take $N := \Gamma_{h(n-1)}$. This allows us to view $R/\mathfrak{m}^n$ as a module over Λ via the natural ring homomorphism $\Lambda \to \breve{\mathfrak{o}}[\Gamma/N]$. The natural maps $R/\mathfrak{m}^{n+1} \to R/\mathfrak{m}^n$ are Λ-equivariant and provide $R \simeq \varprojlim_n R/\mathfrak{m}^n$ itself with the structure of a Λ-module. In fact, this construction makes R a pseudo-compact module over Λ, i.e. R is a complete Hausdorff topological Λ-module possessing a basis of open neighborhoods of zero (the ideals $\mathfrak{m}^n$) consisting of Λ-submodules such that the corresponding quotient modules are of finite length.

Lemma 3.2. *If n is a non-negative integer and if $0 \leq i \leq h-1$ then the $\breve{\mathfrak{o}}$-submodule $T_i^n(R)$ of R is Λ-stable.*

Proof. According to Corollary 2.5 the $\breve{\mathfrak{o}}$-submodule $T_i^n(R)$ of R is closed, hence is complete for the induced topology. As a consequence, the natural map

$$T_i^n(R) \longrightarrow \varprojlim_{m \geq 0} T_i^n(R)/(T_i^n(R) \cap \mathfrak{m}^m)$$

is bijective (cf. [27], Theorem 22.3 and Lemma 22.1). By the construction of the Λ-module structure on R it therefore suffices to see that $T_i^n(R)$ is Γ-stable. This follows from (13). □

Following [29], section 1, we endow the $\breve{K}$-vector space $R_K := R \otimes_{\mathfrak{o}} K \simeq R \otimes_{\breve{\mathfrak{o}}} \breve{K}$ with the finest locally convex topology over $\breve{K}$ for which the inclusion $R \subset R_K$ is continuous when R is endowed with its $\mathfrak{m}$-adic topology. An $\breve{\mathfrak{o}}$-lattice L of R_K is open for this topology if and only if $R \cap aL$ is open in R for any element $a \in \breve{\mathfrak{o}} \smallsetminus \{0\}$.

For any element $F \in \mathcal{H}$ we also denote by F its natural image in $\mathcal{H} \otimes_{\mathfrak{o}} K$, viewed as a $\breve{K}$-linear endomorphism of R_K. We also set $\Lambda_K := \Lambda \otimes_{\mathfrak{o}} K \simeq \Lambda \otimes_{\breve{\mathfrak{o}}} \breve{K}$.

Proposition 3.3. *The locally convex $\breve{K}$-vector space R_K is Hausdorff, complete and induces the $\mathfrak{m}$-adic topology on R. For any integer $n \geq 0$ the $\breve{K}$-linear endomorphism T_1^n of R_K is continuous and injective. Its image is a closed, Λ_K-stable $\breve{K}$-subspace of R_K. If $h \neq 1$ we have $T_1^{n+1}(R_K) \subsetneqq T_1^n(R_K)$. In this case, the Λ_K-module R_K is not topologically of finite length.*

Proof. Let (u) be the ideal of R_K generated by $u_1, \ldots, u_{h-1}$. For any integer $m \geq 0$ consider the $\breve{\mathfrak{o}}$-lattice $L_m := \pi^m R + (u)^m$ of R_K. If $a \in \breve{\mathfrak{o}} \smallsetminus \{0\}$ then $aL_m \cap R \supseteq \mathfrak{m}^{2m+v(a)}$, so that L_m is open in R_K. Further, $\cap_{m \geq 0} L_m = \{0\}$, so that R_K is Hausdorff. Since the $\mathfrak{m}$-adic topology of R is obviously finer than the one induced by R_K, it suffices to see that any power $\mathfrak{m}^m$ of the maximal ideal $\mathfrak{m}$ of R contains a subset of the form $R \cap L$ for some open lattice L of R. This is clear from $R \cap L_m \subseteq \mathfrak{m}^m$.

Since R is complete and since multiplication with π is a homeomorphism from R_K to itself, it follows that any subset of R_K of the form $\pi^m R$, $m \in \mathbb{Z}$, is complete, as well. As in the proof of [29], Lemma 1.4, one can deduce that the locally convex $\breve{K}$-vector space R_K is complete. In fact, any Cauchy net admits a subnet contained in a set of the form $\pi^{-m} R$ for some integer m.

The injectivity of T_1^n follows from Corollary 2.5 and the flatness of R over $\breve{\mathfrak{o}}$. By definition of the topology of R_K, the $\breve{K}$-linear endomorphism T_1^n of R_K is continuous if and only if its restriction $T_1^n : R \to R_K$ to R is. The latter, however, is the composition of the maps $T_1^n : R \to R$ and $R \hookrightarrow R_K$, so that the claim follows from Corollary 2.5.

We now show that the $\breve{K}$-subspace $T_1^n(R_K)$ of R_K is closed. Denote by N the $\breve{\mathfrak{o}}$-submodule of R consisting of all power series $f(u) = \sum_{\alpha \in \mathbb{N}^{h-1}} d_\alpha u^\alpha$ for which $d_\alpha = 0$ whenever all of $\alpha_1, \ldots, \alpha_{h-1}$ are divisible by q^n in $\mathbb{N}$. We claim that N is a closed $\breve{\mathfrak{o}}$-module complement of $T_1^n(R)$ in R. It is clear that N is a closed $\breve{\mathfrak{o}}$-submodule of R. Both N and $R/T_1^n(R)$ are π-adically separated, complete and torsion-free $\breve{\mathfrak{o}}$-modules (cf. Corollary 2.5). To prove the claim, it therefore suffices to see that the natural map $N \to R/T_1^n(R)$ is bijective after reduction modulo $\pi\breve{\mathfrak{o}}$. This follows immediately from Theorem 2.1.

It follows from [27], Theorem 22.3, that the continuous bijection $N \oplus T_1^n(R) \to R$ of pseudo-compact $\breve{\mathfrak{o}}$-modules is a homeomorphism. In particular, the projection $\mathrm{pr}_N : R \to R$ onto N is continuous. This can also be proven directly by showing that $\mathfrak{m}^m = (\mathfrak{m}^m \cap N) \oplus (\mathfrak{m}^m \cap T_1^n(R))$ for any non-negative integer m. Now the $\breve{K}$-linear extension $(\mathrm{pr}_N)_K$ of pr_N has kernel $T_1^n(R_K)$. In order to see that $T_1^n(R_K)$ is closed in R_K it therefore suffices to see that $(\mathrm{pr}_N)_K$ is a continuous endomorphism of R_K. As above, this follows from the fact that its restriction to R is the composition of the continuous maps $\mathrm{pr}_N : R \to R$ and $R \hookrightarrow R_K$.

The two final assertions of the proposition follow directly from Corollary 2.5. $\square$

Remark 3.4. One can show that the actions of Γ and $\mathcal{H}$ on R are in fact the $\breve{\mathfrak{o}}$-linear extensions of $\mathfrak{o}_h$-linear actions on $\mathfrak{o}_h[[u_1, \ldots, u_{h-1}]]$. Our rather ad hoc proof of parts of Proposition 3.3 can then be simplified, using the methods of [29] for the locally compact field K_h. In particular, one can deduce that R_K is the continuous $\breve{K}$-linear dual of a continuous unitary representation of Γ on a $\breve{K}$-Banach space V. Further, V admits an ascending sequence of closed Γ-stable $\breve{K}$-subspaces V_n, $n \geq 0$, whose continuous $\breve{K}$-linear duals are isomorphic to $R_K/T_1^n(R_K)$.

Remark 3.5. Assume the characteristic of K to be zero. If $\mathfrak{H}$ denotes the set of all K-rational hyperplanes in the $(h-1)$-dimensional projective space $\mathbb{P}_K^{h-1}$ over K, then $\Omega_K^h := \mathbb{P}_K^{h-1} \smallsetminus \bigcup_{H \in \mathfrak{H}} H$ is a rigid analytic K-variety known as *Drinfeld's symmetric space of dimension $h-1$ over K*. Its ring of global sections $\mathcal{O}(\Omega_K^h)$ is a K-Fréchet space with a natural action of the group $\mathrm{GL}_h(K)$ which is dual to a locally analytic representation in the sense of [28], section 3. By work of Orlik and Strauch, the $\mathrm{GL}_h(K)$-representation $\mathcal{O}(\Omega_K^h)$ is topologically of finite length (cf. [24], Corollary 7.6). The exact analog of this representation is the Γ-representation $\mathcal{O}(\mathrm{Spf}(R)^{\mathrm{rig}})$ which is dual to a locally analytic representation, as well, provided $K = \mathbb{Q}_p$ (cf. [20], Theorem 3.3). Although the precise relation to its continuous Γ-subrepresentation R_K is currently unclear, the latter is not topologically of finite length unless $h = 1$ (cf. Proposition 3.3). This is due to the appearance of the Hecke algebra $\mathcal{H}$ which is not relevant in Drinfeld's setting. In fact, in the latter situation the spherical Hecke algebra is $K[D^*/\mathfrak{o}_D^*] \simeq K[T_0, T_0^{-1}]$ with T_0 acting trivially.

In the most basic case where $K = \mathbb{Q}_p$ and $h = 2$ the results of section 1 allow us to prove at least that the Λ-modules $T_1^n(R)/T_1^{n+1}(R)$ are finitely generated for any integer $n \geq 0$.

Theorem 3.6. *If $h = 2$ and if $K = \mathbb{Q}_p$ then the Λ-module $T_1^n(R)/T_1^{n+1}(R)$ is finitely generated for any integer $n \geq 0$.*

Proof. It suffices to see that $T_1^n(R)/T_1^{n+1}(R)$ is finitely generated over $\Lambda_1 := \breve{\mathfrak{o}}[[\Gamma_1]]$. Note that $T_1^n(R)/T_1^{n+1}(R)$ is a pseudo-compact Λ-module (cf. Corollary 2.5, Lemma 3.2 and [27], Theorem 22.3). Further, as we shall recall below, $\Gamma_1 = 1 + \Pi\mathfrak{o}_D$ is a pro-p group. Therefore, the ring Λ_1 is a local $\breve{\mathfrak{o}}$-algebra whose maximal ideal is generated by p and finitely many elements of the form $\gamma - 1$, $\gamma \in \Gamma_1$ (cf. [27], Propositions 19.5 and 19.7). According to [5], Corollary 1.5, it suffices to see that the k^{sep}-vector space $(T_1^n(\overline{R})/T_1^{n+1}(\overline{R}))_{\Gamma_1}$ of Γ_1-coinvariants of $T_1^n(\overline{R})/T_1^{n+1}(\overline{R})$ is finite dimensional.

Note that T_1^n induces a k^{sep}-linear bijection $\overline{R}/T_1(\overline{R}) \to T_1^n(\overline{R})/T_1^{n+1}(\overline{R})$ which is Γ_1-equivariant if the action on the right is changed by an automorphism of Γ_1 (cf. (13)). We may therefore restrict to the case $n = 0$. Considering the short exact sequence

$$0 \longrightarrow k^{\mathrm{sep}} \longrightarrow \overline{R} \longrightarrow \overline{\mathfrak{m}} \longrightarrow 0$$

of Γ-equivariant homomorphisms of pseudo-compact Λ-modules, we may further replace $\overline{R}$ by $\overline{\mathfrak{m}}$. Note that $T_1(\overline{\mathfrak{m}}) \subseteq \overline{\mathfrak{m}}$ and even $T_1(\overline{\mathfrak{m}}) \subseteq \overline{\mathfrak{m}}^p$ by Theorem 2.1.

We will prove that $(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))_{\Gamma_1}$ is one dimensional over k^{sep} and is generated by the class of $u := u_1$. For this it suffices to show that the map $\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}) \to \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2$ induces an isomorphism in Γ_1-coinvariants. Namely, the action of Γ_1 on $\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2$ is trivial, as follows from Theorem 1.16 and the proof of Proposition 3.1.

We claim that it suffices to prove that for any integer $n \geq 2$ the natural map

$$[(\overline{\mathfrak{m}}^n + T_1(\overline{\mathfrak{m}}))/(\overline{\mathfrak{m}}^{n+1} + T_1(\overline{\mathfrak{m}}))]_{\Gamma_1} \longrightarrow [\overline{\mathfrak{m}}/(\overline{\mathfrak{m}}^{n+1} + T_1(\overline{\mathfrak{m}}))]_{\Gamma_1} \tag{16}$$

is the zero map. Indeed, by the right exactness of the functor $(\cdot)_{\Gamma_1}$ this would imply that for any integer $n \geq 2$ the natural map

$$\overline{\mathfrak{m}}/(\overline{\mathfrak{m}}^{n+1} + T_1(\overline{\mathfrak{m}})) \to \overline{\mathfrak{m}}/(\overline{\mathfrak{m}}^n + T_1(\overline{\mathfrak{m}}))$$

induces an isomorphism of Γ_1-coinvariants. This in turn would imply

$$\begin{aligned}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))_{\Gamma_1} &\simeq [\varprojlim_{n\geq 2} \overline{\mathfrak{m}}/(\overline{\mathfrak{m}}^n + T_1(\overline{\mathfrak{m}}))]_{\Gamma_1} \\ &\simeq \varprojlim_{n\geq 2} [\overline{\mathfrak{m}}/(\overline{\mathfrak{m}}^n + T_1(\overline{\mathfrak{m}}))]_{\Gamma_1} \simeq [\overline{\mathfrak{m}}/(\overline{\mathfrak{m}}^2 + T_1(\overline{\mathfrak{m}}))]_{\Gamma_1} \\ &\simeq (\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2)_{\Gamma_1} \simeq \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2.\end{aligned}$$

Here the second isomorphism follows from [5], Lemma 4.2 (ii) and Corollary 4.3 (ii), and the fourth isomorphism comes from the fact that $T_1(\overline{\mathfrak{m}}) \subseteq \overline{\mathfrak{m}}^2$.

We will now show that the map (16) is indeed the zero map for any integer $n \geq 2$. Note that $(\overline{\mathfrak{m}}^n + T_1(\overline{\mathfrak{m}}))/(\overline{\mathfrak{m}}^{n+1} + T_1(\overline{\mathfrak{m}}))$ is of k^{sep}-dimension 1 if n is not divisible by p (and then is generated by the class of u^n) and is of dimension 0 if $n \equiv 0 \mod p$. We therefore need to show that $u^n \in \overline{\mathfrak{m}}(\Gamma_1) + T_1(\overline{\mathfrak{m}}) + \overline{\mathfrak{m}}^{n+1}$ if $n \not\equiv 0 \mod p$. Here $\overline{\mathfrak{m}}(\Gamma_1)$ denotes the kernel of the natural surjection $\overline{\mathfrak{m}} \to \overline{\mathfrak{m}}_{\Gamma_1}$. For the rest of the proof assume that $n \geq 2$ is not divisible by p.

If $n \not\equiv 1 \mod p$ then we let $\gamma := 1 + \Pi \in \Gamma_1$. To simplify the notation we write $\gamma(u)$ instead of $\overline{\gamma(u)}$, as we did in section 1. According to Theorem 1.16 we have $(\gamma - 1)(u^{n-1}) \equiv -(n-1)u^n \mod \overline{\mathfrak{m}}^{n+1}$. Since $n - 1 \neq 0$ in k^{sep}, we have $u^n \in \overline{\mathfrak{m}}^{n-1}(\Gamma_1) + \overline{\mathfrak{m}}^{n+1}$, as desired.

If $n \equiv 1 \mod p$ write $n = jp + 1$ with $j \geq 1$. Let us first assume $j = 1$. Let ξ be an arbitrary element of μ_{p^2-1}. Set $\gamma := 1 + \Pi\xi \in \Gamma_1$. Note that according to Theorem 1.16 we have $(\gamma - 1)(u) \equiv \sum_{i=1}^{p-1} (-\xi)^i u^{i+1} + \xi^p u^{p+1} \mod \overline{\mathfrak{m}}^{p+2}$. Choosing p pairwise distinct elements $\xi_1, \ldots, \xi_p \in \mu_{p^2-1}$, viewed also as elements of k_2 by reduction modulo p, the vectors $v_i := (\xi_i, (\xi_i)^2, \ldots, (\xi_i)^p)$, $1 \leq i \leq p$, are a basis of the k_2-vector space k_2^p, as follows from the well-known formula of the Vandermonde determinant. In particular, there are coefficients $\lambda_1, \ldots, \lambda_p \in k_2 \subset k^{\mathrm{sep}}$ such that $\sum_{i=1}^p \lambda_i v_i$ is the p-th standard unit vector of k_2^p. Setting $\gamma_i := 1 + \Pi\xi_i \in \Gamma_1$, our above calculation shows that

$$\sum_{i=1}^{p} \lambda_i(\gamma_i - 1)(u) \equiv u^{p+1} \mod \overline{\mathfrak{m}}^{p+2}.$$

This implies $u^{p+1} \in \overline{\mathfrak{m}}(\Gamma_1) + \overline{\mathfrak{m}}^{p+2}$, as desired.

Now assume $n = jp + 1$ with $j \geq 2$. Let us first treat the case $p \neq 2$. There is an integer $r \geq 0$ such that $n - p - 2 = rp + p - 1$. If $\gamma = 1 + \Pi\xi \in \Gamma_1$ is as before, it follows from Theorem 1.16 that

$$\gamma(u)(1 + \xi u) \equiv u + 2\xi^p u^{p+1} - \xi^{p+1}u^{p+2} + (\xi^{p+2} - \xi)u^{p+3} \mod \overline{\mathfrak{m}}^{p+4}$$

(cf. Remark 1.17 to see that we use the assumption $K = \mathbb{Q}_p$ here). As a consequence, a direct computation shows that

$$\gamma(u^{p-1}) = \gamma(u)^{p-1} \equiv u^{p-1} + \xi u^p - 3\xi^p u^{2p-1} - 2\xi^{p+1}u^{2p} + \xi u^{2p+1} \mod \overline{\mathfrak{m}}^{2p+2}.$$

Using $p > 2$, we obtain

$$\begin{aligned} &\gamma(u^{rp+p-1}) = \gamma(u^p)^r\gamma(u)^{p-1} \equiv (u^{rp} - r\xi^p u^{(r+1)p})\gamma(u^{p-1}) \\ \equiv\ & u^{(r+1)p-1} + \xi u^{(r+1)p} - (r+3)\xi^p u^{(r+2)p-1} \\ & -(r+2)\xi^{p+1}u^{(r+2)p} + \xi u^{(r+2)p+1} \mod \overline{\mathfrak{m}}^{(r+2)p+2}. \end{aligned} \tag{17}$$

A Vandermonde argument similar to the one above shows that

$$u^{n-p-1} + u^n \in \overline{\mathfrak{m}}^{n-p-2}(\Gamma_1) + \overline{\mathfrak{m}}^{n+1}.$$

Since $n - p - 1 = (r+1)p$ is divisible by p, Theorem 2.1 implies that $u^{n-p-1} \in T_1(\overline{\mathfrak{m}})$, completing the proof if $p \neq 2$.

If $p = 2$, let us first assume $n = 2j + 1$ with j odd. It follows from Theorem 1.16 that

$$\begin{aligned} \gamma(u^{n-2}) &= \gamma(u^2)^{j-1}\gamma(u) \equiv (u^2 + \xi^2u^4)^{j-1}(u + \xi u^2 + \xi^2u^3) \\ &\equiv u^{2(j-1)+1} + \xi u^{2j} + j\xi^2u^{2j+1} \mod \overline{\mathfrak{m}}^{n+1}. \end{aligned}$$

Since the image of j in k^{sep} is non-zero, we obtain $u^n \in \overline{\mathfrak{m}}^{n-2}(\Gamma_1) + T_1(\overline{\mathfrak{m}})$, as before.

If $p = 2$ and $n \geq 2$ with $n \equiv 1 \mod 4$, we write $n = i + j8$ with $j \geq 0$ and $i \in \{5, 9\}$. For $i = 5$ we compute

$$\gamma(u)^{1+j8} \equiv u^{j8}\gamma(u) \mod \overline{\mathfrak{m}}^{n+1} \equiv u^{j8}(u + \xi u^2 + \xi^2u^3 + \xi u^5) \mod \overline{\mathfrak{m}}^{n+1},$$

using that $(j + \ell)8 > n$ for any integer $\ell \geq 1$. As in the case $p > 2$ this implies $u^n = u^{5+j8} \in \overline{\mathfrak{m}}^{n-2p}(\Gamma_1) + T_1(\overline{\mathfrak{m}}) + \overline{\mathfrak{m}}^{n+1}$ because $u^{2+j8} \in T_1(\overline{\mathfrak{m}})$.

If $i = 9$ we have $n - 6 = 3 + j8$ and compute

$$\gamma(u)^{3+j8} \equiv \gamma(u)^3(u^{j8} + j\xi^2u^{(j+1)8}) \mod \overline{\mathfrak{m}}^{n+1} \equiv u^{j8}\gamma(u)^3 \mod \overline{\mathfrak{m}}^{n+1},$$

because $\gamma(u)^3 \in \overline{\mathfrak{m}}^3$ and $3 + (j+1)8 = n + 2$. A direct computation, using the enhanced approximation of $\gamma(u)$ in Remark 1.18, shows that

$$\begin{aligned} \gamma(u)^3 = \gamma(u)\gamma(u)^2 &\equiv (u + \xi u^2 + \xi^2u^3 + \xi u^5 + u^7)(u^2 + \xi^2u^4 + \xi u^6) \mod \overline{\mathfrak{m}}^{10} \\ &\equiv u^3 + \xi u^4 + u^6 + \xi u^7 + \xi^2u^8 + u^9 \mod \overline{\mathfrak{m}}^{10}. \end{aligned}$$

As above, this implies $u^n = u^{9+j8} \in \overline{\mathfrak{m}}^{n-3p}(\Gamma_1) + T_1(\overline{\mathfrak{m}}) + \overline{\mathfrak{m}}^{n+1}$ because $u^{6+j8} \in T_1(\overline{\mathfrak{m}})$. $\square$

Corollary 3.7. *Assume $h = 2$ and $K = \mathbb{Q}_p$. If n is a non-negative integer then the Λ-module $T_1^n(R/\breve{\mathfrak{o}})/T_1^{n+1}(R/\breve{\mathfrak{o}})$ is generated by the class of $u_1^{p^n}$.*

Proof. Note that the reduction of $R/\breve{\mathfrak{o}}$ modulo $\pi\breve{\mathfrak{o}}$ is Λ- and $\mathcal{H}$-equivariantly isomorphic to $\overline{\mathfrak{m}}$. According to [5], Corollary 1.5, it suffices to show that the k^{sep}-vector space $(T_1^n(\overline{\mathfrak{m}})/T_1^{n+1}(\overline{\mathfrak{m}}))_{\Gamma_1}$ is generated by the class of $u_1^{p^n}$. This was shown in the proof of Theorem 3.6. $\square$

Remark 3.8. As seen in (13), there is an outer automorphism σ_1 of Γ, extending to an $\breve{\mathfrak{o}}$-linear ring automorphism σ_1 of Λ, such that $T_1 \cdot \lambda = \sigma_1(\lambda) \cdot T_1$ as endomorphisms of R for all $\lambda \in \Lambda$. By Proposition 2.8, $R/\breve{\mathfrak{o}}$ therefore is a module over the *twisted power series ring* $\Lambda[[T_1; \sigma_1]]$. The latter consists of all formal power series $\sum_{n=0}^{\infty} \lambda_n T_1^n$ with $\lambda_n \in \Lambda$ and multiplication defined by

$$(\sum_{n=0}^{\infty} \lambda'_n T_1^n) \cdot (\sum_{m=0}^{\infty} \lambda_m T_1^m) := \sum_{i=0}^{\infty} (\sum_{n+m=i} \lambda'_n \sigma_1^n(\lambda_m)) T_1^i.$$

It follows from Corollary 3.7 and a topological version of the Nakayama lemma applied to the local pseudo-compact subring $\Lambda_1[[T_1; \sigma_1]]$ of $\Lambda[[T_1; \sigma_1]]$ (cf. [5], Corollary 1.5), that $R/\breve{\mathfrak{o}}$ is finitely generated over $\Lambda_1[[T_1; \sigma_1]]$ and $\Lambda[[T_1; \sigma_1]]$, provided $h = 2$ and $K = \mathbb{Q}_p$. A generator is given by the class of u_1.

The computations of Theorem 3.6 can be generalized so as to compute the graded pieces of the $\mathfrak{m}_{\Lambda_1}$-adic filtration of $\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}})$. This is the content of the subsequent proposition and of its corollary. For simplicity, we restrict to the case $p \neq 2$.

Proposition 3.9. *Assume $h = 2$ and $K = \mathbb{Q}_p$ with $p \geq 3$. Let r be a non-negative integer. If $1 \leq j \leq p-1$ then*

$$\mathfrak{m}_{\Lambda_1}^{r(p-1)+j} \cdot \overline{\mathfrak{m}} + T_1(\overline{\mathfrak{m}}) = \overline{\mathfrak{m}}^{2rp+p+j} + \sum_{\ell=j+1}^{p-1} k^{sep} u_1^{2rp+\ell} + T_1(\overline{\mathfrak{m}}).$$

In particular,

$$\begin{aligned} \mathfrak{m}_{\Lambda_1}^{r(p-1)+1} \cdot \overline{\mathfrak{m}} + T_1(\overline{\mathfrak{m}}) &= \overline{\mathfrak{m}}^{2rp+2} + T_1(\overline{\mathfrak{m}}) \quad \text{and} \\ \mathfrak{m}_{\Lambda_1}^{(r+1)(p-1)} \cdot \overline{\mathfrak{m}} + T_1(\overline{\mathfrak{m}}) &= \overline{\mathfrak{m}}^{2(r+1)p-1} + T_1(\overline{\mathfrak{m}}) \end{aligned}$$

for any non-negative integer r.

Proof. Set $u := u_1$. Assume the assertion to be true for $r(p-1)+j$ with $1 \leq j \leq p-1$ (which it is if $r = 0$ and $j = 1$, as follows from Corollary 3.7). We will then prove it for $r(p-1)+j+1$.

Let us first assume $j = p-1$, so that $r(p-1)+j+1 = (r+1)(p-1)+1$. It follows from [27], Proposition 26.5 and from Lemma 3.12 below, that Γ_1 is topologically generated by elements of the form $\gamma = 1 + \Pi\xi$ and $\gamma = 1 + \xi p$ with $\xi \in \mu_{p^2-1}$. By [27], Proposition 19.5, the ideal $\mathfrak{m}_{\Lambda_1}$ is generated by p and the corresponding elements $\gamma - 1$. By our computation of $(\gamma-1)(u^{p-1})$ in the proof of Theorem 3.6, as well as by Theorem 1.19, we have $\mathfrak{m}_{\Lambda_1} \cdot \overline{\mathfrak{m}}^{2(r+1)p-1} \subseteq \overline{\mathfrak{m}}^{2(r+1)p+2} + T_1(\overline{\mathfrak{m}})$. Further, as in Theorem 3.6, one can prove that equality holds, and hence that

$\mathfrak{m}_{\Lambda_1}^{(r+1)(p-1)+1} \cdot \overline{\mathfrak{m}} + T_1(\overline{\mathfrak{m}}) = \overline{\mathfrak{m}}^{2(r+1)p+2} + T_1(\overline{\mathfrak{m}})$, as required.

Now assume $j < p-1$. We have $\overline{\mathfrak{m}}^{2(r+1)p-1} \subseteq \mathfrak{m}_{\Lambda_1}^{r(p-1)+j} \cdot \overline{\mathfrak{m}} + T_1(\overline{\mathfrak{m}})$. As above, this yields $\overline{\mathfrak{m}}^{2(r+1)p+2} \subseteq \mathfrak{m}_{\Lambda_1}^{r(p-1)+j+1} \cdot \overline{\mathfrak{m}} + T_1(\overline{\mathfrak{m}})$. Further, $u^{2pr+p-1} \in \mathfrak{m}_{\Lambda_1}^{r(p-1)+j} \cdot \overline{\mathfrak{m}} + T_1(\overline{\mathfrak{m}})$. If $\gamma = 1 + \Pi\xi \in \Gamma_1$ for some element $\xi \in \mu_{p^2-1}$, then

$$\begin{aligned}\gamma(u^{2rp+p-1}) &\equiv u^{2rp+p-1} + \xi u^{(2r+1)p} - (2r+3)\xi^p u^{2(r+1)p-1} \\ &\quad -(2r+2)\xi^{p+1}u^{2(r+1)p} + \xi u^{2(r+1)p+1} \mod \overline{\mathfrak{m}}^{2(r+1)p+2}\end{aligned}$$

by (17). By the Vandermonde argument used before, we obtain that $u^{2(r+1)p+1}$ and hence that $\overline{\mathfrak{m}}^{2(r+1)p+1}$ is contained in $\mathfrak{m}_{\Lambda_1}^{r(p-1)+j+1} \cdot \overline{\mathfrak{m}} + T_1(\overline{\mathfrak{m}})$.

Further, $u^{2rp+p+p-2} \in \mathfrak{m}_{\Lambda_1}^{r(p-1)+j} \cdot \overline{\mathfrak{m}} + T_1(\overline{\mathfrak{m}})$, by assumption. Applying a suitable element $\gamma - 1 \in \mathfrak{m}_{\Lambda_1}$, we obtain $u^{2rp+p+p-1} \in \mathfrak{m}_{\Lambda_1}^{r(p-1)+j+1} \cdot \overline{\mathfrak{m}} + T_1(\overline{\mathfrak{m}})$. Going down step by step, we obtain $\overline{\mathfrak{m}}^{2rp+p+j+1} \subseteq \mathfrak{m}_{\Lambda_1}^{r(p-1)+j+1} \cdot \overline{\mathfrak{m}} + T_1(\overline{\mathfrak{m}})$. It now remains to see that

$$\mathfrak{m}_{\Lambda_1} \cdot \sum_{\ell=j+1}^{p-1} k^{\mathrm{sep}} u^{2rp+\ell} \equiv \sum_{\ell=j+2}^{p-1} k^{\mathrm{sep}} u^{2rp+\ell} \mod \overline{\mathfrak{m}}^{2rp+p+j+1} + T_1(\overline{\mathfrak{m}}).$$

Let $\xi \in \mu_{p^2-1}$. If $\gamma = 1 + \xi p$ and $n \geq 2pr + j + 1$ then Theorem 1.19 implies $(\gamma - 1)(u^n) \in \overline{\mathfrak{m}}^{n+p+1} \subseteq \overline{\mathfrak{m}}^{2pr+p+j+1}$. If $\gamma = 1 + \Pi\xi$ and $j + 1 \leq \ell \leq p-1$ then write $\ell = p - i$ with $1 \leq i \leq p - j - 1$. According to Theorem 1.16 we have

$$\begin{aligned}\gamma(u^{2pr+\ell}) &\equiv u^{2pr}\gamma(u)^\ell \equiv u^{2pr}\frac{(1+\xi u)^i}{1+\xi^p u^p}u^\ell \\ &\equiv (1+\xi u)^i u^{2pr+\ell} \mod \overline{\mathfrak{m}}^{2pr+p+j+1}.\end{aligned}$$

This shows $\mathfrak{m}_{\Lambda_1} \cdot \sum_{\ell=j+1}^{p-1} k^{\mathrm{sep}} u^{2rp+\ell} \subseteq \sum_{\ell=j+2}^{p-1} k^{\mathrm{sep}} u^{2rp+\ell} + \overline{\mathfrak{m}}^{2rp+p+j+1} + T_1(\overline{\mathfrak{m}})$. Since $(\gamma-1)(u^{2rp+\ell}) \equiv -\ell\xi u^{2rp+\ell+1} \mod \overline{\mathfrak{m}}^{2pr+\ell+2}$ with $\ell \not\equiv 0 \mod p$, a downward induction as above shows that conversely

$$\sum_{\ell=j+2}^{p-1} k^{\mathrm{sep}} u^{2rp+\ell} \subseteq \mathfrak{m}_{\Lambda_1} \cdot \sum_{\ell=j+1}^{p-1} k^{\mathrm{sep}} u^{2rp+\ell} + \overline{\mathfrak{m}}^{2rp+p+j+1} + T_1(\overline{\mathfrak{m}}).$$

This completes the proof. □

Remark 3.10. Assume $K = \mathbb{Q}_p$, $h = 2$, and let n be a positive integer. Modulo $p\Lambda_1$, the ideal $\mathfrak{m}_{\Lambda_1}^{p^n}$ is generated by the maximal ideal of the local ring $\Lambda(\Gamma_{2n+1}) = \Lambda(\Gamma_1^{p^n})$. Proposition 3.9 shows that if $\gamma \in \Gamma_{2n+1}$, then the power series $(\gamma-1)(u_1)$ must generically have u_1-order $\sum_{i=0}^n 2p^i$. This is in accordance with the result [6], Theorem 2, of Chai.

As an immediate consequence of Proposition 3.9 we obtain the following result.

Corollary 3.11. *Assume $h = 2$ and $K = \mathbb{Q}_p$ with $p \geq 3$. For any integer $i \geq 0$ let*

$$\mathrm{gr}^i(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}})) := [\mathfrak{m}_{\Lambda_1}^i \cdot (\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))]/[\mathfrak{m}_{\Lambda_1}^{i+1} \cdot (\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))].$$

(i) *If $i = 0$ then the k^{sep}-vector space $\mathrm{gr}^i(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))$ is one dimensional. A k^{sep}-basis is given by the class of u_1.*

(ii) *If $i > 0$ then the k^{sep}-vector space $\mathrm{gr}^i(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))$ is two dimensional. Write $i = r(p-1) + j$ with $r \geq 0$ and $1 \leq j \leq p-1$. If $j \neq p-1$ then a k^{sep}-basis of $\mathrm{gr}^i(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))$ is given by the classes of $u_1^{2rp+j+1}$ and $u_1^{2rp+p+j}$. If $j = p-1$ then a k^{sep}-basis is given by the classes of $u_1^{2(r+1)p-1}$ and $u_1^{2(r+1)p+1}$.* $\square$

Let h and K be arbitrary again and define

$$\mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}})) := \bigoplus_{i=0}^{\infty} [\mathfrak{m}_{\Lambda_1}^i \cdot (\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))]/[\mathfrak{m}_{\Lambda_1}^{i+1} \cdot (\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))]$$

as above. The action of the center $Z := \mathfrak{o}^*$ of Γ is trivial on R (cf. [11], Proposition 14.13). Therefore, $\mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))$ naturally is a module over the graded k^{sep}-algebra

$$\mathrm{gr}(\overline{\Lambda(\Gamma_1/Z_1)}) := \bigoplus_{i=0}^{\infty} \overline{\mathfrak{m}}_{\Lambda(\Gamma_1/Z_1)}^i / \overline{\mathfrak{m}}_{\Lambda(\Gamma_1/Z_1)}^{i+1}.$$

Here $Z_1 := \Gamma_1 \cap Z$, $\overline{\Lambda(\Gamma_1/Z_1)} := \Lambda(\Gamma_1/Z_1)/\pi\Lambda(\Gamma_1/Z_1)$ and $\overline{\mathfrak{m}}_{\Lambda(\Gamma_1/Z_1)}$ denotes the maximal ideal of the local ring $\overline{\Lambda(\Gamma_1/Z_1)}$.

In a special case, fundamental results of Lazard allow us to explicitly describe the structure of the ring $\mathrm{gr}(\overline{\Lambda(\Gamma_1/Z_1)})$. A recent exposition of the necessary techniques was given in [27], Part B.

For the rest of this article assume $K = \mathbb{Q}_p$ with $p > h+1$. Recall that v_D denotes the valuation on the $\mathbb{Q}_p$-division algebra D, extending the p-adic valuation v on $\mathbb{Q}_p$. In particular, $v_D(\delta) \geq v_D(\Pi) = \frac{1}{h} > \frac{1}{p-1}$ for any element $\delta \in \Pi\mathfrak{o}_D$. Consider the map $\omega : \Gamma_1 \smallsetminus \{1\} \to (\frac{1}{p-1}, \infty) \subset \mathbb{R}$, defined by $\omega(\gamma) := v_D(\gamma - 1)$. As in [27], Example 23.2, one shows that ω is a *p-valuation* on Γ_1 in the sense of [27], page 169.

If $i \geq 1$, and if $\Gamma_i := 1 + \Pi^i\mathfrak{o}_D$ is as before, then $\Gamma_i = \{\gamma \in \Gamma_1 \mid \omega(\gamma) \geq \frac{i}{h}\}$ and $\Gamma_{i+1} = \{\gamma \in \Gamma_1 \mid \omega(\gamma) > \frac{i}{h}\}$. In the notation of [27], page 170, this means $\Gamma_i = (\Gamma_1)_{\frac{i}{h}}$ and $\Gamma_{i+1} = (\Gamma_1)_{\frac{i}{h}+}$. It is a general property of p-valued groups which can be checked directly here, that $[\Gamma_i, \Gamma_j] \subseteq \Gamma_{i+j}$ and $\Gamma_i^p \subseteq \Gamma_{i+h} \subseteq \Gamma_{i+1}$. Therefore, $\mathrm{gr}^i(\Gamma_1) := \Gamma_i/\Gamma_{i+1}$ is an abelian group of exponent p, i.e. a k-vector space (note that $k = \mathbb{F}_p$ since $K = \mathbb{Q}_p$). In fact, in our situation the structure of $\mathrm{gr}^i(\Gamma_1)$ can be made more explicit. Namely, the map $(1 + \Pi^i\delta \mapsto \delta + \Pi\mathfrak{o}_D) : \Gamma_i \to \mathfrak{o}_D/\Pi\mathfrak{o}_D \simeq k_h$ induces an isomorphism of k-vector spaces $\mathrm{gr}^i(\Gamma_1) \simeq k_h$ for all integers $i \geq 1$ (cf. [25], 1.4.4 Proposition 1.8).

According to [27], Lemma 23.4, Lemma 23.5 and Proposition 25.3, the graded k-vector space

$$\mathfrak{g} := \bigoplus_{i=1}^{\infty} \mathrm{gr}^i(\Gamma_1) = \bigoplus_{i=1}^{\infty} \Gamma_i/\Gamma_{i+1}$$

becomes a Lie algebra over the polynomial ring $k[t]$ in the variable t by setting

$$\begin{aligned}[\gamma\Gamma_{i+1}, \gamma'\Gamma_{j+1}] &:= \gamma\gamma'\gamma^{-1}(\gamma')^{-1}\Gamma_{i+j+1} \text{ for } \gamma \in \Gamma_i, \gamma' \in \Gamma_j, \quad \text{and} \\ t \cdot \gamma\Gamma_{i+1} &:= \gamma^p\Gamma_{i+h+1} \text{ for } \gamma \in \Gamma_i.\end{aligned}$$

Lemma 3.12. *The natural map $k[t] \otimes_k (\bigoplus_{i=1}^h \operatorname{gr}^i(\Gamma_1)) \to \mathfrak{g}$ is an isomorphism of $k[t]$-modules.*

Proof. The assertion is equivalent to the claim that for any integer $i \geq 1$ the map $(\gamma\Gamma_{i+1} \mapsto \gamma^p\Gamma_{i+h+1}) : \operatorname{gr}^i(\Gamma_1) \to \operatorname{gr}^{i+h}(\Gamma_1)$ is bijective. That it is injective, follows from one of the axioms of a p-valuation, viz. $\omega(\gamma^p) = \omega(\gamma) + 1$.

If $\gamma = 1 + \Pi^{i+h}\delta = 1 + \Pi^i p\delta \in \Gamma_{i+h}$ with $\delta \in \mathfrak{o}_D^*$, then $\tilde{\gamma} := 1 + \Pi^i\delta \in \Gamma_i$ satisfies

$$\tilde{\gamma}^p = 1 + p\Pi^i\delta + \sum_{j=2}^{p-1} \binom{p}{j} (\Pi^i\delta)^j + (\Pi^i\delta)^p \quad \text{in} \quad D.$$

If $2 \leq j \leq p-1$ then $v_D(\binom{p}{j}(\Pi^i\delta)^j) = 1 + \frac{ij}{h} > 1 + \frac{i}{h} = v_D(\Pi^i p)$. Further, $v_D((\Pi^i\delta)^p) = \frac{pi}{h} = \frac{p-1}{h}i + \frac{i}{h} > 1 + \frac{i}{h}$ by our assumption on p. The above explicit form of the isomorphism $\Gamma_{i+h}/\Gamma_{i+h+1} \simeq \mathfrak{o}_D/\Pi\mathfrak{o}_D$ then implies that $\tilde{\gamma}^p \equiv \gamma \mod \Gamma_{i+h+1}$. $\square$

For any $1 \leq i \leq h$ let $(\gamma_{ij})_{1\leq j\leq h}$ be a family of elements of Γ_i whose images in Γ_i/Γ_{i+1} form a k-basis. It follows from Lemma 3.12 and [27], Proposition 26.5, that for any fixed ordering, the family $(\gamma_{ij})_{1\leq i,j\leq h}$ is an ordered basis of the p-valued group (Γ_1, ω) in the sense of [27], page 182. Setting $b_{ij} := \gamma_{ij} - 1 \in \mathfrak{m}_{\Lambda_1}$ and $b^\alpha := \prod_{i,j} b_{ij}^{\alpha_{ij}}$, it is explained in [27], section 28, that any element $\lambda \in \Lambda_1 = \Lambda(\Gamma_1)$ admits a unique expansion of the form $\lambda = \sum_{\alpha\in\mathbb{N}^{h\times h}} c_\alpha b^\alpha$ with $c_\alpha \in \breve{\mathfrak{o}}$.

For any non-negative real number ν, we let J_ν denote the closure of the $\breve{\mathfrak{o}}$-submodule of Λ_1 generated by all elements of the form $p^\ell(h_1 - 1) \cdot \ldots \cdot (h_s - 1)$ with $\ell, s \geq 0$, $h_1, \ldots, h_s \in \Gamma_1$ and $\ell + \omega(h_1) + \ldots + \omega(h_s) \geq \nu$. According to [27], page 197, each J_ν is an open, two-sided ideal of Λ_1. Note that if i is the unique non-negative integer satisfying $\frac{i-1}{h} < \nu \leq \frac{i}{h}$ then $J_\nu = J_{\frac{i}{h}}$. As a consequence, $J_{\nu+} := \bigcup_{\nu'>\nu} J_{\nu'} = J_{\frac{i+1}{h}}$.

Recall that $\overline{\mathfrak{m}}_{\Lambda(\Gamma_1/Z_1)}$ denotes the maximal ideal of the local ring $\overline{\Lambda(\Gamma_1/Z_1)} = \Lambda(\Gamma_1/Z_1)/p\Lambda(\Gamma_1/Z_1)$.

Lemma 3.13. *For any integer $i \geq 0$ the image $\overline{J}_{\frac{i}{h}}$ of the ideal $J_{\frac{i}{h}}$ of Λ_1 under the natural ring homomorphism $\Lambda_1 \to \overline{\Lambda(\Gamma_1/Z_1)}$ is equal to $\overline{\mathfrak{m}}^i_{\Lambda(\Gamma_1/Z_1)}$.*

Proof. Since $J_0 = \Lambda_1$ (cf. [27], page 197), the case $i = 0$ is clear.

The maximal ideal $\mathfrak{m}_{\Lambda_1}$ of Λ_1 is a closed $\breve{\mathfrak{o}}$-submodule containing the elements $p^\ell(h_1 - 1) \cdot \ldots \cdot (h_s - 1)$ for all $\ell, s \geq 0$, $h_1, \ldots, h_s \in \Gamma_1$. This implies $J_{\frac{1}{h}} \subseteq \mathfrak{m}_{\Lambda_1}$. Conversely, $\mathfrak{m}_{\Lambda_1}$ is generated by p and b_{ij}, $1 \leq i, j \leq h$ (cf. [27], Proposition 19.5). Since $\omega(\gamma_{ij}) = \frac{i}{h} \geq \frac{1}{h}$ for all i, j, all of these are contained in $J_{\frac{1}{h}}$. Thus, $J_{\frac{1}{h}} = \mathfrak{m}_{\Lambda_1}$. Since the image of $\mathfrak{m}_{\Lambda_1}$ under $\Lambda_1 \to \overline{\Lambda(\Gamma_1/Z_1)}$ is precisely $\overline{\mathfrak{m}}_{\Lambda(\Gamma_1/Z_1)}$, this proves the lemma in the case $i = 1$.

If $i \geq 1$ then $\mathfrak{m}^i_{\Lambda_1} = J^i_{\frac{1}{h}} \subseteq J_{\frac{i}{h}}$, whence $\overline{\mathfrak{m}}^i_{\Lambda(\Gamma_1/Z_1)} \subseteq \overline{J}_{\frac{i}{h}}$. It remains to prove the reverse inclusion.

Consider the descending central series $(C^{(m)})_{m\geq 0}$ of Γ_1 defined by $C^{(0)} := \Gamma_1$ and $C^{(m+1)} := [\Gamma_1, C^{(m)}]$ for $m \geq 0$. We claim that $\gamma - 1 \in \mathfrak{m}^{m+1}_{\Lambda_1}$ for any $\gamma \in C^{(m)}$. This is clear for $m = 0$. Assume the assertion to be correct for $m \geq 0$ and let $\gamma \in C^{(m+1)}$. There are elements $\gamma_1, \ldots, \gamma_n \in \Gamma_1$ and $\delta_1, \ldots, \delta_n \in C^{(m)}$ such that $\gamma = [\gamma_1, \delta_1] \cdot \ldots \cdot [\gamma_n, \delta_n]$ is the product of the commutators $[\gamma_j, \delta_j] := \gamma_j \delta_j \gamma_j^{-1} \delta_j^{-1}$. Since

$$\gamma - 1 = [\gamma_1, \delta_1]([\gamma_2, \delta_2] \cdot \ldots \cdot [\gamma_n, \delta_n] - 1 + 1 - [\delta_1, \gamma_1]),$$

an inductive argument allows us to assume $n = 1$. In this case the assertion follows from

$$[\gamma_1, \delta_1] - 1 = \gamma_1 \delta_1((\gamma_1^{-1} - 1)(\delta_1^{-1} - 1) - (\delta_1^{-1} - 1)(\gamma_1^{-1} - 1))$$

and the induction hypothesis.

Note that $C^{(m)} \subseteq \Gamma_{m+1}$. According to [25], 1.4.4 Proposition 1.8 and the remark following [25], Theorem 1.9, the composition of the maps $C^{(m)} \hookrightarrow \Gamma_{m+1} \to \Gamma_{m+1}/\Gamma_{m+2} \simeq k_h$ is surjective for $0 \leq m < h - 1$. For $m = h - 1$ its image is $\ker(\mathrm{tr}_{k_h|k})$, where $\mathrm{tr}_{k_h|k} : k_h \to k$ denotes the trace map. Note that $k_h = k \oplus \ker(\mathrm{tr}_{k_h|k})$ because $\mathrm{tr}_{k_h|k}$ is surjective and because $\mathrm{tr}_{k_h|k}(\alpha) = h \cdot \alpha \neq 0$ for all $\alpha \in k^*$ by our assumption $h < p - 1$. Note also that $k \subseteq k_h$ coincides with the image of $Z_1 = 1 + p\mathfrak{o} \subseteq \Gamma_h$ under $\Gamma_h \to \Gamma_h/\Gamma_{h+1} \simeq k_h$. It follows that the p-valued group (Γ_1, ω) admits an ordered basis $(\gamma_{rs})_{1\leq r,s\leq h}$ such that $\gamma_{rs} \in C^{(r-1)}$ for all $1 \leq r, s \leq h$ with $(r, s) \neq (h, h)$ and such that $\gamma_{hh} = 1 + p \in Z_1$.

As before, we set $b_{rs} := \gamma_{rs} - 1$ for all r, s. It follows from [27], Theorem 28.3 (ii), that any element $\lambda \in J_{\frac{i}{h}}$ has the property that its expansion $\lambda = \sum_\alpha c_\alpha b^\alpha$ satisfies $v(c_\alpha) + \sum_{r,s} \alpha_{rs}\omega(\gamma_{rs}) \geq \frac{i}{h}$ for any $\alpha \in \mathbb{N}^{h\times h}$. If $c_\alpha \in p\breve{\mathfrak{o}}$ or if $\alpha_{hh} > 0$ then $c_\alpha b^\alpha$ maps to zero in $\overline{\Lambda(\Gamma_1/Z_1)}$. Otherwise, $\sum_{(r,s)\neq(h,h)} \alpha_{rs}\frac{r}{h} \geq \frac{i}{h}$, i.e. $\sum_{(r,s)\neq(h,h)} r\alpha_{rs} \geq i$. In this case $c_\alpha b^\alpha \in \mathfrak{m}^i_{\Lambda_1}$ because $b_{rs} \in \mathfrak{m}^r_{\Lambda_1}$, as was shown above. As a consequence, λ maps to $\overline{\mathfrak{m}}^i_{\Lambda(\Gamma_1/Z_1)}$, as required. $\square$

It follows from the proof of Lemma 3.13 that

$$\overline{\mathfrak{g}} := \mathfrak{g}/(t \cdot \mathfrak{g})$$

is an h^2-dimensional nilpotent Lie algebra over k with k-basis $(\gamma_{ij}\Gamma_{i+1})_{i,j}$. Denote by $\overline{\mathfrak{z}}$ the one dimensional central Lie subalgebra of $\overline{\mathfrak{g}}$ generated by the element $\mathfrak{z} := \gamma_{hh}\Gamma_{h+1}$. By abuse of notation we shall also write $\overline{\mathfrak{g}}/\overline{\mathfrak{z}}$ for the Lie algebra $\overline{\mathfrak{g}}/\overline{\mathfrak{z}} \otimes_k k^{\mathrm{sep}}$ over k^{sep}.

Corollary 3.14. *Denoting by $U(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}) := U_{k^{sep}}(\overline{\mathfrak{g}}/\overline{\mathfrak{z}})$ the universal enveloping algebra of $\overline{\mathfrak{g}}/\overline{\mathfrak{z}}$, there is an isomorphism*

$$U(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}) \simeq \mathrm{gr}(\overline{\Lambda(\Gamma_1/Z_1)})$$

of graded k^{sep}-algebras.

Proof. According to [27], Theorem 28.3, the maps $\gamma\Gamma_{i+1} \mapsto (\gamma-1)+J_{\frac{i+1}{h}}$ induce an isomorphism

$$k^{\mathrm{sep}}[t] \otimes_{k[t]} U_{k[t]}(\mathfrak{g}) \xrightarrow{\sim} \bigoplus_{i\geq 0} J_{\frac{i}{h}}/J_{\frac{i+1}{h}}$$

of graded k^{sep}-algebras, sending t to $p + J_{\frac{h+1}{h}}$. It gives rise to an isomorphism

$$U(\overline{\mathfrak{g}}) \xrightarrow{\sim} \bigoplus_{i\geq 0} \overline{J}_{\frac{i}{h}}/\overline{J}_{\frac{i+1}{h}}$$

by reduction modulo t. As a consequence of Lemma 3.13, there is a surjective homomorphism $\bigoplus_{i\geq 0} \overline{J}_{\frac{i}{h}}/\overline{J}_{\frac{i+1}{h}} \to \mathrm{gr}(\overline{\Lambda(\Gamma_1/Z_1)})$, whose kernel contains the image of $\mathfrak{z} \cdot U(\overline{\mathfrak{g}})$. We thus obtain a surjective homomorphism

$$U(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}) \simeq U(\overline{\mathfrak{g}})/\mathfrak{z}U(\overline{\mathfrak{g}}) \longrightarrow \mathrm{gr}(\overline{\Lambda(\Gamma_1/Z_1)})$$

of graded k^{sep}-algebras that we claim to be bijective. By the Poincaré-Birkhoff-Witt Theorem, it suffices to see that for any integer $i \geq 0$ the elements b^α with $\alpha_{hh} = 0$ and $\tau(\alpha) := \sum_{r,s} r\alpha_{rs} = i$ are k^{sep}-linearly independent in $\overline{\mathfrak{m}}^i_{\Lambda(\Gamma_1/Z_1)}/\overline{\mathfrak{m}}^{i+1}_{\Lambda(\Gamma_1/Z_1)}$.

Assume $\lambda = \sum_{\alpha_{hh}=0,\tau(\alpha)=i} c_\alpha b^\alpha \in \overline{\mathfrak{m}}^{i+1}_{\Lambda(\Gamma_1/Z_1)}$ with coefficients $c_\alpha \in k^{\mathrm{sep}}$, not all of which are zero. Viewing $\lambda \in \overline{\Lambda}_1$, this is equivalent to the existence of an element $\lambda' \in \ker(\overline{\Lambda}_1 \to \overline{\Lambda(\Gamma_1/Z_1)})$ such that $\lambda + \lambda' \in \overline{J}_{\frac{i+1}{h}}$ (cf. Lemma 3.13). Note that Z_1 is a p-valued group in its own right with ordered basis γ_{hh}. In particular, $\overline{\mathfrak{m}}_{\Lambda(Z_1)} = b_{hh}\overline{\Lambda(Z_1)}$. It follows from [35], Proposition 7.1.2 (c), that the kernel of the natural map $\overline{\Lambda}_1 \to \overline{\Lambda(\Gamma_1/Z_1)}$ is the closed ideal $b_{hh}\overline{\Lambda}_1$. Writing $\lambda' = \sum_\beta d_\beta b^\beta$, [27], Theorem 28.3 (ii), implies

$$\frac{i+1}{h} \leq \min\{\frac{i}{h}, \inf_\beta\{\frac{\tau(\beta)}{h} \mid d_\beta \neq 0\}\},$$

which is impossible. $\square$

Remark 3.15. For uniform pro-p groups, results as in Corollary 3.14 are true in much greater generality (cf. [35], Theorem 8.7.10). We point out, however, that the p-valued group Γ_1 is not uniform for any $h \geq 2$, and that the filtrations $(\overline{J}_{\frac{i}{h}})_{i\geq 0}$ and $(\overline{\mathfrak{m}}^i_{\Lambda_1})_{i\geq 0}$ of $\overline{\Lambda}_1$ do not coincide.

According to Corollary 3.14, we may view $\mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))$ as a module over $U(\overline{\mathfrak{g}}/\overline{\mathfrak{z}})$. If $h = 2$, the precise structure of this module is given by Theorem 3.16 below. Let us first introduce some notation.

Assume $h = 2$, choose $\xi \in \mu_{p^2-1} \smallsetminus \mu_{p-1}$ and set $\gamma_{11} := 1 + \Pi$, $\gamma_{12} := 1 + \Pi\xi$, $\gamma_{21} := \gamma_{11}\gamma_{12}\gamma_{11}^{-1}\gamma_{12}^{-1}$ and $\gamma_{22} := 1 + p$. We claim that these elements form an ordered basis of (Γ_1, ω). Computing

$$\begin{aligned}
\gamma_{11}\gamma_{12}\gamma_{11}^{-1}\gamma_{12}^{-1} &\equiv (1+\Pi)(1+\Pi\xi)(1-\Pi+p)(1-\Pi\xi+\xi^{1+p}p) \\
&\equiv (1+\Pi(1+\xi)+\xi p)(1-\Pi(1+\xi)+(1+\xi+\xi^{1+p})p) \\
&\equiv 1+(\xi-\xi^p)p \mod \Gamma_3,
\end{aligned}$$

we have $\omega(\gamma_{11}) = \omega(\gamma_{12}) = \frac{1}{2}$ and $\omega(\gamma_{21}) = \omega(\gamma_{22}) = 1$. Since the images of 1 and ξ in k_2 form a basis over k, as do the images of 1 and $\xi - \xi^p$, the claim follows from Lemma 3.12 and [27], Proposition 26.5. We denote by $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{h}$ and $\mathfrak{z}$ the images of γ_{11}, γ_{12}, γ_{21} and γ_{22} in $\mathrm{gr}^1(\Gamma_1)$, $\mathrm{gr}^1(\Gamma_1)$, $\mathrm{gr}^2(\Gamma_1)$ and $\mathrm{gr}^2(\Gamma_1)$, respectively. It follows from Lemma 3.12 that these four elements form a k^{sep}-basis of $\overline{\mathfrak{g}} \otimes_k k^{\mathrm{sep}}$. They satisfy

$$[\mathfrak{x}, \mathfrak{y}] = \mathfrak{h} \quad \text{and} \quad [\mathfrak{x}, \mathfrak{h}] = [\mathfrak{x}, \mathfrak{z}] = [\mathfrak{y}, \mathfrak{h}] = [\mathfrak{y}, \mathfrak{z}] = [\mathfrak{h}, \mathfrak{z}] = [\mathfrak{h}, \mathfrak{h}] = [\mathfrak{z}, \mathfrak{z}] = 0.$$

Theorem 3.16. *Assume $h = 2$ and $K = \mathbb{Q}_p$ with $p > 3$. The left ideal I of $U(\overline{\mathfrak{g}}/\overline{\mathfrak{z}})$ generated by $\mathfrak{h}$ and $\xi\mathfrak{x} - \mathfrak{y}$ is a two-sided ideal. There is a non-split exact sequence of $U(\overline{\mathfrak{g}}/\overline{\mathfrak{z}})$-modules*

$$0 \longrightarrow U(\overline{\mathfrak{g}}/\overline{\mathfrak{z}})/I \longrightarrow \mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}})) \longrightarrow U(\overline{\mathfrak{g}}/\overline{\mathfrak{z}})/I \longrightarrow 0.$$

Proof. Since $\mathfrak{h}$ is contained in the center of $\overline{\mathfrak{g}}$, the left ideal of $U(\overline{\mathfrak{g}}/\overline{\mathfrak{z}})$, generated by $\mathfrak{h}$, is a two-sided ideal. Further, the ring $U(\overline{\mathfrak{g}}/\overline{\mathfrak{z}})/(\mathfrak{h}) \simeq U(\overline{\mathfrak{g}}/(k^{\mathrm{sep}}\mathfrak{h} + k^{\mathrm{sep}}\mathfrak{z}))$ is commutative because the Lie algebra $\overline{\mathfrak{g}}/(k^{\mathrm{sep}}\mathfrak{h} + k^{\mathrm{sep}}\mathfrak{z})$ is. This implies the first assertion.

We continue by explicitly computing the action of $\mathfrak{h}$ on $\mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))$. Note that $\mathfrak{h}$ corresponds to $(1 + (\xi - \xi^p)p) - 1 \in \overline{\mathfrak{m}}^2_{\Lambda(\Gamma_1/Z_1)}/\overline{\mathfrak{m}}^3_{\Lambda(\Gamma_1/Z_1)}$. Choosing $\zeta \in \mu_{p^2-1} \subset \mathfrak{o}_2^* \subset \mathfrak{o}_2$ whose reduction modulo $p\mathfrak{o}_2$ is equal to $\xi - \xi^p$, $\mathfrak{h}$ is equal to the class of $\gamma - 1$, where $\gamma := 1 + \zeta p$. Note that $\zeta \notin \mu_{p-1}$, so that the image of $\eta := \zeta^p - \zeta$ in k^{sep} is non-zero.

The element $\mathfrak{h}$ defines a graded endomorphism of degree 2 of $\mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))$. Setting $\mathrm{gr}^i := \mathrm{gr}^i(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))$, we need to compute $\mathfrak{h} : \mathrm{gr}^i \to \mathrm{gr}^{i+2}$ for any non-negative integer i. First assume $i > 0$ and write $i = r(p-1) + j$ with integers $r \geq 0$ and $1 \leq j \leq p-1$. According to the form of gr^i, as given in Corollary 3.11, we have to distinguish several cases. Set $u := u_1$ and note first that we have

$$(\gamma - 1)(u^n) \equiv n\eta u^{n+p+1} + (n\eta + \frac{n(n+1)}{2}\eta^2)u^{n+2p+2} \mod \overline{\mathfrak{m}}^{n+2p+3}$$

for any integer $n \geq 1$ (cf. Theorem 1.19).

If $1 \leq j < p-3$ then $i + 2 = r(p-1) + j + 2$ with $j + 2 < p - 1$. Thus,

$$\begin{aligned} \mathrm{gr}^i &= k^{\mathrm{sep}}u^{2rp+j+1} + k^{\mathrm{sep}}u^{2rp+p+j} \qquad \text{and} \\ \mathrm{gr}^{i+2} &= k^{\mathrm{sep}}u^{2rp+j+3} + k^{\mathrm{sep}}u^{2rp+p+j+2} \end{aligned}$$

by Corollary 3.11. Using the above approximation of $(\gamma - 1)(u^n)$, we find

$$\mathfrak{h} \cdot u^{2rp+j+1} = (j+1)\eta u^{2rp+p+j+2} \quad \text{and} \quad \mathfrak{h} \cdot u^{2rp+p+j} = 0 \text{ in } \mathrm{gr}^{i+2}.$$

If $j = p - 3$ then $i + 2 = r(p-1) + p - 1$, so that

$$\begin{aligned} \mathrm{gr}^i &= k^{\mathrm{sep}}u^{2rp+p-2} + k^{\mathrm{sep}}u^{2(r+1)p-3} \qquad \text{and} \\ \mathrm{gr}^{i+2} &= k^{\mathrm{sep}}u^{2(r+1)p-1} + k^{\mathrm{sep}}u^{2(r+1)p+1} \end{aligned}$$

by Corollary 3.11. In this case we obtain

$$\mathfrak{h}\cdot u^{2rp+p-2} = -2\eta u^{2(r+1)p-1} \quad \text{and} \quad \mathfrak{h}\cdot u^{2(r+1)p-3} = 0 \text{ in } \mathrm{gr}^{i+1}.$$

If $j = p-2$ then $i+2 = (r+1)(p-1)+1$, so that

$$\begin{array}{rcll} \mathrm{gr}^i &=& k^{\mathrm{sep}}u^{2rp+p-1} + k^{\mathrm{sep}}u^{2(r+1)p-2} & \text{and} \\ \mathrm{gr}^{i+2} &=& k^{\mathrm{sep}}u^{2(r+1)p+2} + k^{\mathrm{sep}}u^{2(r+1)p+p+1}. & \end{array}$$

By the above approximation of $(\gamma-1)(u^n)$ we have

$$\begin{array}{rcll} \mathfrak{h}\cdot u^{2rp+p-1} &=& -\eta u^{2(r+1)p} - \eta u^{2(r+1)p+p+1} = -\eta u^{2(r+1)p+p+1} & \text{and} \\ \mathfrak{h}\cdot u^{2(r+1)p-2} &=& -2\eta u^{2(r+1)p+p-1} = 0 \quad \text{in}\, \mathrm{gr}^{i+2}. & \end{array}$$

If $j = p-1$ then $i+2 = (r+1)(p-1)+2$, so that

$$\begin{array}{rcll} \mathrm{gr}^i &=& k^{\mathrm{sep}}u^{2(r+1)p-1} + k^{\mathrm{sep}}u^{2(r+1)p+1} & \text{and} \\ \mathrm{gr}^{i+2} &=& k^{\mathrm{sep}}u^{2(r+1)p+3} + k^{\mathrm{sep}}u^{2(r+1)p+p+2}. & \end{array}$$

In this case, we find

$$\begin{array}{rcl} \mathfrak{h}\cdot u^{2(r+1)p-1} &=& -\eta u^{2(r+1)p+p} = 0 \quad \text{and} \\ \mathfrak{h}\cdot u^{2(r+1)p+1} &=& \eta u^{2(r+1)p+p+2} \quad \text{in}\, \mathrm{gr}^{i+2}. \end{array}$$

Finally, we consider the case $i = 0$, in which

$$\mathrm{gr}^0 = k^{\mathrm{sep}}u \quad \text{and} \quad \mathrm{gr}^2 = k^{\mathrm{sep}}u^3 + k^{\mathrm{sep}}u^{p+2}.$$

We have $\mathfrak{h}\cdot u = \eta u^{p+2}$ in gr^2.

Altogether, we obtain $\ker(\mathfrak{h}) = \bigoplus_{i\geq 1}(\mathrm{gr}^i \cap \ker(\mathfrak{h}))$ where $\mathrm{gr}^i \cap \ker(\mathfrak{h})$ is the one-dimensional k^{sep}-vector space generated by $u^{2rp+p+j}$ if $i = r(p-1)+j$ is written as before. In particular, $u^{p+1} \in \ker(\mathfrak{h})$. Further, $\ker(\mathfrak{h})$ is a $U(\overline{\mathfrak{g}}/\overline{\mathfrak{z}})$-submodule of $\mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))$ because $\mathfrak{h}$ is central in $\overline{\mathfrak{g}}/\overline{\mathfrak{z}}$. We claim that the map

$$\psi := (\delta \mapsto \delta\cdot u^{p+1}) : U(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}) \to \ker(\mathfrak{h})$$

is surjective with kernel I. By construction, $\mathfrak{h} \in \ker(\psi)$ so that ψ factors through $U(\overline{\mathfrak{g}}/\overline{\mathfrak{z}})/(\mathfrak{h}) \simeq k^{\mathrm{sep}}[\mathfrak{x}, \mathfrak{y}]$.

Let θ be an arbitrary (p^2-1)-th root of unity, set $\gamma := 1+\Pi\theta \in \Gamma_1$, fix a positive integer i, and consider the k^{sep}-linear map $(\gamma-1) : \mathrm{gr}^i \cap \ker(\mathfrak{h}) \to \mathrm{gr}^{i+1} \cap \ker(\mathfrak{h})$. Write $i = r(p-1)+j$ as above. If $j < p-1$ then

$$\mathrm{gr}^i \cap \ker(\mathfrak{h}) = k^{\mathrm{sep}}u^{2rp+p+j} \quad \text{and} \quad \mathrm{gr}^{i+1} \cap \ker(\mathfrak{h}) = k^{\mathrm{sep}}u^{2rp+j+1}.$$

Note that $(\gamma-1)(u^{2rp+p+j}) = -j\theta u^{2rp+p+j+1}$ by Theorem 1.16. If $j = p-1$ then

$$\mathrm{gr}^i \cap \ker(\mathfrak{h}) = k^{\mathrm{sep}}u^{2(r+1)p-1} \quad \text{and} \quad \mathrm{gr}^{i+1} \cap \ker(\mathfrak{h}) = k^{\mathrm{sep}}u^{2(r+1)p+p+1}.$$

By (17) we have

$$\begin{array}{rcl} \gamma(u^{2(r+1)p-1}) &\equiv& u^{2(r+1)p-1} + \theta u^{2(r+1)p} - (2r+4)\theta^p u^{2(r+1)p+p-1} \\ && -(2r+3)\theta^{p+1}u^{2(r+1)p+p} + \theta u^{2(r+1)p+p+1} \end{array}$$

modulo $\overline{\mathfrak{m}}^{2(r+1)p+p+2}$. This implies $(\gamma-1)(u^{2(r+1)p-1}) = \theta u^{2(r+1)p+p+1}$ in $\mathrm{gr}^{i+1} \cap \ker(\mathfrak{h})$, and thus $\xi\mathfrak{x} - \mathfrak{y} \in \ker(\psi)$. In particular, the restriction of ψ to $k^{\mathrm{sep}}[\mathfrak{x}]$ is still surjective. Since $\psi : k^{\mathrm{sep}}[\mathfrak{x}] \to \ker(\mathfrak{h})$ is a graded homomorphism and since the graded pieces are all of dimension one on both sides, the restriction of ψ to $k^{\mathrm{sep}}[\mathfrak{x}]$ is bijective. Since the inclusion $k^{\mathrm{sep}}[\mathfrak{x}] \to k^{\mathrm{sep}}[\mathfrak{x}, \mathfrak{y}]$ induces a bijection $k^{\mathrm{sep}}[\mathfrak{x}] \simeq k^{\mathrm{sep}}[\mathfrak{x}, \mathfrak{y}]/(\xi\mathfrak{x} - \mathfrak{y})$, this proves the claim.

Now consider the quotient $\overline{\mathrm{gr}} := \mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))/\ker(\mathfrak{h})$. By our above computations, $\overline{\mathrm{gr}} = \bigoplus_{i\geq 0} \overline{\mathrm{gr}}^i$, where $\overline{\mathrm{gr}}^i := \mathrm{gr}^i / (\mathrm{gr}^i \cap \ker(\mathfrak{h}))$ is the one dimensional k^{sep}-vector space generated by the class of u, if $i = 0$, by the class of $u^{2rp+j+1}$, if $i = r(p-1)+j$ with $1 \leq j < p-1$, and by the class of $u^{2(r+1)p+1}$, if $i = (r+1)(p-1)$. Our computations also show that $\mathfrak{h}$ acts trivially on $\overline{\mathrm{gr}}$.

Let $\gamma := 1 + \Pi\theta \in \Gamma_1$ be as above. We will explicitly compute that action of the k^{sep}-linear endomorphism $(\gamma - 1) : \overline{\mathrm{gr}}^i \to \overline{\mathrm{gr}}^{i+1}$ for any $i \geq 0$. For $i = 0$ we have $(\gamma-1)(u) = -\theta u^2$ by Theorem 1.16. If $i = r(p-1)+j$ with $1 \leq j < p-2$ then $(\gamma-1)(u^{2rp+j+1}) = -(j+1)\theta u^{2rp+j+2}$ by the same reference. If $j = p-2$ then

$$\overline{\mathrm{gr}}^i = k^{\mathrm{sep}} u^{2rp+p-1} \quad \text{and} \quad \overline{\mathrm{gr}}^{i+1} = k^{\mathrm{sep}} u^{2(r+1)p+1}.$$

As in the proof of Proposition 3.9, we obtain $(\gamma-1)(u^{2rp+p-1}) = \theta u^{2(r+1)p+1}$. Finally, if $j = p-1$, then

$$\overline{\mathrm{gr}}^i = k^{\mathrm{sep}} u^{2(r+1)p+1} \quad \text{and} \quad \overline{\mathrm{gr}}^{i+1} = k^{\mathrm{sep}} u^{2(r+1)p+2}.$$

As in the case $j < p-2$ we conclude that $(\gamma-1)(u^{2(r+1)p+1}) = -\theta u^{2(r+1)p+2}$.

As above, this shows that the $U(\overline{\mathfrak{g}}/\mathfrak{z})$-linear map $(\delta \mapsto \delta \cdot u) : U(\overline{\mathfrak{g}}/\mathfrak{z}) \to \overline{\mathrm{gr}}$ induces an isomorphism $U(\overline{\mathfrak{g}}/\mathfrak{z})/I \simeq \overline{\mathrm{gr}}$. Thus, we obtain an exact sequence

$$0 \longrightarrow U(\overline{\mathfrak{g}}/\mathfrak{z})/I \longrightarrow \mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}})) \longrightarrow U(\overline{\mathfrak{g}}/\mathfrak{z})/I \longrightarrow 0,$$

as required. That it is non-split follows from the fact that $\mathfrak{h}$ does not act trivially on $\mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))$. In fact, the kernel of $\mathfrak{h}$ is the left copy of $U(\overline{\mathfrak{g}}/\mathfrak{z})/I$ in the above presentation. $\square$

If M is a k^{sep}-linear representation of the Lie algebra $\overline{\mathfrak{g}}/\mathfrak{z}$ then we denote by $\mathrm{H}_\bullet(\overline{\mathfrak{g}}/\mathfrak{z}, M) := \mathrm{Tor}_\bullet^{U(\overline{\mathfrak{g}}/\mathfrak{z})}(k^{\mathrm{sep}}, M)$ and $\mathrm{H}^\bullet(\overline{\mathfrak{g}}/\mathfrak{z}, M) := \mathrm{Ext}^\bullet_{U(\overline{\mathfrak{g}}/\mathfrak{z})}(k^{\mathrm{sep}}, M)$ the Lie algebra homology and cohomology groups of M, respectively. The former can be computed using the standard complex $(\bigwedge^\bullet(\overline{\mathfrak{g}}/\mathfrak{z}) \otimes_{k^{\mathrm{sep}}} M, \partial_\bullet)$, whereas the latter can be computed using the standard complex $(\mathrm{Hom}_{k^{\mathrm{sep}}}(\bigwedge^\bullet(\overline{\mathfrak{g}}/\mathfrak{z}), M), \delta^\bullet)$. In particular, $\mathrm{H}_i(\overline{\mathfrak{g}}/\mathfrak{z}, M) = \mathrm{H}^i(\overline{\mathfrak{g}}/\mathfrak{z}, M) = 0$ for all $i > \dim(\overline{\mathfrak{g}}/\mathfrak{z})$.

Corollary 3.17. *Assume $h = 2$ and $K = \mathbb{Q}_p$ with $p > 3$.*

- *(i) The k^{sep}-vector space $\mathrm{H}_0(\overline{\mathfrak{g}}/\mathfrak{z}, \mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}})))$ is one dimensional, generated by the class of u_1.*
- *(ii) The k^{sep}-vector space $\mathrm{H}_1(\overline{\mathfrak{g}}/\mathfrak{z}, \mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}})))$ is two dimensional, generated by the classes of $\mathfrak{h} \otimes u_1$ and $(\xi\mathfrak{x} - \mathfrak{y}) \otimes u_1^{p+1}$.*
- *(iii) The k^{sep}-vector space $\mathrm{H}_2(\overline{\mathfrak{g}}/\mathfrak{z}, \mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}})))$ is one dimensional, generated by the class of $(\xi\mathfrak{x} - \mathfrak{y}) \wedge \mathfrak{h} \otimes u_1^{p+1}$.*

(iv) If $i \geq 3$ then $\mathrm{H}_i(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, \mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))) = 0$.

Proof. We first compute the homology of the $\overline{\mathfrak{g}}/\overline{\mathfrak{z}}$-representation $M := U(\overline{\mathfrak{g}}/\overline{\mathfrak{z}})/I$ appearing in Theorem 3.16. Using the relations $\mathfrak{h} \cdot M = 0$ and $(\xi\mathfrak{x} - \mathfrak{y}) \cdot M = 0$, one finds that $\partial_3 : \bigwedge^3(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}) \otimes_{k^{\mathrm{sep}}} M \to \bigwedge^2(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}) \otimes_{k^{\mathrm{sep}}} M$ is given by

$$\begin{aligned} \partial_3(\mathfrak{x} \wedge \mathfrak{y} \wedge \mathfrak{h} \otimes m) &= \mathfrak{y} \wedge \mathfrak{h} \otimes \mathfrak{x}m - \mathfrak{x} \wedge \mathfrak{h} \otimes \mathfrak{y}m + \mathfrak{x} \wedge \mathfrak{y} \otimes \mathfrak{h}m \\ &= (\mathfrak{y} - \xi\mathfrak{x}) \wedge \mathfrak{h} \otimes \mathfrak{x}m. \end{aligned}$$

Using $[\mathfrak{x}, \mathfrak{y}] = \mathfrak{h}$ in $\overline{\mathfrak{g}}/\overline{\mathfrak{z}}$, one finds that $\partial_2 : \bigwedge^2(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}) \otimes_{k^{\mathrm{sep}}} M \to \bigwedge^1(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}) \otimes_{k^{\mathrm{sep}}} M$ is given by

$$\begin{aligned} & \partial_2(\mathfrak{x} \wedge \mathfrak{y} \otimes m_1 + \mathfrak{x} \wedge \mathfrak{h} \otimes m_2 + \mathfrak{y} \wedge \mathfrak{h} \otimes m_3) \\ = \ & -\mathfrak{h} \otimes m_1 - \mathfrak{x} \otimes \mathfrak{y}m_1 + \mathfrak{y} \otimes \mathfrak{x}m_1 - \mathfrak{x} \otimes \mathfrak{h}m_2 + \mathfrak{h} \otimes \mathfrak{x}m_2 - \mathfrak{y} \otimes \mathfrak{h}m_3 + \mathfrak{h} \otimes \mathfrak{y}m_3 \\ = \ & (\mathfrak{y} - \xi\mathfrak{x}) \otimes \mathfrak{x}m_1 - \mathfrak{h} \otimes (m_1 - \mathfrak{x}m_2 - \xi\mathfrak{x}m_3). \end{aligned}$$

Finally, $\partial_1 : \bigwedge^1(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}) \otimes M \to M$ is given by $\partial_1(\mathfrak{w} \otimes m) = \mathfrak{w}m$ for all $\mathfrak{w} \in \overline{\mathfrak{g}}/\overline{\mathfrak{z}}$, $m \in M$.

Note that the natural map $k^{\mathrm{sep}}[\mathfrak{x}] \to M = U(\overline{\mathfrak{g}}/\overline{\mathfrak{z}})/I$ is bijective. In particular, $\mathfrak{x}$ defines an injective endomorphism of M. It follows that $\mathrm{H}_3(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, M) = \ker(\partial_3) = 0$. Similarly, one sees that

$$\mathfrak{x} \wedge \mathfrak{y} \otimes m_1 + \mathfrak{x} \wedge \mathfrak{h} \otimes m_2 + \mathfrak{y} \wedge \mathfrak{h} \otimes m_3 \in \ker(\partial_2)$$

if and only if $m_1 = 0$ and $m_2 + \xi m_3 = 0$. By our above computation,

$$\mathrm{im}(\partial_3) = \{\mathfrak{x} \wedge \mathfrak{h} \otimes m_2 + \mathfrak{y} \wedge \mathfrak{h} \otimes m_3 \mid m_2, m_3 \in \mathfrak{x}M \text{ and } m_2 + \xi m_3 = 0\}.$$

Since $M/\mathfrak{x}M$ is one dimensional, generated by the image m_0 of $1 \in U(\overline{\mathfrak{g}}/\overline{\mathfrak{z}})$, we obtain $\mathrm{H}_2(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, M) = k^{\mathrm{sep}}((\mathfrak{y} - \xi\mathfrak{x}) \wedge \mathfrak{h} \otimes m_0)$. Further, we have

$$\ker(\partial_1) = (I \cap (\overline{\mathfrak{g}}/\overline{\mathfrak{z}})) \otimes M = k^{\mathrm{sep}}(\mathfrak{y} - \xi\mathfrak{x}) \otimes M + k^{\mathrm{sep}}\mathfrak{h} \otimes M.$$

From our above computation of ∂_2 we obtain that

$$\mathrm{H}_1(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, M) = k^{\mathrm{sep}}((\mathfrak{y} - \xi\mathfrak{x}) \otimes m_0) + k^{\mathrm{sep}}(\mathfrak{h} \otimes m_0)$$

is two dimensional. Finally, $\mathrm{H}_0(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, M) = M/\mathfrak{x}M = k^{\mathrm{sep}}m_0$.

Consider the long exact homology sequence associated with the short exact sequence of $\overline{\mathfrak{g}}/\overline{\mathfrak{z}}$-representations in Theorem 3.16. We denote by $\delta^i : \mathrm{H}_i(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, M) \to \mathrm{H}_{i-1}(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, M)$ the associated connecting homomorphisms. They are defined by choosing k^{sep}-linear sections $t_\bullet : \bigwedge^\bullet(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}) \otimes \mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}})) \to \bigwedge^\bullet(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}) \otimes M$ (resp. $s_\bullet : \bigwedge^\bullet(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}) \otimes M \to \bigwedge^\bullet(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}) \otimes \mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))$) of the homomorphism of complexes $\bigwedge^\bullet(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}) \otimes M \to \bigwedge^\bullet(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}) \otimes \mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))$ (resp. $\bigwedge^\bullet(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}) \otimes \mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}})) \to \bigwedge^\bullet(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}) \otimes M$), and by letting δ^i be the map induced by $t_{i-1} \circ \partial_i \circ s_i$.

Under the natural map $\mathrm{H}_0(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, M) \to \mathrm{H}_0(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, \mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}})))$, the class of m_0 maps to the class of u_1^{p+1}, which is trivial (cf. Corollary 3.11). Thus, there is an exact sequence

$$\mathrm{H}_1(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, \mathrm{gr}(\overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}}))) \longrightarrow \mathrm{H}_1(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, M) \xrightarrow{\delta^1} \mathrm{H}_0(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, M) \longrightarrow 0.$$

Here $\delta^1(\mathfrak{h}\otimes m_0)$ is the class of $\mathfrak{h}u_1$ in $\mathrm{H}_0(\overline{\mathfrak{g}}/\mathfrak{z}, M)$, which is trivial because $\mathfrak{h}$ corresponds to $\gamma_{21}-1 \in \overline{\mathfrak{m}}_{\Lambda(\Gamma_1/Z_1)}$, where $\gamma_{21} = 1+\zeta p$ satisfies $(\gamma_{21}-1)(u_1) \in \overline{\mathfrak{m}}^{p+2}$ by Theorem 1.19. Further, $\delta^1((\mathfrak{y} - \xi\mathfrak{x}) \otimes m_0)$ is the class of $(\mathfrak{y} - \xi\mathfrak{x})u_1 = (\xi^p - \xi)u_1^{p+1}$, which is non-zero in $\mathrm{H}_0(\overline{\mathfrak{g}}/\mathfrak{z}, M) = k^{\mathrm{sep}}m_0 = k^{\mathrm{sep}}u_1^{p+1}$. Thus, $\ker(\delta^1) = k^{\mathrm{sep}}(\mathfrak{h}\otimes m_0)$.

Similarly, $\delta^2((\mathfrak{y} - \xi\mathfrak{x}) \wedge \mathfrak{h} \otimes m_0)$ is the class of

$$\mathfrak{h} \otimes (\mathfrak{y} - \xi\mathfrak{x})u_1 - (\mathfrak{y} - \xi\mathfrak{x}) \otimes \mathfrak{h}u_1 = (\xi^p - \xi)\mathfrak{h} \otimes u_1^{p+1} = (\xi^p - \xi)\mathfrak{h} \otimes m_0$$

in $\mathrm{H}_1(\overline{\mathfrak{g}}/\mathfrak{z}, M)$, which is non-zero. A straightforward analysis of the long exact homology sequence completes the proof. □

Let H be a compact p-adic analytic group (i.e. a compact Lie group over $\mathbb{Q}_p$). The theory of continuous (co)homology of H with coefficients in compact continuous H-modules was developed in [34], relying in large parts on the foundational work [22] of Lazard. We need to consider the parallel situation of pseudo-compact $\Lambda(H)$-modules, i.e. that of complete Hausdorff topological $\Lambda(H)$-modules M which possess a basis of open neighborhoods of zero consisting of $\Lambda(H)$-submodules $(M_j)_j$ such that the quotients M/M_j are of finite length over $\Lambda(H)$. Instead of developing the general formalism of continuous (co)homology of such modules, we will give an ad hoc definition and rely on the fundamental finiteness properties of the ring $\Lambda(H)$ to prove the necessary properties of our (co)homology functors.

Thus, for any pseudo-compact $\Lambda(H)$-module M we simply set

$$\mathrm{H}_\bullet(H, M) := \mathrm{Tor}_\bullet^{\Lambda(H)}(\breve{\mathfrak{o}}, M) \quad \text{and} \quad \mathrm{H}^\bullet(H, M) := \mathrm{Ext}^\bullet_{\Lambda(H)}(\breve{\mathfrak{o}}, M),$$

the torsion and extension groups being computed in the category of all (abstract) $\Lambda(H)$-modules. Here $\breve{\mathfrak{o}}$ denotes the pseudo-compact $\Lambda(H)$-module carrying the trivial action of H.

The ring $\Lambda(H)$ being noetherian (cf. [27], Theorem 27.1 and Theorem 33.4), $\breve{\mathfrak{o}}$ admits a resolution by finitely generated free $\Lambda(H)$-modules. By [5], Lemma 2.1 (ii), it follows that the above torsion groups coincide with those in [5], section 4, computed with respect to the complete tensor product $\hat{\otimes}_{\Lambda(H)}$. Thus, [5], Corollary 4.3 (ii), implies that the functors $\mathrm{H}_\bullet(H, \cdot\,)$ commute with projective limits of pseudo-compact $\Lambda(H)$-modules.

As in [34], Theorem 3.7.2, one can prove an analogous statement for the cohomology functors $\mathrm{H}^\bullet(H, \cdot\,)$. Since this will be of importance later, we will sketch a proof. As a consequence of our arguments, the above cohomology groups coincide with those computed by means of continuous cochains (cf. also [22], Chapitre V, Théorème 3.2.7, and [23], Proposition 5.2.14). For the homology groups, the analogous statement is proved in [5], Lemma 4.2 (ii).

Lemma 3.18. *Let H be a compact p-adic analytic group. If the pseudo-compact $\Lambda(H)$-module M is the projective limit of a projective system $(M_j)_{j\in J}$ of pseudo-compact $\Lambda(H)$-modules M_j, then the natural map*

$$\mathrm{H}^i(H, M) \to \varprojlim_{j\in J} \mathrm{H}^i(H, M_j)$$

is bijective for all $i \geq 0$.

Proof. According to [27], Proposition 22.5, the Jacobson radical of $\Lambda(H)$ is open. By construction of the pseudo-compact topology of $\Lambda(H)$, any $\Lambda(H)$-module of finite length is therefore of finite length over $\breve{\mathfrak{o}}$. As a consequence, any pseudo-compact $\Lambda(H)$-module is a pseudo-compact $\breve{\mathfrak{o}}$-module via restriction of scalars.

Let N be an arbitrary pseudo-compact $\Lambda(H)$-module. If m is a positive integer then $\mathrm{Hom}_{\Lambda(H)}(\Lambda(H)^m, N)$ is $\breve{\mathfrak{o}}$-equivariantly isomorphic to N^m, hence is a pseudocompact $\breve{\mathfrak{o}}$-module. This construction is functorial in the sense that if $f : N \to N'$ is a continuous homomorphism of pseudo-compact $\Lambda(H)$-modules, then the induced $\breve{\mathfrak{o}}$-linear map $\mathrm{Hom}_{\Lambda(H)}(\Lambda(H)^m, N) \to \mathrm{Hom}_{\Lambda(H)}(\Lambda(H)^m, N')$ is continuous. In fact, the induced map $N^m \to (N')^m$ is just the m-fold direct sum of f. Further, if m' is another positive integer, and if $g : \Lambda(H)^m \to \Lambda(H)^{m'}$ is a $\Lambda(H)$-linear map, then the induced $\breve{\mathfrak{o}}$-linear map $\mathrm{Hom}_{\Lambda(H)}(\Lambda(H)^{m'}, N) \to \mathrm{Hom}_{\Lambda(H)}(\Lambda(H)^m, N)$ is continuous. In fact, the induced map $N^{m'} \to N^m$ is just given by left multiplication with an $(m \times m')$-matrix with coefficients in $\Lambda(H)$, so that the assertion follows from the fact that N is a topological module over $\Lambda(H)$.

Now choose a resolution $P^\bullet \to \breve{\mathfrak{o}} \to 0$ of $\breve{\mathfrak{o}}$ by finitely generated free $\Lambda(H)$-modules. By [27], Theorem 22.3, the continuous $\breve{\mathfrak{o}}$-linear maps in the complex $\mathrm{Hom}_{\Lambda(H)}(P^\bullet, N)$ of pseudo-compact $\breve{\mathfrak{o}}$-modules have closed images. Thus, the cohomology groups $\mathrm{H}^\bullet(H, N)$ are pseudo-compact over $\breve{\mathfrak{o}}$, as well.

Coming back to our original situation, it follows from the universal property of the projective limit and the constructions above that the natural $\breve{\mathfrak{o}}$-linear map

$$\mathrm{Hom}_{\Lambda(H)}(P^i, M) \longrightarrow \varprojlim_{j \in J} \mathrm{Hom}_{\Lambda(H)}(P^i, M_j)$$

is a topological isomorphism for any $i \geq 0$.

For varying $j \in J$, the complexes $\mathrm{Hom}_{\Lambda(H)}(P^\bullet, M_j)$ form a projective system of complexes of continuous $\breve{\mathfrak{o}}$-linear maps between pseudo-compact $\breve{\mathfrak{o}}$-modules. Since the category of pseudo-compact $\breve{\mathfrak{o}}$-modules has exact projective limits (cf. [27], Theorem 22.3 (iv)), we have

$$\begin{aligned} \mathrm{H}^i(H, M) &\simeq \mathrm{H}^i(\mathrm{Hom}_{\Lambda(H)}(P^\bullet, \varprojlim_{j \in J} M_j)) \simeq \mathrm{H}^i(\varprojlim_{j \in J} \mathrm{Hom}_{\Lambda(H)}(P^\bullet, M_j)) \\ &\simeq \varprojlim_{j \in J}(\mathrm{H}^i \, \mathrm{Hom}(P^\bullet, M_j)) \simeq \varprojlim_{j \in J} \mathrm{H}^i(H, M_j) \end{aligned}$$

for any $i \geq 0$. □

The following result constitutes the main step in the proof of Theorem 3.20 below.

Theorem 3.19. *Assume $h = 2$ and $K = \mathbb{Q}_p$ with $p > 3$. For any integer $i \geq 0$ we have* $\mathrm{H}_i(\Gamma, (R/\breve{\mathfrak{o}})/T_1(R/\breve{\mathfrak{o}})) = \mathrm{H}^i(\Gamma, (R/\breve{\mathfrak{o}})/T_1(R/\breve{\mathfrak{o}})) = 0$.

Proof. Set $M := (R/\breve{\mathfrak{o}})/T_1(R/\breve{\mathfrak{o}})$ and $\overline{M} := M/pM \simeq \overline{\mathfrak{m}}/T_1(\overline{\mathfrak{m}})$. We claim that it suffices to prove $\mathrm{H}_i(\Gamma, \overline{M}) = \mathrm{H}^i(\Gamma, \overline{M}) = 0$ for all $i \geq 0$. To see this, consider the long exact (co)homology sequence associated with the short exact sequence

$$0 \longrightarrow M \xrightarrow{p} M \longrightarrow \overline{M} \longrightarrow 0$$

(cf. Corollary 2.5, noting that $T_1(\breve{\mathfrak{o}}) = \breve{\mathfrak{o}}$). Under the above vanishing assumption, we have $p\,\mathrm{H}_i(\Gamma, M) = \mathrm{H}_i(\Gamma, M)$ and $p\,\mathrm{H}^i(\Gamma, M) = \mathrm{H}^i(\Gamma, M)$ for all $i \geq 0$. As seen above, the $\breve{\mathfrak{o}}$-modules $\mathrm{H}_i(\Gamma, M)$ and $\mathrm{H}^i(\Gamma, M)$ are pseudo-compact. Therefore, [5], Lemma 1.4, implies that $\mathrm{H}_i(\Gamma, M) = \mathrm{H}^i(\Gamma, M) = 0$, as required.

Note that there are natural isomorphisms

$$\mathrm{H}_\bullet(\Gamma, \overline{M}) \simeq \mathrm{Tor}_\bullet^{\overline{\Lambda}}(k^{\mathrm{sep}}, \overline{M}) \quad \text{and} \quad \mathrm{H}^\bullet(\Gamma, \overline{M}) \simeq \mathrm{Ext}_{\overline{\Lambda}}^\bullet(k^{\mathrm{sep}}, \overline{M}),$$

stemming from the fact that $\breve{\mathfrak{o}}$ admits a free resolution $P^\bullet \to \breve{\mathfrak{o}} \to 0$ over Λ which is $\breve{\mathfrak{o}}$-linearly split, hence remains exact after reduction modulo $p\breve{\mathfrak{o}}$ (cf. [22], Chapitre V, (2.2) for the case of a p-valued group, as well as [22], Chapitre V, (3.2.6) and the splitting assertion (3.1.6) in the general case).

If $Z := \mathfrak{o}^*$ denotes the center of Γ, then there are Hochschild-Serre spectral sequences

$$\begin{aligned} \mathrm{H}_r(\Gamma/Z, \mathrm{H}_s(Z, \overline{M})) &\Longrightarrow \mathrm{H}_{r+s}(\Gamma, \overline{M}) \quad \text{and} \\ \mathrm{H}^r(\Gamma/Z, \mathrm{H}^s(Z, \overline{M})) &\Longrightarrow \mathrm{H}^{r+s}(\Gamma, \overline{M}). \end{aligned}$$

Using that our (co)homology groups commute with projective limits, the existence of these spectral sequences can be established by using the Hochschild-Serre spectral sequences for discrete modules over finite groups, as well as [5], Corollary 4.3. Alternatively, one can use the fact that Λ is topologically projective over $\Lambda(Z)$ (cf. [5], Lemma 4.5) and imitate the proof of [18], Theorem 6.8.

Since the action of Z on $\overline{M}$ is trivial, we have $\mathrm{H}_\bullet(Z, \overline{M}) \simeq \mathrm{H}_\bullet(Z, k^{\mathrm{sep}}) \otimes_{k^{\mathrm{sep}}} \overline{M}$ and $\mathrm{H}^\bullet(Z, \overline{M}) \simeq \mathrm{H}^\bullet(Z, k^{\mathrm{sep}}) \otimes_{k^{\mathrm{sep}}} \overline{M}$. For the homology groups, this is immediate. For the cohomology groups, the assertion follows from Lemma 3.18 together with the facts that $\overline{M}$ is the projective limit of finite dimensional k^{sep}-vector spaces and that $\mathrm{H}^\bullet(Z, k^{\mathrm{sep}})$ is finite dimensional, as well. As a consequence, $\mathrm{H}_\bullet(Z, \overline{M})$ and $\mathrm{H}^\bullet(Z, \overline{M})$ are $\Lambda(\Gamma/Z)$-isomorphic to finite direct sums of copies of $\overline{M}$. Therefore, it suffices to prove $\mathrm{H}_i(\Gamma/Z, \overline{M}) = \mathrm{H}^i(\Gamma/Z, \overline{M}) = 0$ for all $i \geq 0$.

Set $Z_1 := \Gamma_1 \cap Z$. Since the finite group $(\Gamma/Z)/(\Gamma_1/Z_1) \simeq \mu_{p^2-1}/\mu_{p-1}$ has order $p+1$, which is prime to p, another application of the Hochschild-Serre spectral sequences shows that

$$\begin{aligned} \mathrm{H}_\bullet(\Gamma/Z, \overline{M}) &\simeq \mathrm{H}_\bullet(\Gamma_1/Z_1, \overline{M})_{(\Gamma/Z)/(\Gamma_1/Z_1)} \quad \text{and} \\ \mathrm{H}^\bullet(\Gamma/Z, \overline{M}) &\simeq \mathrm{H}^\bullet(\Gamma_1/Z_1, \overline{M})^{(\Gamma/Z)/(\Gamma_1/Z_1)}. \end{aligned}$$

Let us now treat the homology groups first. Note that with respect to its maximal adic filtration, the ring $S := \overline{\Lambda(\Gamma_1/Z_1)}$ is complete with noetherian graded ring (cf. Corollary 3.14). As in the proof of [16], I.7.2 Corollary 2, the finitely generated, maximal adically filtered S-module $\overline{M}$ admits a strict resolution $L^\bullet \to \overline{M} \to 0$ by finitely generated filtered free S-modules L^i, $i \geq 0$, such that the associated complex $\mathrm{gr}(L^\bullet) \to \mathrm{gr}(\overline{M}) \to 0$ is an exact resolution of $\mathrm{gr}(\overline{M})$ by finitely generated free $\mathrm{gr}(S)$-modules $\mathrm{gr}(L^i)$.

We endow the complex $k^{\mathrm{sep}} \otimes_S L^\bullet$ with the tensor product filtration. Its morphisms are of degree zero. According to [16], Chapter III, §1, this filtered complex gives rise to a spectral sequence with E^1_{i+1}-term $\mathrm{Tor}_i^{\mathrm{gr}(S)}(k^{\mathrm{sep}}, \mathrm{gr}(\overline{M}))$ (cf. [16], III.1.1 Observation 1 and Lemma I.6.14). Note that this k^{sep}-vector space is isomorphic to $\mathrm{H}_i(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, \mathrm{gr}(\overline{M}))$ by Corollary 3.14.

According to [16], III.1.1 Remark 3, the E^∞_{i+1}-term of this spectral sequence is the graded k^{sep}-vector space associated with a certain filtration on

$$\mathrm{H}_i(k^{\mathrm{sep}} \otimes_S L^\bullet) \simeq \mathrm{Tor}_i^S(k^{\mathrm{sep}}, \overline{M}) \simeq \mathrm{H}_i(\Gamma_1/Z_1, \overline{M}).$$

In fact, together with the filtration of k^{sep}, also that of $k^{\mathrm{sep}} \otimes_S L^\bullet$ and hence that of $\mathrm{H}_i(\Gamma_1/Z_1, \overline{M})$ is discrete in the sense of [16], Definition I.2.4, with exactly one jump. It follows from its very definition that the spectral sequence degenerates in E^1. As a consequence, $\mathrm{H}_i(\Gamma_1/Z_1, \overline{M}) \simeq \mathrm{H}_i(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, \mathrm{gr}(\overline{M}))$ over k^{sep} for all $i \geq 0$. This is also proved in [13], Theorem 3.3'.

The group Γ/Z acts on the complex $k^{\mathrm{sep}} \otimes_S L^\bullet$ through its conjugation action on the free modules L^i and the trivial action on k^{sep}. Note that this changes the chain maps but stabilizes the homology groups. The induced action on $\mathrm{H}_i(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, \mathrm{gr}(\overline{M}))$ is the one coming from the adjoint action of Γ on $\mathfrak{g} = \bigoplus_{i\geq 1} \Gamma_i/\Gamma_{i+1}$ and the natural action on $\mathrm{gr}(\overline{M})$. If $\zeta \in \mu_{p^2-1} \subset \Gamma$ then

$$\begin{aligned}
\mathrm{Ad}(\zeta)(\mathfrak{x}) &\equiv \zeta(1+\Pi)\zeta^{-1}\Gamma_2 \equiv (1+\Pi\zeta^{p-1})\Gamma_2 = \zeta^{p-1}\mathfrak{x}, \\
\mathrm{Ad}(\zeta)(\mathfrak{y}) &\equiv \zeta(1+\Pi\xi)\zeta^{-1}\Gamma_2 \equiv (1+\Pi\zeta^{p-1}\xi)\Gamma_2 = \zeta^{p-1}\mathfrak{y} \quad \text{and} \\
\mathrm{Ad}(\zeta)(\mathfrak{h}) &\equiv \zeta(1+(\xi-\xi^p)p)\zeta^{-1}\Gamma_3 = (1+(\xi-\xi^p)p)\Gamma_3 = \mathfrak{h}.
\end{aligned}$$

Since we are free in our choice of $\xi \in \mu_{p^2-1} \setminus \mu_{p-1}$, we may assume $\xi^p = -\xi$ in k_2 by lifting $\theta^p - \theta$ to μ_{p^2-1} with $\theta \in k_2 \setminus k$. Under the identification $\mathrm{gr}^1(\Gamma_1) \simeq k_2$ we then have

$$\begin{aligned}
\zeta^{p-1}\mathfrak{x} = \zeta^{p-1} &= \frac{1}{2}(\zeta^{p-1} + \zeta^{p(p-1)}) + \frac{1}{2}\frac{\zeta^{p-1} - \zeta^{p(p-1)}}{\xi}\xi \\
&= \frac{1}{2}(\zeta^{p-1} + \zeta^{p(p-1)})\mathfrak{x} + \frac{1}{2}\frac{\zeta^{p-1} - \zeta^{p(p-1)}}{\xi}\mathfrak{y}
\end{aligned}$$

with $\frac{1}{2}(\zeta^{p-1} + \zeta^{p(p-1)}), \frac{1}{2}(\zeta^{p-1} - \zeta^{p(p-1)})\xi^{-1} \in k$. Similarly,

$$\zeta^{p-1}\mathfrak{y} = \zeta^{p-1}\xi = \frac{1}{2}(\zeta^{p-1} - \zeta^{p(p-1)})\xi\mathfrak{x} + \frac{1}{2}(\zeta^{p-1} + \zeta^{p(p-1)})\mathfrak{y}$$

with $\frac{1}{2}(\zeta^{p-1} - \zeta^{p(p-1)})\xi, \frac{1}{2}(\zeta^{p-1} + \zeta^{p(p-1)}) \in k$. Using that $\zeta^{p(p-1)} = \zeta^{1-p}$, a direct computation shows that $\mathrm{Ad}(\zeta)(\xi\mathfrak{x} - \mathfrak{y}) = \zeta^{1-p} \cdot (\xi\mathfrak{x} - \mathfrak{y})$ in $\bigwedge^1(\overline{\mathfrak{g}}/\overline{\mathfrak{z}})$. Note that $\zeta^{p-1}\mathfrak{x} \in k_2 \simeq \mathrm{gr}^1(\Gamma_1)$ is different from $\zeta^{p-1} \cdot \mathfrak{x} \in \overline{\mathfrak{g}} \otimes_k k^{\mathrm{sep}}$, the tensor product being taken over k.

By Lemma 1.1 we have $\zeta(u) = \zeta^{p-1}u$ and $\zeta(u^{p+1}) = u^{p+1}$. Together with the above computations, our results in Corollary 3.17 show that all $(\Gamma/Z)/(\Gamma_1/Z_1)$-representations $\mathrm{H}_i(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, \mathrm{gr}(\overline{M}))$ are finite direct sums of non-trivial characters. Thus, $\mathrm{H}_i(\Gamma/Z, \overline{M}) \simeq \mathrm{H}_i(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, \mathrm{gr}(\overline{M}))_{(\Gamma/Z)/(\Gamma_1/Z_1)} = 0$ for all $i \geq 0$.

The reasoning for the cohomology groups is similar. We choose a strict resolution $P^\bullet \to k^{\mathrm{sep}} \to 0$ by finitely generated, filtered free S-modules P^i. The induced complex $\mathrm{gr}(P^\bullet) \to k^{\mathrm{sep}} \to 0$ is a resolution of k^{sep} by finitely generated free $\mathrm{gr}(S)$-modules $\mathrm{gr}(P^i)$. By [16], I.2.5 and Proposition I.6.6, the k^{sep}-vector spaces $\mathrm{Hom}_S(P^i, \overline{M})$ are filtered by degree. As above, we obtain a spectral sequence with initial terms

$$E^1_{i+1} \simeq \mathrm{Ext}^i_{\mathrm{gr}(S)}(k^{\mathrm{sep}}, \mathrm{gr}(\overline{M})) \simeq \mathrm{H}^i(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, \mathrm{gr}(\overline{M}))$$

(cf. [16], III.1.1 Observation 1 and Lemma I.6.9, as well as our Corollary 3.14). By [16], Proposition I.6.7, the filtration of our complex is separated, so that the E^∞_{i+1}-term of the spectral sequence is the graded k^{sep}-vector space associated with a certain filtration on $\mathrm{Ext}^i_S(k^{\mathrm{sep}}, \overline{M}) \simeq \mathrm{H}^i(\Gamma_1/Z_1, \overline{M})$ (cf. [16], III.1.1 Remark 3).

Since $\mathrm{gr}(\overline{M})$ is a finitely generated $U(\overline{\mathfrak{g}}/\overline{\mathfrak{z}})$-module (cf. Theorem 3.16), it follows from Poincaré duality below that the initial terms of the spectral sequence are finite dimensional k^{sep}-vector spaces almost all of which are zero. Therefore, the spectral sequence is finitely convergent, i.e. $E^\infty_i = E^n_i$ for some $n \geq 1$ and all i. This implies that $\mathrm{H}^i(\Gamma_1/Z_1, \overline{M})$ admits a filtration whose associated graded pieces are subquotients of $\mathrm{H}^i(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, \mathrm{gr}(\overline{M}))$. By the naturality of this construction under the action of Γ/Z, it suffices to show that $\mathrm{H}^i(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, \mathrm{gr}(\overline{M}))^{(\Gamma/Z)/(\Gamma_1/Z_1)} = 0$ for all $i \geq 0$.

Fix an integer $i \geq 0$. We recall from [17], Chapter VI, Theorem 6.10, the construction of the Poincaré duality isomorphism

$$\mathrm{H}_{3-i}(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, \mathrm{gr}(\overline{M}) \otimes_{k^{\mathrm{sep}}} (\bigwedge^3(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}))^*) \simeq \mathrm{H}^i(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, \mathrm{gr}(\overline{M})). \tag{18}$$

It is induced by the k^{sep}-linear isomorphisms

$$\bigwedge^{3-\bullet}(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}) \otimes_{k^{\mathrm{sep}}} \mathrm{gr}(\overline{M}) \otimes_{k^{\mathrm{sep}}} (\bigwedge^3(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}))^* \longrightarrow \mathrm{Hom}_{k^{\mathrm{sep}}}(\bigwedge^{\bullet}(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}), \mathrm{gr}(\overline{M})),$$

given by sending $\delta \otimes m \otimes \varepsilon$ to the linear map $(\delta' \mapsto \varepsilon(\delta \wedge \delta') \cdot m)$. This explicit formula shows that the duality isomorphism (18) is Γ/Z-equivariant.

Note that the adjoint action of $\overline{\mathfrak{g}}/\overline{\mathfrak{z}}$ on $(\bigwedge^3(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}))^*$ is trivial. This can be checked directly and also follows from the fact that any nilpotent Lie algebra is unimodular. On the other hand, our above computations show that $\zeta \in \mu_{p^2-1}$ acts on $(\bigwedge^3(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}))^* = (k^{\mathrm{sep}}(\mathfrak{x} \wedge \mathfrak{y} \wedge \mathfrak{h}))^*$ through the trivial character. Therefore, Corollary 3.17 implies that $\mathrm{H}^i(\overline{\mathfrak{g}}/\overline{\mathfrak{z}}, \mathrm{gr}(\overline{M}))$ is a finite direct sum of non-trivial characters of $(\Gamma/Z)/(\Gamma_1/Z_1)$. This completes the proof. $\square$

We note that the above spectral sequences, relating Lie algebra and group (co)homology, are also considered in [34], Theorem 5.1.12.

As pointed out in the introduction, the following result is predicted by Hopkins' chromatic splitting conjecture. In greater generality, it was first proved by Shimomura, Yabe and Behrens, using methods from algebraic topology (cf. [1], Theorem 7.7).

Theorem 3.20. *Assume $h = 2$ and $K = \mathbb{Q}_p$ with $p > 3$. For any integer $i \geq 0$ we have $\mathrm{H}_i(\Gamma, R/\breve{\mathfrak{o}}) = \mathrm{H}^i(\Gamma, R/\breve{\mathfrak{o}}) = 0$. Equivalently, the Γ-equivariant inclusion $\breve{\mathfrak{o}} \to R$ induces isomorphisms $\mathrm{H}_i(\Gamma, \breve{\mathfrak{o}}) \simeq \mathrm{H}_i(\Gamma, R)$ and $\mathrm{H}^i(\Gamma, \breve{\mathfrak{o}}) \simeq \mathrm{H}^i(\Gamma, R)$ for all $i \geq 0$.*

Proof. It follows from Corollary 2.5, Proposition 2.8 and [27], Lemma 22.1, that the homomorphism

$$R/\breve{\mathfrak{o}} \longrightarrow \varprojlim_{n \geq 1} (R/\breve{\mathfrak{o}})/T_1^n(R/\breve{\mathfrak{o}})$$

of pseudo-compact Λ-modules is a topological isomorphism. Since our (co)homology groups commute with projective limits of pseudo-compact Λ-modules (cf. Lemma 3.18 and the remarks preceding it), it suffices to prove the analogous statement for $(R/\breve{\mathfrak{o}})/T_1^n(R/\breve{\mathfrak{o}})$, where n is an arbitrary positive integer. By dévissage, we are further reduced to the analogous statement for $T_1^{n-1}(R/\breve{\mathfrak{o}})/T_1^n(R/\breve{\mathfrak{o}})$. By Theorem 2.6, T_1^{n-1} induces an $\breve{\mathfrak{o}}$-linear topological isomorphism

$$(R/\breve{\mathfrak{o}})/T_1(R/\breve{\mathfrak{o}}) \to T_1^{n-1}(R/\breve{\mathfrak{o}})/T_1^n(R/\breve{\mathfrak{o}}).$$

By (13) this isomorphism becomes Γ-equivariant, if the action on the left is pulled back via an outer automorphism of Γ. Since the (co)homology groups of Γ for this twisted action on $(R/\breve{\mathfrak{o}})/T_1(R/\breve{\mathfrak{o}})$ are canonically isomorphic to the original ones, the theorem follows from Theorem 3.19. $\square$

Remark 3.21. If $K = \mathbb{Q}_p$ with $p > 2$ and if $h = p - 1$ then the so-called *Tate-Farrell cohomology* of Γ with coefficients in R was considered in [33]. In addition to the fact that the Tate-Farrell cohomology and the continuous group cohomology agree only in large degrees, our methods are completely different from those of [33]. In fact, Theorem 3.20 follows from a profound analysis of the structure of R as a Λ-module which is not discussed in [loc.cit.].

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