UNIVERSITÄT DUISBURG ESSEN

Open-Minded

Master thesis

Condensed Mathematics

Florian Leptien
Mathematics
3014072

Supervised by
Prof. Dr. Jan Kohlhaase

Date of Submission:
September 10, 2024

Faculty of mathematics

Contents

1 Introduction

In *Lecture Notes on Condensed Mathematics* [7], Dustin Clausen and Peter Scholze raise the question *"How to do algebra when rings/groups/modules carry a topology"*. The goal of this work is to give a detailed overview of many concepts needed to answer this question. A key concept is that of an abelian category which generalizes the category of modules over a ring. Many concepts, like exact sequences, kernels, cokernels, and derived functors are naturally formulated in the language of abelian categories. In turn, abelian categories have their place in many mathematical areas. In fact, abelian categories have well-behaved exactness properties which implies that morphisms have desirable properties. For example, the property of being an isomorphism in an abelian category can be measured by the kernel and the cokernel of a morphism. We will see that topological structures often do not mix well with algebra and thereby motivate a new approach to tackle this problem. For example, topological abelian groups do not form an abelian category, as we will recall by having a closer look at one of the motivating examples in [7] (cf. Example 2.17). As stated in [7], another downside of mixing algebra and topology is the fact that *"for a topological group G, a short exact sequence of continuous G-modules does not in general give long exact sequences of continuous group cohomology groups. More abstractly, the theory of derived categories does not mix well with topological structures"*. In their work [7] Dustin Clausen and Peter Scholze aim to present *"a unified approach to the problem of doing algebra when rings/modules/groups/... carry a topology, and resolve those and other foundational problems"*.

Roughly speaking, condensed mathematics redefines topological abelian groups as *condensed abelian groups*, by embedding them in an abelian category. More precisely, a *condensed abelian group* is a functor from the opposite category of profinite sets to the category of abelian groups that satisfies a certain sheaf condition. Consequently, every Hausdorff topological abelian group can be viewed as a condensed abelian group by associating the functor that assigns to each profinite set the abelian group of continuous maps from the profinite set to the topological abelian group. As it turns out, the category of condensed abelian groups forms a particularly nice abelian category that allows us to study topological abelian groups within the framework of an abelian category and all the machinery that comes with it. More generally, one can define *condensed sets/rings/modules/...*, redefining topological spaces/rings/modules/... .

Not surprisingly, we start this work in Section 2 by recalling some general facts about abelian categories before we have a detailed look at projective objects and (projective-) generators in abelian categories that we generalize to families of (projective-)generators. This generalization will play a crucial role when we end this work in Section 8 by showing that condensed abelian groups form a nice abelian category. In particular, this allows us to give a different proof than in [7] of the fact that the category of κ-condensed abelian groups is generated by projective objects (cf. Remark 8.28). Here, κ is a *strong limit cardinal* used to cut off the category of profinite sets to avoid set theoretic issues such as the associated functor category not being locally small. In between, in Section 3 we introduce the concept of a site and the associated category of sheaves, a so-called topos, as a generalization of sheaves on a topological space. This lays the basis for introducing κ-condensed sets in Section 4, which are in some sense the building blocks for condensed sets in Section 7 where we get rid of the cut off cardinal κ. In contrast to many other related works we solve arising set theoretic issues in a very concrete manner by showing that the involved categories (e.g. κ-small profinite sets) are actually equivalent to small categories. For example, this ensures that the involved categories of sheaves are locally small. Moreover, this ensures that we can take a certain limit in the proof of the very important result (cf. Theorem 5.18) that the category of κ-condensed sets/groups/modules/... is equivalent to the category of sheaves of sets/groups/modules/... on the site of κ-small *extremally disconnected sets*. We give a detailed proof of this result. In fact, a complete proof did not seem to exist in the literature, except for a few erroneous arguments, e.g. in Version 1 of [2], [25] and [23]. This equivalence is used in Section 8 to prove many of the good properties of the category of κ-condensed abelian groups. A crucial tool are Stone–Čech compactifications of discrete spaces. While we are mainly interested in the abstract properties of these spaces we will make the construction very explicit in Section 5. This allows us to construct a counterexample to one of the erroneous arguments mentioned above (cf. Example 5.30). In Section 6 we will show that certain topological spaces that come up in practice embed fully faithfully into κ-condensed sets. This implies that passing to the condensed setting comes at no loss. Lastly, we point out that the authors of [2] have included an argument from one of our proofs of Theorem 5.18 into Version 2 of their preprint.

Acknowledgments

First and foremost, I would like to express my gratitude to Jan Kohlhaase for supervising this thesis. Among the many reasons why I asked him to be my supervisor, perhaps the most important was his positive influence on my mathematical skills throughout the years, for which I am deeply grateful. Despite his demanding schedule as the current Dean of the Faculty, he constantly provided me with invaluable feedback to improve this thesis.

Moreover, I would like to thank Jochen Heinloth for agreeing to review this thesis. I also want to thank all the members of the faculty for making my studies a great experience. To name just a few, I am particularly grateful to Georg Hein who always had an open door for me and took the time to enthusiastically explain math to me. I am equally grateful to Riccardo Tosi, Giulio Marazza and Niklas Müller for the time they spent with me discussing topics related to this thesis. Furthermore, I would like to thank Xavier Xarles for the valuable correspondence we had regarding the preprint [2].

Finally, I would like to thank my girlfriend Annika for her endless support throughout the work on this thesis, that kept me sane when proofs didn't want to work out.

2 Abelian categories as motivation

2.1 Basics on Abelian categories

In this section we want to briefly introduce abelian categories. In doing so, we lay the groundwork for κ-condensed abelian groups introduced in section 8. The presented material is mostly based on the book *Categories and Sheaves* by Masaki Kashiwara and Pierre Shapira [17].

Definition 2.1. A *pre-additive category* is a category $\mathcal{C}$ such that for any two objects X and Y of $\mathcal{C}$, $\operatorname{Hom}(X, Y)$ is endowed with a structure of an abelian group and the composition map $\circ$ is bilinear.

Example 2.2. The category of abelian groups is pre-additive for the pointwise addition of group homomorphisms.

Proposition 2.3. *Let $\mathcal{C}$ be a pre-additive category and let X be an object of $\mathcal{C}$. The following statements are equivalent:*

(i) *X is an initial object, i.e. for any object Z in $\mathcal{C}$ there is exactly one morphism $X \to Z$.*

(ii) *X is a terminal object, i.e. for any object Z in $\mathcal{C}$ there is exactly one morphism $Z \to X$.*

(iii) *$id_X = 0_X : X \to X$ where 0_X is the zero element of the abelian group $\operatorname{Hom}(X, X)$.*

(iv) *The abelian group $\operatorname{Hom}(X, X)$ is the zero group.*

In particular, any initial object (or any terminal object) in $\mathcal{C}$ is a zero object, i.e. an object that is both initial and terminal.

Proof. If X is initial, there is exactly one morphism $X \to X$ and thus, $\operatorname{Hom}(X, X) = 0$. This shows that (i) implies (iv). The implication (iv) $\Rightarrow$ (iii) is clear. Let us now show that (iii) implies (ii). Let $f : Z \to X$ be any morphism and note that

$$\begin{aligned} 0_X \circ f &= (0_X + 0_X) \circ f \\ &= 0_X \circ f + 0_X \circ f \end{aligned}$$

and hence, $0_X \circ f = 0$ where 0 is the zero element of the abelian group $\operatorname{Hom}(Z, X)$. Then

$$\begin{aligned} f &= id_X \circ f \\ &= 0_X \circ f \\ &= 0 \end{aligned}$$

and thus, X is terminal. For the last implication notice that a terminal object in $\mathcal{C}$ is an initial object in the pre-additive category $\mathcal{C}^{\text{op}}$ and hence is terminal in $\mathcal{C}^{\text{op}}$ by the implication (i) $\Rightarrow$ (ii). But a terminal object in $\mathcal{C}^{\text{op}}$ is an initial object in $\mathcal{C}$. $\square$

Example 2.4. In the category of rings with unit whose morphisms are ring homomorphisms that preserve the unit, the integers $\mathbb{Z}$ are an initial object that is not terminal. Indeed, for example there is no morphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$. Of course, the terminal object in this category is the zero ring. In particular, by Proposition 2.3, the category is not a pre-additive category.

Definition 2.5. A category $\mathcal{C}$ is called *additive* if its pre-additive and admits finite products.

One can show that in an additive category any finite product is also a finite coproduct (cf. Corollary 8.2.4 in [17]). In particular, if $\mathcal{C}$ is additive so is $\mathcal{C}^{\mathrm{op}}$. Moreover, the empty product is a zero object which we typically denote by 0.

Example 2.6. If R is a ring, the category $\mathbf{Mod}(R)$ of left R-modules and the full subcategory $\mathbf{Mod}_{\mathrm{fin}}(R)$ of finitely generated left R-modules are additive categories.

From now on until stated otherwise, $\mathcal{C}$ will always denote an additive category.

Definition 2.7. A *sequence* in $\mathcal{C}$ is a family $(X_i, f_i)_{i\in\mathbb{Z}}$ where X_i is an object of $\mathcal{C}$ and $f_i : X_i \to X_{i+1}$ is a morphism in $\mathcal{C}$.

Example 2.8. If M is a left R-module and if $N \subseteq M$ is a submodule, then there is a finite sequence in $\mathbf{Mod}(R)$ given by

$$0 \to N \hookrightarrow M \to M/N \to 0.$$

Before we can discuss what it means for such a sequence to be *exact* we need to introduce some familiar concepts in terms of category theory.

Definition 2.9. Let $f : X \to Y$ be a morphism in $\mathcal{C}$.

(i) The *kernel* of f, if it exists, is the fiber product of $X \times_Y 0$ of $X \xrightarrow{f} Y \leftarrow 0$. It is denoted by $\ker(f)$. Equivalently, $\ker(f)$ is the *equalizer* of the parallel arrows $f, 0 : X \rightrightarrows Y$.

(ii) The *cokernel* of f, if it exists, is the kernel of f in $\mathcal{C}^{\mathrm{op}}$. It is denoted by $\operatorname{coker}(f)$. Equivalently, $\operatorname{coker}(f)$ is the *coequalizer* of the parallel arrows $f, 0 : X \rightrightarrows Y$.

Hence, a kernel of f is a pair (K, h), where K is an object of $\mathcal{C}$ and $h : K \to X$ is a morphism in $\mathcal{C}$ such that $f \circ h = 0$ and which satisfies the following universal property:

If (K', h') is another pair such that $f \circ h' = 0$, then there exists a unique morphism $u : K' \to K$ such that the following diagram commutes:

$$\begin{tikzcd} & K' \arrow[dl, dotted, "\exists! u"'] \arrow[d, "h'"] \arrow[dr, "0"] & \\ K \arrow[r, "h"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

As a direct consequence of this universal property, one can directly check that the morphism h is a *monomorphism*.

Moreover the cokernel of f is a pair (C, k) where C is an object of $\mathcal{C}$ and $k : Y \to C$ is a morphism in $\mathcal{C}$ such that $k \circ f = 0$ and which satisfies the following universal property:

If (C', k') is another pair such that $k' \circ f = 0$, then there exists a unique morphism $u : C \to C'$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "0"'] & Y \arrow[r, "k"] \arrow[d, "k'"] & C \arrow[dl, dotted, "\exists! u"] \\ & C' & \end{tikzcd}$$

Likewise, as a direct consequence of this universal property, one can directly check that the morphism k is an *epimorphism*.

Example 2.10. Let R be a ring with unit and let $\mathbf{Mod}(R)$ be the category of left R-modules. The kernel of a morphism $f : M \to N$ in $\mathbf{Mod}(R)$ is the left R-submodule $f^{-1}(0) \hookrightarrow M$ and the cokernel of f is the left R-quotient module $N \to N/f(M)$ of N. In particular, any morphism in $\mathbf{Mod}(R)$ has a kernel, this is in general not true for the full subcategory $\mathbf{Mod}_{\text{fin}}(R)$ of finitely generated left R-modules. Indeed, if R is a non-noetherian ring there is an ideal $I \subseteq R$ that is not finitely generated. Then the natural morphism $R \to R/I$ has no kernel in $\mathbf{Mod}_{\text{fin}}(R)$ as I is not finitely generated. The reason is that the inclusion $\mathbf{Mod}_{\text{fin}}(R) \subseteq \mathbf{Mod}(R)$ preserves kernels if they exist, i.e. if $f : M \to N$ is a morphism in $\mathbf{Mod}_{\text{fin}}(R)$ with kernel (K, h), then f has a kernel in $\mathbf{Mod}(R)$ and they are isomorphic. To see this, assume we have a morphism $h' : K' \to M$ in $\mathbf{Mod}(R)$ such that $f \circ h' = 0$. Then K' is the union of its finitely generated submodules, let $\iota_U : U \hookrightarrow K'$ be one of those. In particular, we get a morphism $h' \circ \iota_U : U \to M$ in $\mathbf{Mod}_{\text{fin}}(R)$ such that $f \circ h' \circ \iota_U = 0$ and we see that $h' \circ \iota_U = h \circ u$ for some unique $u : U \to K$. If $\iota_{U'} : U' \hookrightarrow K'$ is another inclusion, the morphisms $u : U \to K$ and $u' : U' \to K$ agree on $U \cap U'$. In particular, we can glue them to a unique morphism $v : K' \to K$ such that $h' = h \circ v$. Hence, the kernel of f is preserved.

If kernels and cokernels exist, they are useful to measure key properties of a morphism.

Lemma 2.11. *Let $f : X \to Y$ be a morphism in $\mathcal{C}$ and assume that f has a kernel $h : \ker(f) \to X$ and a cokernel $k : Y \to \operatorname{coker}(f)$. Then the following statements are true:*

(i) $\ker(f) \cong 0$ *if and only if f is a monomorphism.*

(ii) $h : \ker(f) \to X$ *is an isomorphism if and only if $f = 0$.*

(iii) $\operatorname{coker}(f) \cong 0$ *if and only if f is an epimorphism.*

(iv) $k : Y \to \operatorname{coker}(f)$ *is an isomorphism if and only if $f = 0$.*

Proof. Let us start with (i). Suppose $\ker(f) \cong 0$, then $h = 0$. Let $g_1, g_2 : Z \to X$ be morphisms in $\mathcal{C}$ such that $f \circ g_1 = f \circ g_2$. Then we have that $f \circ (g_1 - g_2) = 0$. By the universal property of the kernel of f there exists a unique morphism $u : Z \to \ker(f)$ such that

$$g_1 - g_2 = h \circ u = 0 \circ u = 0.$$

Hence, $g_1 = g_2$. For the converse assume that f is a monomorphism. Let $g : Z \to \ker(f)$ be any morphism. Then

$$f \circ h \circ g = 0 = f \circ 0.$$

Since f is a monomorphism, this implies that $h \circ g = 0$. On the other hand, $0 = h \circ 0$ and since h is a monomorphism as well, this implies that $g = 0$. Thus, $\ker(f)$ is terminal, by Proposition 2.3 also initial and hence the zero object. As for (ii), if $h : \ker(f) \to X$ is an isomorphism, it follows that

$$f = f \circ h \circ h^{-1} = 0 \circ h^{-1} = 0.$$

Conversely, if $f = 0$, then $f = f \circ id_X = 0$ and by the universal property of the kernel $h : \ker(f) \to X$ there is a unique morphism $u : X \to \ker(f)$ such that $h \circ u = id_X$. Clearly, $h \circ u \circ h = h = h \circ id_{\ker(f)}$. Since h is a monomorphism, this implies that $u \circ h = id_{\ker(f)}$. Hence, $h : \ker(f) \to X$ is an isomorphism. The statements (iii) and (iv) follow by duality from (i) and (ii). $\square$

Definition 2.12. Let $f : X \to Y$ be a morphism in $\mathcal{C}$ admitting a kernel and a cokernel.

(i) The *image* of f, if it exists, is the kernel of $k : Y \to \mathrm{coker}(f)$. It is denoted by $\mathrm{im}(f)$.

(ii) The *coimage* of f, if it exists, is the image of f in $\mathcal{C}^{\mathrm{op}}$. It is denoted by $\mathrm{coim}(f)$. Equivalently, the coimage is the cokernel of $h : \ker(f) \to X$.

Hence, an image of f is a pair (I, m), where I is an object of $\mathcal{C}$ and $m : I \to Y$ is a morphism in $\mathcal{C}$ such that $k \circ m = 0$ and which satisfies the following universal property:

If (I', m') is another pair such that $k \circ m' = 0$, then there exists a unique morphism $u : I' \to I$ such that the following diagram commutes:

$$\begin{tikzcd}
 & I' \arrow[dl, dotted, "\exists! u"'] \arrow[d, "m'"] \arrow[dr, "0"] & \\
I \arrow[r, "m"] & Y \arrow[r, "k"] & \mathrm{coker}(f)
\end{tikzcd}$$

As a direct consequence of this universal property there exists a unique morphism $e : X \to I$ such that $f = m \circ e$. Moreover, the morphism m is a monomorphism.

Furthermore, the coimage of f is a pair (C, n) where C is an object of $\mathcal{C}$ and $n : X \to C$ is a morphism in $\mathcal{C}$ such that $n \circ h = 0$ and which satisfies the following universal property:

$$\begin{tikzcd}
\ker(f) \arrow[r, "h"] \arrow[dr, "0"'] & X \arrow[r, "n"] \arrow[d, "n'"] & C \arrow[dl, dotted, "\exists! u"] \\
 & C' &
\end{tikzcd}$$

As a direct consequence of this universal property, there exists a unique morphism $\tilde{f} : C \to Y$ such that $f = \tilde{f} \circ n$. Moreover, the morphism n is an epimorphism.

Example 2.13. Let R be a ring. In the category $\mathbf{Mod}(R)$ of left R-modules the image of a morphism $f : M \to N$ is the left R-submodule $f(M) \hookrightarrow N$ and the coimage of f is the left R-quotient module $M \to M/f^{-1}(0)$.

Suppose $\mathcal{C}$ is an additive category that admits all kernels and cokernels and suppose $f : X \to Y$ is a morphism in $\mathcal{C}$. Consider the following commutative diagram:

$$\begin{tikzcd} \ker(f) \arrow[r, "h"] & X \arrow[r, "f"] \arrow[d, "n"'] \arrow[dr, dotted, "\exists! e"] & Y \arrow[r, "k"] & \operatorname{coker}(f) \\ & \operatorname{coim}(f) \arrow[r, dotted, "\exists! u"] & \operatorname{im}(f) \arrow[u, "m"'] & \end{tikzcd}$$

Here, the dotted arrows are obtained as follows. We have $k \circ f = 0$ and hence, by the universal property of the image, we get a unique morphism $e : X \to \operatorname{im}(f)$ such that $f = m \circ e$. Then we have that

$$m \circ 0 = 0 = f \circ h = m \circ e \circ h,$$

such that $0 = e \circ h$ since m is a monomorphism. By the universal property of $\operatorname{coim}(f)$ there exists a unique morphism $u : \operatorname{coim}(f) \to \operatorname{im}(f)$ such that $e = u \circ n$.

Moreover, the morphism u is the unique morphism such that the square is commutative. This follows from the fact that n is an epimorphism and that m is a monomorphism. In particular, if u is an epimorphism (or even an isomorphism), the morphism e is an epimorphism.

Definition 2.14. An additive category $\mathcal{C}$ is *abelian* if it satisfies the following conditions:

(i) Any morphism admits a kernel and a cokernel.

(ii) Any morphism f in $\mathcal{C}$ is *strict*, i.e. the natural morphism $\operatorname{coim}(f) \to \operatorname{im}(f)$ is an isomorphism.

Abelian categories have many desirable properties. For example, as a direct consequence of the definition, in an abelian category one has fiber products and fibered coproducts. Indeed, let $f : X \to Z$ and $g : Y \to Z$ be morphisms in $\mathcal{C}$. Since $\mathcal{C}$ is additive, the product $X \times Y$ with the natural projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ exists in $\mathcal{C}$. We obtain two morphisms:

$$X \times Y \xrightarrow{\pi_X} X \xrightarrow{f} Z$$
$$X \times Y \xrightarrow{\pi_Y} Y \xrightarrow{g} Z.$$

Since $\mathcal{C}$ is abelian, the kernel $h : \ker(f \circ \pi_X - g \circ \pi_Y) \to X \times Y$ exists. We define the object

$$X \times_Z Y := \ker(f \circ \pi_X - g \circ \pi_Y),$$

the fiber product projections are given by $\tilde{f} := \pi_X \circ h$ and $\tilde{g} := \pi_Y \circ h$. Let us check that they have the desired universal property. By construction, h equalizes $f \circ \pi_X$ and $g \circ \pi_Y$ whence $f \circ \tilde{f} = g \circ \tilde{g}$.

Now consider a commutative diagram:

$$\begin{tikzcd} T \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

By the universal property of $X \times Y$ we obtain a commutative diagram:

$$\begin{tikzcd} & T \arrow[dl, "p"'] \arrow[d, "\exists! t"'] \arrow[dr, "q"] & \\ X & X \times_Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

Then

$$\begin{aligned}(f \circ \pi_X - g \circ \pi_Y) \circ t &= f \circ \pi_X \circ t - g \circ \pi_Y \circ t \\ &= f \circ p - g \circ q \\ &= 0,\end{aligned}$$

and by the universal property of $h : X \times_Z Y = \ker(f \circ \pi_X - g \circ \pi_Y) \to X \times Y$ we obtain a unique morphism $u : T \to X \times_Z Y$ such that $t = h \circ u$. In particular, u is the unique morphism such that the desired diagram commutes:

$$\begin{tikzcd} T \arrow[dr, dotted, "\exists! u"] \arrow[drr, bend left, "q"] \arrow[ddr, bend right, "p"'] & & \\ & X \times_Z Y \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{f}"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & Z \end{tikzcd}$$

One can construct the fibered coproduct in the analogous way with the coproduct and the cokernel of a suitable morphism.

Remark 2.15. Abelian categories have the following permanence properties:

(i) If $\{\mathcal{C}_i\}_{i \in I}$ is a small family of abelian categories, i.e. I is any set, then the product category $\prod_{i \in I} \mathcal{C}_i$ is abelian. (Co-)kernels are constructed component wise.

(ii) Let I be a small category. If $\mathcal{C}$ is abelian, the category $\mathcal{C}^I$ of functors from I to $\mathcal{C}$ is abelian. For example, if $F, G : I \to \mathcal{C}$ are two functors and $\eta : F \to G$ is a morphism of functors, define the functor N by $N(X) := \ker(F(X) \to G(X))$. Then N is a kernel of η. The smallness assumption on I is needed to ensure that $\mathrm{Hom}(F, G)$ is a set. Indeed, a morphism $\eta : F \to G$ of functors is a family $(\eta(X))_{X \in I} \in \prod_{X \in I} \mathrm{Hom}(F(X), G(X))$ satisfying certain conditions.

(iii) If $\mathcal{C}$ is abelian, so is $\mathcal{C}^{\mathrm{op}}$ with kernels becoming cokernels and vice versa.

Example 2.16. If R is a ring, then the additive category $\mathbf{Mod}(R)$ of left R-modules is an abelian category. For this note that all kernels and cokernels exist and if $f : M \to N$ is a morphism in $\mathbf{Mod}(R)$, then the canonical morphism

$$\mathrm{coim}(f) = M/f^{-1}(0) \to f(M) = \mathrm{im}(f)$$

is given by $x + \ker(f) \mapsto f(x)$ and is an isomorphism by the first isomorphism theorem for left R-modules. The category $\mathbf{Mod}_{\mathrm{fin}}(R)$ is abelian if and only if R is noetherian.

Example 2.17. Let us now give an example of an additive category admitting all kernels and cokernels but the canonical morphism $\mathrm{coim}(f) \to \mathrm{im}(f)$ is not an isomorphism in general. For this consider the additive category of topological abelian groups. Here the kernel of a morphism $f : A \to B$ is $f^{-1}(0) \hookrightarrow A$ with the subspace topology of A. The image of f is $f(A) \hookrightarrow B$ with the subspace topology of B. The cokernel of f is $B/f(A)$ with the quotient topology of B and the coimage of f is given by $A/\ker(f)$ with the quotient topology of A. The canonical morphism

$$\mathrm{coim}(f) = A/\ker(f) \to f(A) = \mathrm{im}(f)$$

is given by $x+\ker(f) \mapsto f(x)$ and is continuous with respect to the given topologies and by the first isomorphism theorem for groups it is bijective. However, the inverse is in general not continuous. Indeed, let $\mathbb{R}$ be the real numbers with the usual topology and let $\mathbb{R}_{\mathrm{disc}}$ be the real numbers with the discrete topology. Consider $f = \mathrm{id}_{\mathbb{R}} : \mathbb{R}_{\mathrm{disc}} \to \mathbb{R}$. By the above, both the kernel and the cokernel are zero. Moreover, we have that $\mathrm{im}(f) = \mathbb{R}$ and $\mathrm{coim}(f) = \mathbb{R}_{\mathrm{disc}}$. In particular, the natural morphism $\mathrm{coim}(f) \to \mathrm{im}(f)$ is continuous and bijective but the inverse is not continuous, i.e. $\mathrm{coim}(f) \to \mathrm{im}(f)$ is not an isomorphism in the category of topological abelian groups. The same holds for f even though both kernel and cokernel are zero.

As promised, in an abelian category the property of being an isomorphism can be measured by the kernel and cokernel. Recall that a morphism in an additive category is a monomorphism if and only if the kernel is zero and an epimorphism if and only if the cokernel is zero (cf. Lemma 2.11).

Proposition 2.18. *A morphism in an abelian category $\mathcal{C}$ is an isomorphism if and only if it is both an epimorphism and a monomorphism.*

Proof. Any isomorphism is always a monomorphism as well as an epimorphism. Suppose then that we have a morphism $f : X \to Y$ that is both a monomorphism and an epimorphism. Consider the commutative diagram:

$$\begin{tikzcd}
\ker(f) \arrow[r, "h"] & X \arrow[r, "f"] \arrow[d, "n"'] & Y \arrow[r, "k"] & \mathrm{coker}(f) \\
& \mathrm{coim}(f) \arrow[r, "u"] & \mathrm{im}(f) \arrow[u, "m"'] &
\end{tikzcd}$$

As $\mathcal{C}$ is abelian, the morphism u is an isomorphism and hence, it is enough to show that m and n are isomorphisms. Recall that the image $m : \mathrm{im}(f) \to Y$ is the kernel of the cokernel $k : Y \to \mathrm{coker}(f)$. By Lemma 2.11, m is an isomorphism if and only if $k = 0$. Since $k \circ f = 0 = 0 \circ f$ and because f is an epimorphism, $k = 0$. Similarly, we have that n is an isomorphism. Thus, f is an isomorphism. $\square$

Let us now discuss what it means for a sequence $X' \xrightarrow{f} X \xrightarrow{g}$ with $g \circ f = 0$ in an abelian category to be exact. Consider the following diagram:

$$\begin{tikzcd}
& \mathrm{im}(f) \arrow[d] & \\
X' \arrow[ur] \arrow[r, "f"] & X \arrow[r, "g"] & X''
\end{tikzcd}$$

The composition $X' \to \mathrm{im}(f) \to X \to X''$ vanishes. Since the morphism $X' \to \mathrm{Im}(f)$ is an epimorphism, the morphism $\mathrm{Im}(f) \to X \xrightarrow{g} X''$ vanishes. Hence, there is a natural morphism $\mathrm{im}(f) \to \ker(g)$.

Definition 2.19. Let $\mathcal{C}$ be an abelian category and consider a sequence $X' \xrightarrow{f} X \xrightarrow{g} X''$ with $g \circ f = 0$. The sequence is called *exact* if the natural morphism $\mathrm{im}(f) \to \ker(g)$ is an isomorphism. More generally, a sequence $(X_i, f_i)_{i \in \mathbb{Z}}$ with $f_{i+1} \circ f_i = 0$ is called exact if any sequence $X_{n-1} \to X_n \to X_{n+1}$ extracted from it is exact.

Definition 2.20. A functor $F : \mathcal{C} \to \mathcal{C}'$ between abelian categories is called *additive* if the induced map $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$ is additive for any $X, Y \in \mathcal{C}$. An additive functor $F : \mathcal{C} \to \mathcal{C}'$ is called *left exact* if for all short exact sequences

$$0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$$

the sequence

$$0 \to F(X') \xrightarrow{F(f)} F(X) \xrightarrow{F(g)} F(X'')$$

is exact in $\mathcal{C}'$ and we say that F is *right exact* if for all short exact sequences

$$0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$$

the sequence

$$F(X') \xrightarrow{F(f)} F(X) \xrightarrow{F(g)} F(X'') \to 0$$

is exact in $\mathcal{C}'$. Finally, we say that F is *exact* if F is both left and right exact.

2.2 Projective objects and (projective-)generators

In this section we recall the notions of projective objects and generators in an abelian category $\mathcal{C}$. The first part of this section on projective objects and (projective-)generators is based on the lecture notes *Homological algebra* by Sophie Morel [21]. The generalization to collections of (projective-)generators is still inspired by the techniques used in the first part.

Lemma 2.21. *Let $\mathcal{C}$ be an abelian category. For every $X \in \mathcal{C}$ the following functors are both left exact:*

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{C}}(X,-) : \mathcal{C} \to \{\text{abelian groups}\} \\ &Y \mapsto \mathrm{Hom}_{\mathcal{C}}(X,Y) \\ &(f : Y \to Y') \mapsto f_* := (g \mapsto f \circ g) : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,Y') \end{aligned}$$

and

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{C}}(-,X) : \mathcal{C}^{\mathrm{op}} \to \{\text{abelian groups}\} \\ &Y \mapsto \mathrm{Hom}(Y,X) \\ &(f : Y' \to Y) \mapsto f^* := (g \mapsto g \circ f) : \mathrm{Hom}_{\mathcal{C}}(Y,X) \to \mathrm{Hom}_{\mathcal{C}}(Y',X). \end{aligned}$$

Proof. By duality it is enough to show that the functor $\mathrm{Hom}_{\mathcal{C}}(X,-)$ is left exact. Let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a short exact sequence. We need to show that the sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(X,A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{C}}(X,B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{C}}(X,C)$$

is exact. Let $\alpha \in \ker(f_*)$. Then we see that

$$f \circ \alpha = f_*(\alpha) = 0 = f \circ 0.$$

Since f is a monomorphism, we conclude that $\alpha = 0$ and hence, $\ker(f_*) = 0$. Now let $\alpha \in \mathrm{im}(f_*)$, then there exists $\beta \in \mathrm{Hom}(X,A)$ such that

$$\alpha = f_*(\beta) = f \circ \beta.$$

Hence, we have that

$$g_*(\alpha) = g \circ \alpha = g \circ f \circ \beta = 0$$

and thus, $\alpha \in \ker(g_*)$. Let $\beta \in \ker(g_*)$. Then we have that

$$0 = g_*(\beta) = g \circ \beta.$$

Since f is a monomorphism, f is its own image. Since the canonical map $\mathrm{im}(f) \to \ker(g)$ is an isomorphism, we get that β factors through f. Thus, $\beta \in \mathrm{im}(f_*)$. $\square$

Definition 2.22. Let $\mathcal{C}$ be an abelian category. An object $P \in \mathcal{C}$ is called *projective* if the left exact functor $\mathrm{Hom}_{\mathcal{C}}(P,-)$ is exact. We say that $\mathcal{C}$ has *enough projectives* if for every object $X \in \mathcal{C}$ there exists an epimorphism $P \to X$ with P projective.

Remark 2.23. The dual notion of projective is *injective*. Thus, an object $I \in \mathcal{C}$ is injective if the left exact functor $\mathrm{Hom}_{\mathcal{C}}(-, I)$ is exact. An object is projective (resp. injective) in $\mathcal{C}$ if and only if it is injective (resp. projective) in $\mathcal{C}^{\mathrm{op}}$. Hence, all statements about projective objects in $\mathcal{C}$ in this section have a dual version about injective objects in $\mathcal{C}^{\mathrm{op}}$.

Proposition 2.24. *Let $\mathcal{C}$ be an abelian category and let $P \in \mathcal{C}$. Then P is projective if and only if for any $X, Y \in \mathcal{C}$, for any epimorphism $f : X \to Y$ and any morphism $u : P \to Y$, there exists $v : P \to X$ such that $u = f \circ v$.*

Proof. Assume that P is projective. Let $f : X \to Y$ be an epimorphism and let $u : P \to Y$ be any morphism. Furthermore, let $h : \ker(f) \to X$ be the kernel of f. Consider the short exact sequence

$$0 \to \ker(f) \xrightarrow{h} X \xrightarrow{f} Y \to 0.$$

By applying the exact functor $\mathrm{Hom}_{\mathcal{C}}(P, -)$ we obtain the short exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(P, \ker(f)) \xrightarrow{h_*} \mathrm{Hom}_{\mathcal{C}}(P, X) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{C}}(P, Y) \to 0.$$

Since f_* is surjective, it follows that there exists some $v : P \to X$ such that $u = f \circ v$. For the converse, assume that we are given a short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

We need to show that the sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(P, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{C}}(P, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{C}}(P, C) \to 0$$

is exact. We already know that $\mathrm{Hom}_{\mathcal{C}}(P, -)$ is always left exact, so the surjectivity of g_* is missing. Since g is an epimorphism, the assumption on P ensures that g_* is surjective. Hence, P is projective. $\square$

Remark 2.25. In an abelian category the coproduct is usually denoted by $\oplus$ and one speaks of the direct sum. We will refer to direct sums indexed by a set as *small direct sums*.

Here is an element free proof of what is sometimes called the *splitting lemma*.

Proposition 2.26. *Let $\mathcal{C}$ be an abelian category and let*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a short exact sequence in $\mathcal{C}$. The following statements are equivalent:

(i) *There exists $h : C \to B$ such that $g \circ h = id_C$.*

(ii) *There exists $k : B \to A$ such that $k \circ f = id_A$.*

(iii) *There exists $\varphi = (k, g) : B \to A \oplus C$ and there exists $\psi = (f, h) : A \oplus C \to B$ that are mutually inverse.*

Proof. It is clear that (iii) implies (i) and (ii). The statement (ii) implies (iii) is the statement that (i) implies (iii) in the opposite category. Hence it is enough to show that (i) implies (ii) and (iii). By the universal property of the direct sum $A \oplus C$ and its injections ϕ_A and ϕ_C we have a unique morphism $\psi := (f, h) : A \oplus C \to B$ such that $\psi \circ \phi_A = f$ and $\psi \circ \phi_C = h$. Let $a : \ker(g) \to B$ be the kernel of g. By assumption we have that

$$g = g \circ h \circ g$$

and hence

$$g \circ (id_B - h \circ g) = 0.$$

By the universal property of $a : \ker(g) \to B$ there is a unique morphism $k' : B \to \ker(g)$ such that

$$id_B - h \circ g = a \circ k'.$$

Let $m : \mathrm{im}(f) \to B$ be the image of f. Since f is a monomorphism and hence its own image, we have that $f = m \circ u$ where $u : A \to \mathrm{im}(f)$ is an isomorphism. By assumption the natural morphism $n : \mathrm{im}(f) \to \ker(g)$ is an isomorphism. By construction we have a commutative diagram:

$$\begin{tikzcd}
 & \mathrm{im}(f) \arrow[r, "n"] \arrow[d, "m"] & \ker(g) \arrow[dl, "a"] & \\
A \arrow[ur, "u"] \arrow[r, "f"] & B \arrow[rr, "g"] & & C
\end{tikzcd}$$

Set $k := u^{-1} \circ n^{-1} \circ k'$. Then we have that

$$\begin{aligned}
f \circ k &= f \circ u^{-1} \circ n^{-1} \circ k' \\
&= a \circ n \circ u \circ u^{-1} \circ n^{-1} \circ k' \\
&= a \circ k' \\
&= id_B - h \circ g.
\end{aligned}$$

This implies that

$$\begin{aligned}
f \circ k \circ h &= (id_B - h \circ g) \circ h \\
&= h - h \circ g \circ h \\
&= 0.
\end{aligned}$$

Hence, $k \circ h = 0$ because f is a monomorphism. Moreover, we have

$$\begin{aligned}
f \circ k \circ f &= (id_B - h \circ g) \circ f \\
&= f - h \circ g \circ f \\
&= f,
\end{aligned}$$

because $g \circ f = 0$. Again, because f is a monomorphism, this implies that $k \circ f = id_A$, which is (ii). The universal property of $A \times C$ and the canonical projections π_A and π_C induce a unique morphism $\tilde{\varphi} := (k, g) : B \to A \times C$ such that $\pi_A \circ \tilde{\varphi} = k$ and

$\pi_C \circ \tilde{\varphi} = g$. Consider the morphism $\varphi : B \to A \oplus C$ defined as the composition of $\tilde{\varphi}$ and $\phi_A \circ \pi_A + \phi_C \circ \pi_C$, i.e. $\varphi = \phi_A \circ k + \phi_C \circ g$. Then we have

$$\begin{aligned}\psi \circ \varphi &= \psi \circ (\phi_A \circ k + \phi_C \circ g)\\ &= \psi \circ \phi_A \circ k + \psi \circ \phi_C \circ g\\ &= f \circ k + h \circ g\\ &= id_B - h \circ g + h \circ g\\ &= id_B,\end{aligned}$$

and

$$\begin{aligned}\varphi \circ \psi \circ \phi_A &= \varphi \circ f\\ &= (\phi_A \circ k + \phi_C \circ g) \circ f\\ &= \phi_A \circ k \circ f + \phi_C \circ g \circ f\\ &= \phi_A.\end{aligned}$$

Likewise, $\varphi \circ \psi \circ \phi_C = \phi_C$ and hence $\varphi \circ \psi = id_{A \oplus C}$. $\square$

Definition 2.27. Let $\mathcal{C}$ be an abelian category and let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a short exact sequence. We say that the sequence *splits* if it satisfies one of the equivalent conditions from Proposition 2.26.

Example 2.28. Let $\mathcal{C}$ be an abelian category. An object C of $\mathcal{C}$ is projective if and only if any short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ splits. Indeed, assume first that C is projective. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be any short exact sequence. By Proposition 2.24 the identity id_C factors through some $h : C \to B$. Hence, the sequence splits. Conversely, let $X \xrightarrow{g} Y$ and $C \xrightarrow{f} Y$ and suppose that g is an epimorphism. Consider the fiber product $C \times_Y X$ with its projections p_1 and p_2. The projection $C \times_Y X \xrightarrow{p_1} C$ is an epimorphism (cf. Lemma 12.5.13 in [22]) and hence the short exact sequence

$$0 \to \ker(p_1) \to C \times_Y X \xrightarrow{p_1} C \to 0$$

splits, i.e. there is $h : C \to C \times_Y X$ such that $p_1 \circ h = id_C$. Set $\tilde{f} := p_2 \circ h$. Then

$$g \circ \tilde{f} = g \circ p_2 \circ h = f \circ p_1 \circ h = f$$

and this implies by Proposition 2.24 that C is projective.

Lemma 2.29. *Let $\mathcal{C}$ be an abelian category admitting small direct sums and let $(X_i)_{i \in I}$ be a family of objects in $\mathcal{C}$. Then $\bigoplus_{i \in I} X_i$ is projective if and only if all the X_i are.*

Proof. The functor $\mathrm{Hom}_{\mathcal{C}}(\bigoplus_{i \in I} X_i, -)$ is isomorphic to the functor $\prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(X_i, -)$. In the category of abelian groups a product of sequences is exact if and only if each of the sequences is exact. This shows the claim. $\square$

Let R be a commutative ring with 1. We have everything at hand to classify the projective objects in the category of left R-modules.

Example 2.30. The projective objects in the category of left R-modules are exactly direct summands of free R-modules and finitely generated projective R-modules are direct summands of finitely generated free R-modules. In particular, the category of left R-modules has enough projectives. To see all this, note first that R is projective. Indeed, let M be a left R-module, the natural bijections

$$\mathrm{Hom}_R(R, M) \to M, u \mapsto u(1)$$

give rise to a natural isomorphism of the hom-functor $\mathrm{Hom}_R(R, -)$ and the identity functor of the category of left R-modules. Hence, $\mathrm{Hom}_R(R, -)$ is exact and thus R projective. By Lemma 2.29, every free R-module is projective. By the same lemma every direct summand of a free R-module is projective. Now let P be a projective R-module. Then the R-linear map

$$f : \bigoplus_{p\in P} R \to P, e_p \mapsto p$$

is surjective. By Example 2.28 the short exact sequence

$$0 \to \ker(f) \to \bigoplus_{p\in P} R \xrightarrow{f} P \to 0$$

splits. Hence, Proposition 2.26 implies that P is direct summand of the free R-module $\bigoplus_{p\in P} R$. Of course, if P is finitely generated, we can take a finite direct sum.

Definition 2.31. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is called *conservative* if it reflects isomorphisms, i.e. if for any morphism f in $\mathcal{C}$, $F(f)$ being an isomorphism in $\mathcal{D}$ implies that f is an isomorphism.

Example 2.32. The forgetful functor from the category of groups to the category of sets is conservative. Not any forgetful functor is conservative, indeed, the forgetful functor from the category of topological spaces to the category of sets is not conservative. The reason is that not every continuous bijection is a homeomorphism.

Definition 2.33. Let $\mathcal{C}$ be a locally small category. We say that an object X of $\mathcal{C}$ is a *generator* if the functor $\mathrm{Hom}(X, -) : \mathcal{C} \to \{\text{sets}\}$ is conservative and we say that X is a *projective generator* if it is a generator and a projective object.

Example 2.34. (i) A singleton is a generator in the category of sets.

(ii) $\mathbb{Z}$ is a projective generator in the category of abelian groups.

Our next goal is to classify projective generators in an abelian category that admits all direct sums indexed by sets.

Proposition 2.35. *Let $\mathcal{C}$ be an abelian category that admits all small direct sums and let Q be a generator. The following statements are true:*

(i) *The functor* $\mathrm{Hom}(Q, -) : \mathcal{C} \to \{\textit{abelian groups}\}$ *is faithful.*

(ii) *If $X \in \mathcal{C}$, then $X \cong 0$ if and only if* $\mathrm{Hom}(Q, X) = 0$.

(iii) *If $f : X \to Y$ is a morphism of $\mathcal{C}$ and $f_* :$* $\mathrm{Hom}(Q, X) \to \mathrm{Hom}(Q, Y)$ *is surjective, then f is an epimorphism.*

(iv) *For any object X in $\mathcal{C}$ consider the morphism $\bigoplus_{\mathrm{Hom}(Q,X)} Q \to X$ whose composition with the injection corresponding to $f \in \mathrm{Hom}(Q,X)$ is f. Then this morphism is an epimorphism.*

Proof. Let $f : X \to Y$ be a morphism such that $f_* = 0$. Let $h : \ker(f) \to X$ be the kernel of f. Consider the exact sequence

$$0 \to \ker(f) \xrightarrow{h} X \xrightarrow{f} Y.$$

Applying the left exact functor $\mathrm{Hom}(Q,-)$ to the sequence yields an exact sequence

$$0 \to \mathrm{Hom}(Q,\ker(f)) \xrightarrow{h_*} \mathrm{Hom}(Q,X) \xrightarrow{f_*} \mathrm{Hom}(Q,Y).$$

This implies that h_* is a kernel of f_*. Because $f_* = 0$, this means that h_* is an isomorphism. Thus, h is an isomorphism as $\mathrm{Hom}(Q,-)$ is conservative. Hence, we have that $f = 0$ which implies that $\mathrm{Hom}(Q,-)$ is faithful. This shows (i).
If we have that $\mathrm{Hom}(Q,X) = 0$, then the zero morphism $X \to X$ induces an isomorphism $\mathrm{Hom}(Q,X) \to \mathrm{Hom}(Q,X)$. Since $\mathrm{Hom}(Q,-)$ is conservative, this means that the zero morphism $X \to X$ is an isomorphism and thus, $X \cong 0$. Hence, (ii) is true.
Let $f : X \to Y$ be a morphism such that $f_* : \mathrm{Hom}(Q,X) \to \mathrm{Hom}(Q,Y)$ is surjective and let $k : Y \to \mathrm{coker}(f)$ be the cokernel of f. By assumption, if $g \in \mathrm{Hom}(Q,Y)$, there exists $h \in \mathrm{Hom}(Q,X)$ such that $g = f_*(h) = f \circ h$. But then

$$k \circ g = k \circ f \circ h = 0$$

and thus, $k_* = 0$. Since $\mathrm{Hom}(Q,-)$ is faithful by part (i), this means that $k = 0$. Hence, f is an epimorphism. This shows (iii).
Let X be an object in $\mathcal{C}$ and let $g : \bigoplus_{\mathrm{Hom}(Q,X)} Q \to X$ be the morphism of (iv). If $f : Q \to X$ is a morphism, then the commutative diagram

$$\begin{tikzcd} \bigoplus_{\mathrm{Hom}(Q,X)} Q \arrow[r, "g"] & X \\ Q \arrow[u, "\phi_f"] \arrow[ur, "f"'] & \end{tikzcd}$$

gives rise to the commutative diagram

$$\begin{tikzcd} \prod_{\mathrm{Hom}(Q,X)} \mathrm{Hom}(Q,Q) \arrow[r, "g_*"] & \mathrm{Hom}(Q,X) \\ \mathrm{Hom}(Q,Q) \arrow[u, "(\phi_f)_*"] \arrow[ur, "f_*"'] & \end{tikzcd}$$

and we see that

$$f = f_*(id_Q) = g_*((\phi_f)_*(id_Q)).$$

Hence, g is an epimorphism by (iii) and thus, (iv) is true which finishes the proof. $\square$

Proposition 2.36. *Let $\mathcal{C}$ be an abelian category that admits all small direct sums and let Q be an object of $\mathcal{C}$. The following statements are equivalent:*

(i) *Q is a generator.*

(ii) *The functor* $\mathrm{Hom}(Q, -) : \mathcal{C} \to \{$*abelian groups*$\}$ *is faithful.*

(iii) *For any object X of $\mathcal{C}$ there exists a set I and an epimorphism $\bigoplus_I Q \to X$.*

Proof. We have seen (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) in Proposition 2.35. Let us now prove that (ii) implies (i). Suppose that $\mathrm{Hom}(Q, -)$ is faithful and let $f : X \to Y$ be a morphism in $\mathcal{C}$ such that f_* is an isomorphism. Let $h : \ker(f) \to X$ and $k : Y \to \mathrm{coker}(f)$ be the kernel and the cokernel of f. Then $f \circ h = 0$ implies that $f_* \circ h_* = 0$ and hence $h_* = 0$. Because $\mathrm{Hom}(Q, -)$ is faithful, this means that $h = 0$. Similarly $k = 0$ and thus, $\ker(f) \cong 0$ and $\mathrm{coker}(f) \cong 0$. This means that f is an isomorphism, i.e. $\mathrm{Hom}(Q, -)$ is conservative. It remains to show that (iii) implies (ii). Let $f : X \to Y$ be a morphism in $\mathcal{C}$ and suppose that $f_* = 0$. By assumption there exists some set I and an epimorphism $u : \bigoplus_I Q \to X$. We can write $u = (u_i)_{i \in I}$ with morphisms $u_i : Q \to X$. Since $f_* = 0$, any of the compositions $f \circ u_i$ is zero. This implies that $f \circ u = 0$. By assumption u is an epimorphism and hence, $f = 0$. Thus, the functor $\mathrm{Hom}(Q, -)$ is faithful. □

Corollary 2.37. *Let $\mathcal{C}$ be an abelian category that admits all small direct sums and let P be an object of $\mathcal{C}$. The following statements are equivalent:*

(i) *P is a projective generator.*

(ii) *The functor* $\mathrm{Hom}(Q, -) : \mathcal{C} \to \{$*abelian groups*$\}$ *is exact and faithful.*

(iii) *P is projective and for every nonzero object X of $\mathcal{C}$ there exists a nonzero morphism $P \to X$.*

Moreover, if $\mathcal{C}$ has a projective generator, then it has enough projective objects.

Proof. Let us start with (i) implies (ii). For this suppose that P is a projective generator. As P is projective, the functor $\mathrm{Hom}(P, -)$ is exact by definition, it is faithful by Proposition 2.36 because P is a generator.
Assume now that $\mathrm{Hom}(P, -)$ is exact and faithful. Then P is projective by definition and by Proposition 2.36 it is a generator. If X is any nonzero object in $\mathcal{C}$, then $id_X \neq 0$. Because $\mathrm{Hom}(P, -)$ is faithful, this means that $(id_X)_* \neq 0$. Hence, there is $g : P \to X$ that is not the zero morphism. This shows that (ii) implies (i) and that (ii) implies (iii). Finally, we show that (iii) implies (ii). If P is projective, then the functor $\mathrm{Hom}(P, -)$ is exact by definition. Let $f : X \to Y$ be a morphism and suppose that $f \neq 0$. We want to show that $f_* \neq 0$. Let $m : \mathrm{im}(f) \to Y$ be the image of f. There is an epimorphism $e : X \to \mathrm{im}(f)$ such that $f = m \circ e$. Since $f \neq 0$, we have that $m \neq 0$ and hence $\mathrm{im}(f) \neq 0$. By assumption there exists a nonzero morphism $u : P \to \mathrm{im}(f)$. As P is projective, by Proposition 2.24 there exists $v : P \to X$ such that $u = e \circ v$. Then

$$\begin{aligned} f \circ v &= m \circ e \circ v \\ &= m \circ u \\ &\neq 0 \end{aligned}$$

because the image m is a monomorphism and $u \neq 0$. So $f_* \neq 0$ and thus, the functor $\mathrm{Hom}(P, -)$ is faithful. The final statement follows from Proposition 2.36. □

By the above, if Q is a generator, then the functor $\mathrm{Hom}(Q, -)$ is faithful. This means precisely that for any two distinct morphisms $f, g : X \to Y$ there always exists a morphism $h : Q \to X$ such that $f \circ h \neq g \circ h$. Conversely, if Q is an object with that property, then the functor $\mathrm{Hom}(Q, -)$ is faithful. We generalize the notion of a generator to more than one object. Those *generators* will play an important role later on in this work for the category of (κ-)condensed abelian groups, many of the good properties of this category come from the existence of *compact projective generators*. Our goal is thus to give a nice characterization of projective generators.

Definition 2.38. Let $\mathcal{C}$ be a category. A *family of generators* of $\mathcal{C}$ is a set $\mathcal{Q}$ of objects in $\mathcal{C}$ such that whenever we have two distinct morphisms $f, g : X \to Y$ in $\mathcal{C}$, there exists an object Q in the set $\mathcal{Q}$ and a morphism $h : Q \to X$ such that $f \circ h \neq g \circ h$. In this case we also say that $\mathcal{Q}$ *generates* the category $\mathcal{C}$.

Remark 2.39. If the set $\mathcal{Q}$ has just one object Q, then Q is a generator in the usual sense.

Lemma 2.40. *Let $\mathcal{C}$ be an abelian category, $\mathcal{Q}$ be a family of generators and $f : X \to Y$ a morphism in $\mathcal{C}$. Suppose that $f_* : \mathrm{Hom}(Q, X) \to \mathrm{Hom}(Q, Y)$ is surjective for all $Q \in \mathcal{Q}$, then f is an epimorphism.*

Proof. Suppose that f_* is surjective for all $Q \in \mathcal{Q}$ and let $k : Y \to \mathrm{coker}(f)$ be the cokernel of f. As in the proof of Proposition 2.35 (iii) we have that $k_* = 0$ for all $Q \in \mathcal{Q}$. But then for all $Q \in \mathcal{Q}$ and all morphisms $h : Q \to Y$ we have that $k \circ h = 0 \circ h$ and hence, $k = 0$ because $\mathcal{Q}$ is a family of generators. Thus, f is an epimorphism. □

Proposition 2.41. *Let $\mathcal{C}$ be an abelian category that admits all small direct sums and let $\mathcal{Q}$ be a set of objects in $\mathcal{C}$. The following statements are equivalent:*

(i) *$\mathcal{Q}$ is a family of generators.*

(ii) *For any object X in $\mathcal{C}$ consider the morphism $\bigoplus_{Q \in \mathcal{Q}} \bigoplus_{\mathrm{Hom}(Q,X)} Q \to X$ whose composition with the injection corresponding to $f \in \mathrm{Hom}(Q, X)$ is f. Then this morphism is an epimorphism.*

(iii) *For any object X of $\mathcal{C}$ there exists a set I, objects $Q_i \in \mathcal{Q}$ for every $i \in I$ and an epimorphism $\bigoplus_{i \in I} Q_i \to X$.*

Proof. Suppose that (i) is true and let $g : \bigoplus_{Q \in \mathcal{Q}} \bigoplus_{\mathrm{Hom}(Q,X)} Q \to X$ be the morphism from (ii). As in the proof of Proposition 2.35 (iv) one sees that g_* is surjective for all $Q \in \mathcal{Q}$. By the previous lemma this means that g is an epimorphism. This shows (ii). The implication (ii) $\Rightarrow$ (iii) is clear. Let us now assume (iii) holds and assume that we are given two distinct morphisms $f, g : X \to Y$. By assumption there exists a set I, objects $Q_i \in \mathcal{Q}$ for every $i \in I$ and an epimorphism $h : \bigoplus_{i \in I} Q_i \to X$. Write $h = (h_i)_{i \in I}$ with morphisms $h_i : Q_i \to X$. Because h is an epimorphism, we necessarily have that $f \circ h_i \neq g \circ h_i$ for some $i \in I$. This shows (i). □

Corollary 2.42. *Let $\mathcal{C}$ be an abelian category that admits all small direct sums and let $\mathcal{Q}$ be a family of generators of $\mathcal{C}$. Suppose that all $Q \in \mathcal{Q}$ are projective. Then $\mathcal{C}$ admits a projective generator, given by $P := \bigoplus_{Q \in \mathcal{Q}} Q$.*

Proof. P is projective by Lemma 2.29 and P is a generator by Proposition 2.41 (ii) and Proposition 2.36 (iii). □

Corollary 2.43. *Let $\mathcal{C}$ be an abelian category that admits all small direct sums and let $\mathcal{Q}$ be a set of objects in $\mathcal{C}$. The following statements are equivalent:*

(i) *$\mathcal{Q}$ is a set of generators consisting of projective objects.*

(ii) *All elements of $\mathcal{Q}$ are projective and for every nonzero object X of $\mathcal{C}$ there exists $Q \in \mathcal{Q}$ and a nonzero morphism $Q \to X$.*

Proof. Suppose (i) is true. Clearly all $Q \in \mathcal{Q}$ are projective. Let X be a nonzero object of $\mathcal{C}$. By Proposition 2.41 (iii) there exists a set I, objects $Q_i \in \mathcal{Q}$ for every $i \in I$ and an epimorphism $h : \bigoplus_{i \in I} Q_i \to X$. Because X is nonzero, so is the epimorphism h. Write $h = (h_i)_{i \in I}$ with morphisms $h_i : Q_i \to X$, then we necessarily have that one of the $h_i : Q_i \to X$ is nonzero. This shows (ii). Let us now assume that (ii) is true. Of course, all the $Q \in \mathcal{Q}$ are projective and hence we need to show that they generate. Let $g_1, g_2 : X \to Y$ be two distinct morphisms in $\mathcal{C}$. Set $f := g_1 - g_2 : X \to Y$. Clearly f is not the zero morphism. Let $m : \operatorname{im}(f) \to Y$ be the image of f. There is an epimorphism $e : X \to \operatorname{im}(f)$ such that $f = m \circ e$. Since $f \neq 0$, we have that $m \neq 0$ and hence $\operatorname{im}(f) \neq 0$. By assumption there exists a nonzero morphism $u : Q \to \operatorname{im}(f)$ for some $Q \in \mathcal{Q}$. As Q is projective, by Proposition 2.24 there exists $v : Q \to X$ such that $u = e \circ v$. Then

$$\begin{aligned} f \circ v &= m \circ e \circ v \\ &= m \circ u \\ &\neq 0 \end{aligned}$$

because the image m is a monomorphism and $u \neq 0$. In particular this means that $g_1 \circ v \neq g_2 \circ v$ and hence $\mathcal{Q}$ is a set of projective generators. □

2.3 Grothendieck abelian categories and Grothendieck's axioms

Among all abelian categories, perhaps not surprisingly, some of them have better properties than others; Alexander Grothendieck listed some important properties. So-called Grothendieck abelian categories will play an important role in the sequel. They satisfy certain axioms of Grothendieck's list and have a generator. We will briefly introduce them now.

Definition 2.44. A *filtered* category is a non-empty category J such that:

(i) For every two objects j and j' in J there exists an object k in J and two morphisms $j \to k$ and $j' \to k$.

(ii) For every two parallel arrows $u, v : i \to j$ in J there exists an object k in J and a morphism $w : j \to k$ such that $w \circ u = w \circ v$.

A *cofiltered* category is a non-empty category J such that the opposite category is filtered.

The following two axioms make an additive category $\mathcal{C}$ abelian.

(AB1) Any morphism in $\mathcal{C}$ admits a kernel and a cokernel.

(AB2) For every morphism f in $\mathcal{C}$ the natural morphism $\operatorname{coim}(f) \to \operatorname{im}(f)$ is an isomorphism.

Here are the axioms that an abelian category $\mathcal{C}$ may satisfy, a * indicates that this is the dual version.

(AB3) All small direct sums exist.

(AB4) (AB3) is satisfied and direct sums are exact.

(AB5) (AB3) is satisfied and colimits indexed by small filtered categories are exact.

(AB6) (AB3) is satisfied and for any index set J and small filtered categories $I_j, j \in J$, with functors $i \mapsto M_i$ from I_j to $\mathcal{C}$, the natural map

$$\varinjlim_{(i_j \in I_j)_{j\in J}} \prod_{j\in J} M_{i_j} \to \prod_{j\in J} \varinjlim_{i_j \in I_j} M_{i_j}$$

is an isomorphism.

(AB3*) All small products exist.

(AB4*) (AB3*) is satisfied and products are exact.

(AB5*) (AB3*) is satisfied and limits indexed by small filtered categories are exact.

(AB6*) (AB3*) is satisfied and for any index set J and small cofiltered categories $I_j, j \in J$, with functors $i \mapsto M_i$ from I_j to $\mathcal{C}$, the natural map

$$\varprojlim_{(i_j \in I_j)_{j\in J}} \bigoplus_{j\in J} M_{i_j} \to \bigoplus_{j\in J} \varprojlim_{i_j \in I_j} M_{i_j}$$

is an isomorphism.

Definition 2.45. A *Grothendieck abelian category* is an abelian category $\mathcal{C}$ that satisfies (AB5) and has a generator, to wit:

(i) Colimits indexed by small categories exist in $\mathcal{C}$ (AB3).

(ii) If J is a small filtered category, the functor $\varinjlim : \operatorname{Func}(J, \mathcal{C}) \to \mathcal{C}, M \mapsto \varinjlim_J M$ is exact (AB5).

(iii) $\mathcal{C}$ has a generator.

Remark 2.46. Oftentimes one restricts to abelian categories that are $\mathcal{U}$-small for some universe $\mathcal{U}$ in order to avoid set theoretic issues. For us this is of no concern. This approach is carried out in [21]. To mention at least one result that relies on Grothendieck abelian categories, we want to mention the Freyd-Mitchell embedding theorem (cf. Theorem III.3.1 in [21]). The embedding theorem says that one can embed any $\mathcal{U}$-small abelian category into a certain module category. Furthermore, because Grothendieck abelian categories are locally presentable (cf. Corollary 5.2 in [18]), there is a nice version of the adjoint functor theorem, e.g. a functor has a right adjoint if and only if it commutes with all colimits. Indeed, by Theorem 1.58 in [1] locally presentable categories are co-wellpowered. This means we can apply the special adjoint functor theorem (cf. Chapter V.8 in [19]).

3 Sites and topoi

In this section we briefly want to introduces sites and categories of sheaves on a site as a generalization of sheaves on a topological space. For this, let us first recall the definition of a sheaf on the topological space X. For this we denote the category of open subsets of X by $\mathrm{Op}(X)$, the morphisms are inclusions, i.e. there is a unique morphism $U \to V$ if and only if $U \subset V$.

Definition 3.1. A *presheaf* of sets/rings/groups/... on X is a functor

$$\mathcal{F} : \mathrm{Op}(X)^{\mathrm{op}} \to \{\text{sets/rings/groups/...}\}.$$

A *sheaf* of sets on X is a presheaf $\mathcal{F}$ such that for any $U \in \mathrm{Op}(X)$ and any cover $U = \bigcup_i U_i$ with $U_i \in \mathrm{Op}(X)$, the natural map $\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i)$ is the equalizer of the natural maps $\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$.
If $\mathcal{F}$ is a presheaf of rings/groups/..., then $\mathcal{F}$ is a presheaf $\mathcal{F}'$ of sets in a natural way (by forgetting the extra structure). We say that $\mathcal{F}$ is a sheaf of rings/groups/... if $\mathcal{F}'$ is a sheaf of sets.

If we replace $\mathrm{Op}(X)$ by any category $\mathcal{C}$, most of the previous definition still makes sense. However, we need a replacement for coverings. The following definitions are taken from [22] sections 7.6 and 7.7. In particular, the category $\mathcal{C}$ is assumed to be small.

Definition 3.2. Let $\mathcal{C}$ be a category. A *family of morphisms with fixed target* in $\mathcal{C}$ is given by an object $U \in \mathcal{C}$, a set I and for each $i \in I$ a morphism $U_i \to U$ of $\mathcal{C}$ with target U. We use the notation $\{U_i \to U\}_{i \in I}$ to indicate this.
A *site* is given by a category $\mathcal{C}$ and a set $\mathrm{Cov}(\mathcal{C})$ of families of morphisms with fixed target $\{U_i \to U\}_{i \in I}$ called *coverings* of $\mathcal{C}$, satisfying the following axioms

(i) If $V \to U$ is an isomorphism, then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.

(ii) If $\{U_i \xrightarrow{f_i} U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ and for each $i \in I$ we have $\{V_{ij} \xrightarrow{g_{ij}} U_i\}_{j \in J_i} \in \mathrm{Cov}(\mathcal{C})$, then $\{V_{ij} \xrightarrow{f_i \circ g_{ij}} U\}_{i \in I, j \in J_i} \in \mathrm{Cov}(\mathcal{C})$.

(iii) If $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$ is a morphism of $\mathcal{C}$, then $U_i \times_U V$ exists for all $i \in I$ and $\{U_i \times_U V \to V\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$.

By abuse of notation, we usually write $\mathcal{C}$ to indicate this site.

Remark 3.3. In many of the examples to follow $\mathrm{Cov}(\mathcal{C})$ is a proper class and thereby not a set. There are several ways around this, some of them are discussed in section 7.6 in [22]. Moreover, many authors do not exclude *large* categories (i.e. categories that are not necessarily small) from the notion of a site. For example, this is the case in *Sketches of an Elephant: A Topos Theory Compendium* by Peter T. Johnstone [16]. The subsequent definitions still make sense and we can ignore this issue. We will still use the term site even if $\mathrm{Cov}(\mathcal{C})$ is a proper class or the category $\mathcal{C}$ is large. However, in all cases of interest to us, we will be able to replace $\mathcal{C}$ by an equivalent small category which is small (cf. Proposition 4.3). The benefit of this approach is that we can use results from the *Stacks Project* without having to worry about problems that may arise because the category is large or because the coverings form a proper class.

Definition 3.4. Let I be a set. A family of maps of sets $\{S_i \to S\}_{i\in I}$ is called *jointly surjective* if the natural map $\coprod_{i\in I} S_i \to S$ is surjective. A family is called *finite* if the index set I is finite.

Here is an important example of a site as mentioned above. The category is neither small nor do the coverings form a set.

Example 3.5. The category of profinite sets forms a site with finite jointly surjective families of maps as covers. Conditions (i) and (ii) are clear. Let us now verify (iii). First note that if $S' \to S$ and $S'' \to S$ are continuous maps between profinite sets, then $S' \times_S S''$ is a profinite set. Indeed, products of profinite sets are profinite and so are closed subspaces. Finally, $S' \times_S S''$ is closed as it is the preimage of the closed diagonal (S is Hausdorff) in $S \times S$ under the product map $S' \times S'' \to S \times S$. If $\{S_i \to S\}_{i\in I}$ is a finite family of jointly surjective maps and if $S' \to S$ is arbitrary, then we need to check that the family $\{S_i \times_S S' \to S'\}_{i\in I}$ is finite jointly surjective. Set $S'' := \coprod_{i\in I} S_i$. The natural map $\coprod_{i\in I} S_i \times_S S' \to S'$ can be identified with the projection $S'' \times_S S' \to S'$, the latter is surjective as the base change of the surjection $\coprod_{i\in I} S_i \twoheadrightarrow S$. Note that almost the same arguments work for the category of compact Hausdorff spaces with finite jointly surjective families of maps as covers.

We are now able to define the more abstract notion of a (pre-)sheaf.

Definition 3.6. Let $\mathcal{C}$ be a site. A *presheaf* of sets/rings/groups/... on $\mathcal{C}$ is a functor

$$\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \{\text{sets/rings/groups/...}\}.$$

A *sheaf* of sets on $\mathcal{C}$ is a presheaf of sets $\mathcal{F}$ such that for any covering $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$, the natural map $\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i)$ is the equalizer of the natural maps $\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$. A sheaf of rings/groups/... is defined as above.

For completeness, we would like to mention that one can define (pre-)sheaves that take values in more abstract categories. Let $\mathcal{C}$ be a site and let $\mathcal{A}$ be a locally small category. A presheaf $\mathcal{F}$ on $\mathcal{C}$ with values in $\mathcal{A}$ is as usual a functor $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$. For a fixed object $X \in \mathcal{A}$ we get a presheaf of sets $\mathcal{F}_X$ defined by the rule $\mathcal{F}_X(U) := \mathrm{Hom}_{\mathcal{A}}(X, \mathcal{F}(U))$.

Definition 3.7. Let $\mathcal{C}$ be a site, let $\mathcal{A}$ be a locally small category and let $\mathcal{F}$ be a presheaf on $\mathcal{C}$ with values in $\mathcal{A}$. We say that $\mathcal{F}$ is a *sheaf* on $\mathcal{C}$ with values in $\mathcal{A}$ if for all $X \in \mathcal{A}$ the presheaf of sets $\mathcal{F}_X$ is a sheaf of sets.

In all of the above cases, (pre-)sheaves form a category. A morphism of presheaves is a natural transformation of functors and a morphism of sheaves is a morphism of the underlying presheaves. Hence, the category of sheaves is a full subcategory of the category of presheaves. The category of sheaves on a site $\mathcal{C}$ is called a *topos*.
As a general fact (cf. Section 7.10 in [22]), we note that the inclusion functor from the category of sheaves into the category of presheaves admits a left adjoint, namely the sheafification functor.

Let us recall some basic facts about morphisms of sheaves (cf. Section 7.11 in [22]).

Definition 3.8. Let $\mathcal{C}$ be a site, and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of sets.

(i) We say that φ is *injective* if for every object U of $\mathcal{C}$ the map $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective.

(ii) We say that φ is *surjective* if for every U of $\mathcal{C}$ and every section $s \in \mathcal{G}(U)$ there exists a covering $\{U_i \xrightarrow{g_i} U\}_{i \in I}$ such that for all $i \in I$ the restriction $\mathcal{G}(g_i)(s)$ is in the image of $\varphi(U_i) : \mathcal{F}(U_i) \to \mathcal{G}(U_i)$.

Remark 3.9. The injective (resp. surjective) morphisms defined above are exactly the monomorphisms (resp. epimorphisms) of the category of sheaves of sets on $\mathcal{C}$. A morphism of sheaves is an isomorphism if and only if it is both injective and surjective (cf. Lemma 7.11.2 in [22]).

4 κ-Condensed sets

In this section we are going to introduce κ-condensed sets and discuss some basic properties of the category of κ-condensed sets. The exposition closely follows the lecture notes *Condensed Mathematics* of Clausen and Scholze in [7]. A key feature of this section is Proposition 4.3 where we show that the category of κ-small profinite sets is in fact essentially small. This allows us to explicitly avoid set theoretic issues throughout this work - a feature seemingly not shared by related works.

The following definition is only preliminary and will be made precise in 4.2.

Definition 4.1. The pro-étale site $*_{\text{proét}}$ of a point is the category of profinite sets S, with finite jointly surjective families of maps as covers. A *condensed set* is a sheaf of sets on the site $*_{\text{proét}}$. Similarly, a *condensed ring/group/...* is a sheaf of rings/groups/... on the site $*_{\text{proét}}$. Given a condensed set T, the *underlying set* of T is the set $T(*)$. Generally, if $\mathcal{C}$ is any category, the category $\mathrm{Cond}(\mathcal{C})$ of condensed objects of $\mathcal{C}$ is the category of $\mathcal{C}$-valued sheaves on the site $*_{\text{proét}}$.

Definition 4.1 presents set-theoretic problems. The main problem is that the category of profinite sets is large, for example, it is not a good idea to consider functors on all of it because the category of presheaves is not locally small. Indeed, by a result of P. Freyd and R. Street in [12] a category is essentially small if and only if both the category and the category of presheaves is locally small. We fix this issue with a smallness condition. We choose an *uncountable strong limit cardinal* κ. That is, κ has the property that whenever λ is a cardinal such that $\lambda < \kappa$, then also $2^\lambda < \kappa$. We say that a topological space S is κ*-small* whenever $|S| < \kappa$ and we denote by $*_{\kappa\text{-proét}}$ the site of κ-small profinite sets. κ-small profinite sets have many permanence properties, in particular they are closed under finite fiber products and coproducts which we will frequently make use of. Indeed, we have already argued that fiber products of profinite sets are again profinite. If $S_1 \to S$ and $S_2 \to S$ are continuous maps of κ-small profinite sets, their fiber product is a κ-small profinite set as well since we have $|S_1 \times_S S_2| \leq |S_1 \times S_2| < \kappa \otimes \kappa = \kappa$. It is straightforward to check that finite coproducts of κ-small profinite sets are profinite again and they are κ-small because the cardinal bounds work out as well since $|S_1 \coprod S_2| < \kappa \oplus \kappa = \kappa$. The notation for cardinal arithmetic is taken from section 3.6 in [22].

Definition 4.2. *A* κ-condensed set *is a sheaf of sets on the site* $*_{\kappa\text{-proét}}$*. Similarly, a* κ-condensed ring/group/... *is a sheaf of rings/groups/... on the site* $*_{\kappa\text{-proét}}$*. Given a κ-condensed set T, the* underlying set *of T is the set $T(*)$. Generally, if $\mathcal{C}$ is any category, the category* $\mathrm{Cond}_\kappa(\mathcal{C})$ *of κ-condensed objects of $\mathcal{C}$ is the category of $\mathcal{C}$-valued sheaves on the site* $*_{\kappa\text{-proét}}$.

The category of κ-small profinite sets is itself not small but has a small *skeleton*, i.e. a full subcategory whose inclusion into κ-small profinite sets is essentially surjective and such that any two isomorphic objects are already equal.

Proposition 4.3. *The category of κ-small profinite sets has a small skeleton.*

Proof. Fix a set X of cardinality κ. Let λ be a cardinal such that $\lambda < \kappa$ and let S be a profinite set of cardinality λ. We can embed S into X via an injection $\iota_S : S \to X$. We endow X with the topology for which the open sets are X and those of $\iota_S(S)$. This makes ι_S an open immersion and we call the resulting topological space X_S. Any two non-homeomorphic κ-small profinite sets S and S' give rise to distinct topologies on X, in fact, $X_S \cong X_{S'}$ if and only if $S \cong S'$. Assume first that $X_S \cong X_{S'}$. It is then enough to show that $\iota_S(S) \cong \iota_{S'}(S')$ and hence, we may assume that $S, S' \subseteq X$. Suppose that $f : X_S \to X_{S'}$ is a homeomorphism. Denote by τ_S and $\tau_{S'}$ the topologies of S respectively S' (i.e. the topologies of X_S and $X_{S'}$ without the open X). We claim that f restricts to a homeomorphism $S \to S'$. Clearly $S \in \tau_S$. Since f is a homeomorphism, we have that $f(S) \in \tau_{S'}$ because otherwise we would have $f(S) = X$, which is not possible since $|S| < |X|$. Hence, $f(S) \subseteq S'$. By symmetry of the argument, we have that $f^{-1}(S') \subseteq S$ and hence $S' \subseteq f(S)$. Altogether, they are equal and thus, $S \cong S'$. Let us now assume we have a homeomorphism $f : S \to S'$. We may again assume that $S, S' \subseteq X$ because $\iota_S(S) \cong \iota_{S'}(S')$. Since $|S| = |S'| < |X| = \kappa$ and because κ is infinite, we have that $|X \backslash S| = |X \backslash S'|$. Let $g : X \backslash S \to X \backslash S'$ be any bijection. Define $h : X \to X$ as follows, if $x \in S$, set $h(x) := f(x)$, if $x \in X \backslash S$, set $h(x) := g(x)$. The map h is then bijective by construction. We have $h^{-1}(X) = X$ open and if $U \in \tau_{S'}$, there is some $V \in \tau_S$ such that $f(V) = U$. This implies that $h^{-1}(U) = V$ is open. So h is continuous. h is also open, because if $U \in \tau_S$ we have that $h(U) = f(U) \in \tau_{S'}$. Clearly $h(X) = X$ is open. Hence, $h : X_S \to X_{S'}$ is a homeomorphism. Since there are at most 2^{2^κ} different topologies on X, we are done because we can choose one κ-small profinite set S from each isomorphism class and we end up with a small skeleton. $\square$

Of course, the skeleton $\mathcal{C}$ of the category of κ-small profinite sets forms a site with covers given by finite jointly surjective families of maps. In particular, the category of presheaves (e.g. of sets) on the skeleton is locally small. Indeed, if T and T' are presheaves of sets, then a morphism $\eta : T \to T'$ is just a family

$$(\eta_S)_{S \in \mathcal{C}} \in \prod_{S \in \mathcal{C}} \mathrm{Hom}(T(S), T'(S))$$

satisfying certain properties. Hence, the category of presheaves on $\mathcal{C}$ with values in $\mathcal{D}$ is locally small as long as $\mathcal{D}$ is locally small. The next lemma shows in particular that κ-condensed sets are locally small as well.

Lemma 4.4. *The categories of sheaves of sets/rings/groups... on the sites $*_{\kappa\text{-proét}}$ and $\mathcal{C}$ are equivalent by restrictions.*

Proof. Note first that it is enough to treat the case of sets. Let us fix an isomorphism $f_S : S \to S'$ for any κ-small profinite set, i.e. S' is the unique κ-small profinite set in the skeleton $\mathcal{C}$ such that $S \cong S'$. We start by establishing that restriction is fully faithful. Let T and T' be κ-condensed sets and assume that $\eta, \epsilon : T \to T'$ are morphisms of κ-condensed sets such that their restriction to $\mathcal{C}$ agrees. Let S be a κ-small set and $f_S : S \to S'$ be the fixed isomorphism. By naturality, we have two commutative diagrams, one for η, one for ϵ:

$$\begin{tikzcd} T(S') \arrow[r] \arrow[d, "T(f_S)"'] & T'(S') \arrow[d, "T'(f_S)"] \\ T(S) \arrow[r] & T'(S) \end{tikzcd}$$

Since $\eta(S')$ and $\epsilon(S')$ agree, the commutativity of the diagrams implies that $\eta(S)$ and $\epsilon(S)$ agree as well. Now assume that we are given a natural transformation η of the restrictions of T and T'. Let S be a κ-small profinite set. We define

$$\tilde{\eta}(S) := T'(f_S) \circ \eta(S') \circ T(f_S)^{-1}.$$

By definition, $\eta(S) = \tilde{\eta}(S)$ if S was already in the skeleton. Let S_1 and S_2 be κ-small profinite sets with their fixed isomorphisms $f_{S_1} : S_1 \to S'_1$ and $f_{S_2} : S_2 \to S'_2$ and let $g : S_1 \to S_2$ be a continuous map. We want to show that $T'(g) \circ \tilde{\eta}(S_2) = \tilde{\eta}(S_1) \circ T(g)$. By naturality of η for objects of the skeleton we have a commutative diagram:

$$\begin{tikzcd}
T(S'_2) \arrow[r, "\eta(S'_2)"] \arrow[d, "T(f_{S_2}\circ g\circ f_{S_1}^{-1})"'] & T'(S'_2) \arrow[d, "T'(f_{S_2}\circ g\circ f_{S_1}^{-1})"] \\
T(S'_1) \arrow[r, "\eta(S'_1)"] & T'(S'_1)
\end{tikzcd}$$

The commutativity of the diagram implies that $T'(g) \circ \tilde{\eta}(S_2) = \tilde{\eta}(S_1) \circ T(g)$.

Let us now show that restriction is essentially surjective. Let T be a sheaf on $\mathcal{C}$. Let S be a κ-small profinite set, $f_S : S \to S'$ be the fixed isomorphism and define $T'(S) := T(S')$. We claim that this defines a functor. Indeed, let S_1 and S_2 be κ-small profinite sets with their fixed isomorphisms $f_{S_1} : S_1 \to S'_1$ and $f_{S_2} : S_2 \to S'_2$ and let $g : S_1 \to S_2$ be a continuous map. We define $T'(g) := T(f_{S_2} \circ g \circ f_{S_1}^{-1})$ and see that T' is a functor. Let us now check that T' is actually a sheaf. For this let $\{S_i \xrightarrow{g_i} S\}_{i\in I}$ be a cover. We want to show that the diagram

$$T'(S) \to \prod_{i\in I} T'(S_i) \rightrightarrows \prod_{i,j\in I} T'(S_i \times_S S_j)$$

is an equalizer diagram. In fact, by definition of T', the diagram agrees with the equalizer diagram corresponding to the cover $\{S'_i \xrightarrow{f_S\circ g_i\circ f_{S_i}^{-1}} S'\}_{i\in I}$ in the skeleton. Here, $f_{S_i} : S_i \to S'_i$ denote the usual isomorphisms. This shows that T' is a sheaf. It remains to see that the restriction of T' is isomorphic to T. Let S' be a κ-small profinite set in the skeleton and $f_{S'} : S' \to S'$ be the isomorphism. We define

$$\eta(S') := T(f_{S'}) : T'(S') \to T(S'),$$

$\eta(S')$ is then clearly bijective. We show that η is a natural transformation. Let S'_1 and S'_2 be κ-small profinite sets in the skeleton with their isomorphisms $f_{S'_i} : S'_i \to S'_i$ and let $g : S'_1 \to S'_2$ be a continuous map. We then have that

$$\begin{aligned}
T(g) \circ \eta(S'_2) &= T(g) \circ T(f_{S'_2}) \\
&= T(f_{S'_2} \circ g) \\
&= T(f_{S'_2} \circ g \circ f_{S'_1}^{-1} \circ f_{S'_1}) \\
&= T(f_{S'_1}) \circ T(f_{S'_2} \circ g \circ f_{S'_1}^{-1}) \\
&= T(f_{S'_1}) \circ T'(g).
\end{aligned}$$

□

Remark 4.5. The same kind of arguments work for all sites that we will encounter. For us this means that we will not ever mention Lemma 4.4 or Proposition 4.3 again, i.e. for example we will still speak of the site $*_{\kappa\text{-proét}}$.

Proposition 4.6. *A presheaf T of sets/rings/groups/... on $*_{\kappa\text{-pro\'et}}$ is a sheaf of sets/rings/groups/... on $*_{\kappa\text{-pro\'et}}$ if and only if T satisfies the following three conditions:*

(i) $T(\emptyset) = \{*\}$

(ii) *For any two κ-small profinite sets S_1, S_2, the natural map*

$$T(S_1 \coprod S_2) \to T(S_1) \times T(S_2)$$

is a bijection.

(iii) *For any surjection $S' \twoheadrightarrow S$ of κ-small profinite sets with the fiber product $S' \times_S S'$ and its two projections p_1, p_2 to S', the map*

$$T(S) \to \{x \in T(S') : T(p_1)(x) = T(p_2)(x) \in T(S' \times_S S')\}$$

is a bijection.

Proof. Let us first note that it is enough to treat the case of sets. Assume T is a sheaf. Then it is straightforward to see that $T(\emptyset) = \{*\}$. Now let S_1 and S_2 be two κ-small profinite sets. Set $S := S_1 \coprod S_2$ and consider the finite jointly surjective cover $\{S_i \to S\}_{i \in \{1,2\}}$. As T is a sheaf, $T(S) \hookrightarrow T(S_1) \times T(S_2)$ is the equalizer of

$$T(S_1) \times T(S_2) \rightrightarrows T(S_1) \times \{*\} \times \{*\} \times T(S_2) = T(S_1) \times T(S_2)$$

Both arrows are the identity and therefore, $T(S) \to T(S_1) \times T(S_2)$ is a bijection.
If $S' \twoheadrightarrow S$ is a surjection of κ-small profinite sets, consider the cover given by $S' \twoheadrightarrow S$. That the map $T(S) \to \{x \in T(S') : T(p_1)(x) = T(p_2)(x) \in T(S' \times_S S')\}$ is bijective is just the statement that $T(S) \to T(S')$ is the equalizer of $T(S') \rightrightarrows T(S' \times_S S')$, which is true since T is a sheaf by assumption.
The converse is slightly more involved. Let $\{f_i : S_i \to S\}_{i \in I}$ be a finite jointly surjective family. Consider the following diagram:

$$\begin{tikzcd}
T(S) \arrow[r] & \prod_{i\in I} T(S_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j\in I} T(S_i \times_S S_j) \\
T(S) \arrow[r] \arrow[u, "id"] & T(\coprod_{i\in I} S_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[u, "(ii)"'] & T(\coprod_{i\in I} S_i \times_S \coprod_{j\in I} S_j)
\end{tikzcd}$$

The lower row is an equalizer diagram, where we have applied (iii) to the surjection $\coprod_{i\in I} S_i \twoheadrightarrow S$. The second vertical map comes from (ii) and is a bijection which makes the left square commutative. We want to show that the upper row is also an equalizer diagram. For this it is enough to construct a third vertical bijection such that the right squares are commutative. Let us briefly go through the construction. The universal property of the fiber product $\coprod_{i\in I} S_i \times_S \coprod_{j\in I} S_j$ induces maps

$$S_k \times_S S_l \to \coprod_{i\in I} S_i \times_S \coprod_{j\in I} S_j,$$

which in turn induce maps

$$T(\coprod_{i\in I} S_i \times_S \coprod_{j\in I} S_j) \to T(S_k \times_S S_l).$$

Finally, by the universal property of the product $\prod_{i,j\in I} T(S_i \times_S S_j)$, we obtain the desired map

$$T(\coprod_{i\in I} S_i \times_S \coprod_{j\in I} S_j) \to \prod_{i,j\in I} T(S_i \times_S S_j).$$

One can now easily verify that the right squares commute with the constructed map. All that is left to check is the bijectivity. For this notice that we have a natural bijection

$$\coprod_{i,j\in I} S_i \times_S S_j \to \coprod_{i\in I} S_i \times_S \coprod_{j\in I} S_j,$$

that induces a bijection

$$T(\coprod_{i\in I} S_i \times_S \coprod_{j\in I} S_j) \to T(\coprod_{i,j\in I} S_i \times_S S_j).$$

In particular, the following diagram commutes:

$$\begin{tikzcd} T(\coprod_{i\in I} S_i \times_S \coprod_{j\in I} S_j) \arrow[r] \arrow[d] & T(\coprod_{i,j\in I} S_i \times_S S_j) \arrow[dl, "(ii)"] \\ \prod_{i,j\in I} T(S_i \times_S S_j) & \end{tikzcd}$$

The claim follows as the arrow (ii) is bijective by assumption and thus T is a sheaf as desired. $\square$

Remark 4.7. Essentially the same arguments show that the analogous statement is true if we replace the site $*_{\kappa\text{-proét}}$ by the site of κ-small compact Hausdorff spaces.

Definition 4.8. Let X, Y be topological spaces. A surjective map $f : X \to Y$ is called a *quotient map* if $A \subseteq Y$ is closed in Y if and only if $f^{-1}(A) \subseteq X$ is closed in X.

Lemma 4.9. *Any continuous surjection $S' \twoheadrightarrow S$ where S' is compact and S is Hausdorff, is a quotient map. If this is the case, any composite $S' \twoheadrightarrow S \to T$ is continuous if and only if $S \to T$ is.*

Proof. Let $f : S' \to S$ be such a continuous surjection. Consider the equivalence relation $\sim$ on S' that is given by $S' \times_S S'$ where the fiber product is taken with respect to f. Since f is continuous, so is the induced bijection $\tilde{f} : S'/\sim \to S$ where $S'/\sim$ is endowed with the quotient topology. Let $\pi : S' \to S'/\sim$ denote the natural projection. If $A \subset S'/\sim$ is closed, so is $\pi^{-1}(A) \subset S'$ by definition. Closed subsets of compact spaces are compact itself. Hence, $\tilde{f}(A) = f(\pi^{-1}(A)) \subset S$ is compact by continuity of f. As S is a Hausdorff space, $\tilde{f}(A)$ is closed. Thus, $\tilde{f}$ is a homeomorphism which shows that f is a quotient map. $\square$

There is a natural way of passing from topological structures to condensed structures that captures the idea of condensed sets. Indeed, let us quickly explain the intuition. The obvious difference is that we look at functors instead of topological spaces. A topological space T is determined by the datum of its underlying topology whereas the datum of a κ-condensed set T is specified by the datum of the sets $T(S)$ for κ-small profinite sets S. The sets $T(S)$ should be *thought* of as continuous maps from S to

T. The functoriality of T then says that composition of continuous maps $S \to T$ and $X \to S$ gives rise to a continuous map $X \to T$. Condition ii) in Proposition 4.6 says that giving a continuous map $S_1 \coprod S_2 \to T$ is the same as giving two continuous maps $S_1 \to T$ and $S_2 \to T$. Condition iii) in Proposition 4.6 precisely says that given a continuous surjection $f : S' \to S$, a continuous map $S \to T$ is the same as a continuous map $S' \to T$ that is constant on the fibers of f (S' is the quotient of S by the equivalence relation $S' \times_S S'$). The following example makes this precise:

Example 4.10. Let T be any topological space. There is an associated κ-condensed set $\underline{T}$. For this we define the functor $\underline{T}$ as follows:

$$\begin{aligned}
&\underline{T} : \{\kappa\text{-small profinite sets}\}^{\mathrm{op}} \to (\mathrm{Set})\\
&S \mapsto \underline{T}(S) := C(S, T)\\
&(f : S' \to S) \mapsto f^* : C(S, T) \to C(S', T), \alpha \mapsto \alpha \circ f
\end{aligned}$$

Let us check the conditions from Proposition 4.6. First notice that $\underline{T}(\emptyset) = \{*\}$. To verify part ii) of the proposition let S_1 and S_2 be two κ-small profinite sets and denote by ι_j the natural inclusions $S_j \hookrightarrow S_1 \coprod S_2$. The inclusions induce a natural map

$$\underline{T}(S_1 \coprod S_2) \to \underline{T}(S_1) \times \underline{T}(S_2), \alpha \mapsto (\alpha \circ \iota_1, \alpha \circ \iota_2),$$

which is bijective by the universal property of the topological disjoint union $S_1 \coprod S_2$. Let us now verify part iii). For this let $f : S' \to S$ be a surjection of κ-small profinite sets. We need to check that the map

$$\underline{T}(S) \to \{\beta \in \underline{T}(S') : \beta \circ p_1 = \beta \circ p_2 \in \underline{T}(S' \times_S S')\}, \alpha \mapsto \alpha \circ f,$$

is bijective. This follows immediately from Lemma 4.9 and the universal property of S and f. Indeed, if $\beta \in \underline{T}(S')$ such that $\beta \circ p_1 = \beta \circ p_2$, then β factors uniquely as $\beta = \tilde{\beta} \circ f$. Hence, an inverse is given by $\beta \mapsto \tilde{\beta}$.
Of course, if T is a topological ring/group/..., then $\underline{T}$ is a condensed ring/group/... .

The example shows that the fully faithful Yoneda embedding from the category of κ-small profinite sets to the category of presheaves $\{\text{profinite sets}\}^{\mathrm{op}} \to (\mathrm{Set})$ actually gives rise to a fully faithful functor from the category of κ-small profinite sets to κ-condensed sets.

Corollary 4.11. *κ-small profinite sets embed fully faithfully into κ-condensed sets. The embedding is given by the Yoneda embedding $T \mapsto (S \mapsto \underline{T}(S) = C(S, T))$.*

Let us list some basic properties of the category of κ-condensed sets.

Proposition 4.12. *Let $\{T_i\}_{i \in I}$ be κ-condensed sets indexed by a set I. Then the product $\prod_{i \in I} T_i$ exists and is given by the assignment $S \mapsto \prod_{i \in I} T_i(S)$.*

Proof. Assume we are given a set I and for all $i \in I$ a κ-condensed set T_i. We define the presheaf $\prod_{i \in I} T_i := (S \mapsto \prod_{i \in I} T_i(S))$. Given $f : S' \to S$ the induced map $(\prod_{i \in I} T_i)(f) : (\prod_{i \in I} T_i)(S') \to (\prod_{i \in I} T_i)(S)$ is defined componentwise. We check the conditions from Proposition 4.6. It is clear that we have $(\prod_{i \in I} T_i)(\emptyset) = \{*\}$ because all T_i satisfy $T_i(\emptyset) = \{*\}$. Condition (ii) holds since all the T_i satisfy (ii) and thus

$$\begin{aligned}
(\prod_{i\in I} T_i)(S_1 \coprod S_2) &= \prod_{i\in I} T_i(S_1 \coprod S_2)\\
&= \prod_{i\in I}(T_i(S_1) \times T_i(S_2))\\
&= (\prod_{i\in I} T_i(S_1)) \times (\prod_{i\in I} T_i(S_2))\\
&= (\prod_{i\in I} T_i)(S_1) \times (\prod_{i\in I} T_i)(S_2).
\end{aligned}$$

To verify condition (iii) let $f : S' \to S$ be a surjection. As $(\prod_{i\in I} T_i)(f)$ is defined componentwise and because all the T_i satisfy condition (iii) it is clear that $(\prod_{i\in I} T_i)(f)$ gives us the desired bijection. It remains to check that $\prod_{i\in I} T_i$ has the required universal property. Let S be a κ-small profinite set. The projections $\pi_j(S) : \prod_{i\in I} T_i(S) \to T_j(S)$ define natural transformations $\pi_j : \prod_{i\in I} T_i \to T_j$. Indeed, they are clearly natural as $T_j(f) \circ \pi_j(S) = \pi_j(S') \circ (\prod_{i\in I} T_i)(f)$ for any map $f : S' \to S$. Let T be a κ-condensed set and assume that we are given morphisms of κ-condensed sets $g_j : T \to T_j$. If S is a κ-small profinite set, the maps $g_j(S) : T(S) \to T_j(S)$ induce a unique map $g(S) : T(S) \to \prod_{i\in I} T_i(S)$ such that $\pi_j(S) \circ g(S) = g_j(S)$. Hence, for all $j \in I$ and for any $f : S' \to S$ we have that $\pi_j(S') \circ (\prod_{i\in I} T_i)(f) \circ g(S) = \pi_j(S') \circ g(S') \circ T(f)$. But then $(\prod_{i\in I} T_i)(f) \circ g(S) = f(S') \circ T(f)$ which means that g is a morphism of κ-condensed sets. The uniqueness follows from the uniqueness of the $g(S)$.

□

Proposition 4.13. *The category of κ-condensed sets has fiber products. More precisely, given two morphisms of κ-condensed sets $\eta : T_1 \to T$ and $\epsilon : T_2 \to T$, the fiber product $T_1 \times_T T_2$ is given by the assignment $S \mapsto T_1(S) \times_{T(S)} T_2(S)$.*

Proof. Let us first show that the assignment $T_1 \times_T T_2 := (S \mapsto T_1(S) \times_{T(S)} T_2(S))$ defines a sheaf. If S is a κ-small profinite set, we denote by $p_1(S)$ and $p_2(S)$ the projections of the fiber product $T_1(S) \times_{T(S)} T_2(S)$ of $\eta(S)$ and $\epsilon(S)$. Given $f : S' \to S$ we obtain the following commutative diagram:

$$\begin{tikzcd}
T_1(S) \times_{T(S)} T_2(S) \arrow[dr, dotted, "\exists!(T_1\times_T T_2)(f)"] \arrow[drr, bend left, "T_2(f)\circ p_2(S)"] \arrow[ddr, bend right, "T_1(f)\circ p_1(S)"'] & & \\
& T_1(S') \times_{T(S')} T_2(S') \arrow[r, "p_2(S')"] \arrow[d, "p_1(S')"'] & T_2(S') \arrow[d, "\epsilon(S')"] \\
& T_1(S') \arrow[r, "\eta(S')"] & T(S')
\end{tikzcd}$$

The uniqueness of the dotted arrows implies that $T_1 \times_T T_2$ is a presheaf. Let us now check the conditions from 4.6. The first condition is satisfied as $(T_1 \times_T T_2)(\emptyset) = \{*\}$. Let us now check condition (ii). We have that

$$(T_1 \times_T T_2)(S_1 \coprod S_2) = (T_1(S_1) \times T_1(S_2)) \times_{T(S_1)\times T(S_2)} (T_2(S_1) \times T_2(S_2)).$$

We claim that the latter is in bijection with $(T_1 \times_T T_2)(S_1) \times (T_1 \times_T T_2)(S_2)$. Set $X := T_1(S_1) \times T_1(S_2)$, $Y := T_2(S_1) \times T_2(S_2)$ and $Z := T(S_1) \times T(S_2)$ and consider the following commutative diagram.

$$\begin{tikzcd}
(T_1 \times_T T_2)(S_1) \times (T_1 \times_T T_2)(S_2) \arrow[dr, dotted, "\exists ! u"] \arrow[drr, bend left, "{(p_2(S_1),p_2(S_2))}"] \arrow[ddr, bend right, "{(p_1(S_1),p_1(S_2))}"'] & & \\
& X \times_Z Y \arrow[r, "q_2"] \arrow[d, "q_1"'] & Y \arrow[d, "{(\epsilon(S_1),\epsilon(S_2))}"] \\
& X \arrow[r, "{(\eta(S_1),\eta(S_2))}"] & Z
\end{tikzcd}$$

One checks directly that u is the desired bijection. Now let us assume that we are given a surjection $f : S' \to S$. We have to check condition (iii). The map

$$(T_1 \times_T T_2)(f) : T_1(S) \times_{T(S)} T_2(S) \to T_1(S') \times_{T(S')} T_2(S')$$

is clearly injective as $T_1(f)$ and $T_2(f)$ are because T_1 and T_2 satisfy condition (iii). Denote the two projections $S' \times_S S' \to S'$ by q_1 and q_2. We want to check that the image is given by the set

$$\{x \in (T_1 \times_T T_2)(S') : (T_1 \times_T T_2)(q_1)(x) = (T_1 \times_T T_2)(q_2)(x) \in (T_1 \times_T T_2)(S' \times_S S')\}.$$

Let $x = (x_1, x_2) \in (T_1 \times_T T_2)(S')$ such that $(T_1 \times_T T_2)(q_1)(x) = (T_1 \times_T T_2)(q_2)(x)$. This means that

$$(T_1(q_1)(x_1), T_2(q_1)(x_2)) = (T_1(q_2)(x_1), T_2(q_2)(x_2)).$$

Hence, again by condition (iii) for T_j there are $y_j \in T_j(S)$ such that $T_j(f)(y_j) = x_j$. We know that $\eta(S')(x_1) = \epsilon(S')(x_2)$ and we want to show that $\eta(S)(y_1) = \epsilon(S)(y_2)$ because then $(y_1, y_2) \in T_1(S) \times_{T(S)} T_2(S)$ is a preimage for (x_1, x_2) under $(T_1 \times_T T_2)(f)$. As T also satisfies condition (iii) we know that $T(f)$ is injective. We have that

$$\begin{aligned}
(T(f) \circ \eta(S))(y_1) &= (\eta(S') \circ T_1(f))(y_1) \\
&= \eta(S')(x_1) \\
&= \epsilon(S')(x_2) \\
&= (\epsilon(S') \circ T_2(f))(y_2) \\
&= (T(f) \circ \epsilon(S))(y_2),
\end{aligned}$$

which implies that $\eta(S)(y_1) = \epsilon(S)(y_2)$. All that is left to do is to verify that $T_1 \times_T T_2$ has the required universal property. The projections $p_j : T_1 \times_T T_2 \to T_j$ are defined in the obvious way and are clearly natural by construction of $(T_1 \times_T T_2)(f)$ for a map $f : S' \to S$. Assume we are given morphisms of κ-condensed sets $p : X \to T_1$ and $q : X \to T_2$ such that $\eta \circ p = \epsilon \circ q$. If S is a κ-small profinite set, then it is clear that we have a commutative diagram:

$$
\begin{tikzcd}
X(S) \arrow[dr, dotted, "\exists ! u(S)"] \arrow[drr, bend left, "q(S)"] \arrow[ddr, bend right, "p(S)"'] & & \\
& T_1(S) \times_{T(S)} T_2(S) \arrow[r, "p_2(S)"] \arrow[d, "p_1(S)"'] & T_2(S) \arrow[d, "\epsilon(S)"] \\
& T_1(S) \arrow[r, "\eta(S)"'] & T(S)
\end{tikzcd}
$$

This defines the desired morphism $u : X \to T_1 \times_T T_2$. Indeed, the uniqueness is clear, we just need to argue that u is natural. Given $f : S' \to S$ we need to show that $u(S') \circ X(f) = (T_1 \times_T T_2)(f) \circ u(S)$. Consider the following commutative diagram:

$$
\begin{tikzcd}
X(S) \arrow[dr, dotted, "\exists !"] \arrow[drr, bend left, "T_2(f)\circ q(S)"] \arrow[ddr, bend right, "T_1(f)\circ p(S)"'] & & \\
& T_1(S') \times_{T(S')} T_2(S') \arrow[r, "p_2(S')"] \arrow[d, "p_1(S')"'] & T_2(S') \arrow[d, "\epsilon(S')"] \\
& T_1(S') \arrow[r, "\eta(S')"'] & T(S')
\end{tikzcd}
$$

As both $u(S') \circ X(f)$ and $(T_1 \times_T T_2)(f) \circ u(S)$ make the diagram commutative, they must be equal. This finishes the proof. $\square$

Example 4.14. Let $T_1 \to T$ and $T_2 \to T$ be continuous maps. If S is a κ-small profinite set then we have that

$$
\begin{aligned}
(\underline{T_1 \times_T T_2})(S) &= C(S, T_1 \times_T T_2) \\
&= C(S, T_1) \times_{C(S,T)} C(S, T_2) \\
&= \underline{T_1}(S) \times_{\underline{T}(S)} \underline{T_2}(S),
\end{aligned}
$$

naturally in S. Hence $\underline{T_1 \times_T T_2} = \underline{T_1} \times_{\underline{T}} \underline{T_2}$ as κ-condensed sets. Moreover, if $S' \xrightarrow{f} S$ is a surjection of κ-small profinite sets, then S is the quotient of S' by the equivalence relation $S' \times_S S'$. In other words

$$
S' \times_S S' \underset{p_2}{\overset{p_1}{\rightrightarrows}} S' \xrightarrow{f} S
$$

is a coequalizer diagram. Condition (iii) in Proposition 4.6 enforces that the same is true on the level of κ-condensed sets, i.e.

$$
\underline{S'} \times_{\underline{S}} \underline{S'} \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \underline{S'} \xrightarrow{f^*} \underline{S}
$$

is a coequalizer diagram. Indeed, let T be a κ-condensed set and $g : \underline{S'} \to T$ be a morphism of κ-condensed sets such that $g \circ p_1^* = g \circ p_2^*$. Condition (iii) in Proposition 4.6 precisely says that there is a unique morphism $u : \underline{S} \to T$ such that $g = u \circ f^*$ (e.g. $T(S) = \mathrm{Hom}(\underline{S}, T)$). In particular, f^* is an epimorphism.

Let $\{T_i\}_{i\in I}$ be κ-condensed sets indexed by a set I. Consider the assignment

$$(\coprod_{i\in I} T_i)_{\mathrm{pre}} := (S \mapsto \coprod_{i\in I} T_i(S)).$$

Let $f : S' \to S$ and denote by $\phi_j^{\mathrm{pre}}(S')$ the natural inclusions $T_j(S') \to \coprod_{i\in I} T_i(S')$. The compositions

$$T_j(S) \xrightarrow{T_j(f)} T_j(S') \xrightarrow{\phi_j^{\mathrm{pre}}(S')} \coprod_{i\in I} T_i(S')$$

give rise to a natural map

$$(\coprod_{i\in I} T_i)_{\mathrm{pre}}(f) : (\coprod_{i\in I} T_i)_{\mathrm{pre}}(S) \to (\coprod_{i\in I} T_i)_{\mathrm{pre}}(S').$$

This defines a presheaf $(\coprod_{i\in I} T_i)_{\mathrm{pre}}$ of sets, which is not a sheaf in general. Indeed, for two κ-condensed sets T_1 and T_2 we have that

$$(T_1 \coprod T_2)_{\mathrm{pre}}(\emptyset) = T_1(\emptyset) \coprod T_2(\emptyset) = \{*\} \coprod \{*\} \neq \{*\}.$$

Sheafification helps.

Proposition 4.15. *Let $\{T_i\}_{i\in I}$ be κ-condensed sets indexed by a set I. Then the coproduct $\coprod_{i\in I} T_i$ in the category of κ-condensed sets exists and is given by the sheafification of $(\coprod_{i\in I} T_i)_{pre} := (S \mapsto \coprod_{i\in I} T_i(S))$.*

Proof. Assume we are given a set I and for all $i \in I$ a κ-condensed set T_i. We have already defined the presheaf $(\coprod_{i\in I} T_i)_{\mathrm{pre}}$. Let S be a κ-small profinite set. The canonical injections $\phi_j^{\mathrm{pre}}(S) : T_j(S) \to (\coprod_{i\in I} T_i)_{\mathrm{pre}}(S)$ give rise to natural transformations $\phi_j^{\mathrm{pre}} : T_j \to (\coprod_{i\in I} T_i)_{\mathrm{pre}}$ as the naturality is already built into the definition of $(\coprod_{i\in I} T_i)_{\mathrm{pre}}(f)$ for $f : S' \to S$. Let $\coprod_{i\in I} T_i$ be the sheafification of $(\coprod_{i\in I} T_i)_{\mathrm{pre}}$ and let $\iota : (\coprod_{i\in I} T_i)_{\mathrm{pre}} \to \coprod_{i\in I} T_i$ be the canonical natural transformation. These natural transformations give rise to natural transformations ϕ_j defined as the composition

$$T_j \xrightarrow{\phi_j^{\mathrm{pre}}} (\coprod_{i\in I} T_i)_{\mathrm{pre}} \xrightarrow{\iota} \coprod_{i\in I} T_i.$$

Let us now verify the universal property of the coproduct. Assume that we are given morphisms of κ-condensed sets $\eta_j : T_j \to T$. If S is a κ-small profinite set, the maps $\eta_j(S) : T_j(S) \to T(S)$ give rise to unique maps $\eta^{\mathrm{pre}}(S) : \coprod_{i\in I} T_i(S) \to T(S)$ such that $\eta^{\mathrm{pre}}(S) \circ \phi_j^{\mathrm{pre}}(S) = \eta_j(S)$. These maps define a unique natural transformation

$$\eta^{\mathrm{pre}} : (\coprod_{i\in I} T_i)_{\mathrm{pre}} \to T$$

such that $\eta^{\mathrm{pre}} \circ \phi_j^{\mathrm{pre}} = \eta_j$. Indeed, the uniqueness is clear and the naturality is true because we have for all $j \in I$ the equality

$$T(f) \circ \eta^{\mathrm{pre}}(S) \circ \phi_j^{\mathrm{pre}}(S) = \eta^{\mathrm{pre}}(S') \circ (\coprod_{i\in I} T_i)_{\mathrm{pre}}(f) \circ \phi_j^{\mathrm{pre}}(S),$$

and hence $T(f) \circ \eta^{\mathrm{pre}}(S) = \eta^{\mathrm{pre}}(S') \circ (\coprod_{i \in I} T_i)_{\mathrm{pre}}(f)$. By the universal property of sheafification there is a unique morphism of κ-condensed sets $\eta : \coprod_{i \in I} T_i \to T$ such that $\eta^{\mathrm{pre}} = \eta \circ \iota$. We conclude that

$$\eta \circ \phi_j = \eta \circ \iota \circ \phi_j^{\mathrm{pre}} = \eta^{\mathrm{pre}} \circ \phi_j^{\mathrm{pre}} = \eta_j.$$

□

Example 4.16. Let $\{S_i\}_{i \in I}$ be a finite family of κ-small profinite sets. Condition (ii) in Proposition 4.6 implies that there is a natural isomorphism

$$\coprod_{i \in I} \underline{S_i} \cong \underline{\coprod_{i \in I} S_i}.$$

Indeed, if $S := \coprod_{i \in I} S_i$, then by the Yoneda Lemma and because of the previous proposition we have natural isomorphisms for any κ-condensed set T:

$$\begin{aligned} \mathrm{Hom}(\underline{S}, T) &= T(S) \\ &= T(\coprod_{i \in I} S_i) \\ &= \prod_{i \in I} T(S_i) \\ &= \prod_{i \in I} \mathrm{Hom}(\underline{S_i}, T) \\ &= \mathrm{Hom}(\coprod_{i \in I} \underline{S_i}, T). \end{aligned}$$

5 Stone-Čech compactifications and extremally disconnected sets

In this section we are going to introduce *extremally disconnected sets*. We will only be interested in those that are also compact Hausdorff spaces. As we will see, they are closely related to certain Stone-Čech compactifications and have very useful properties. The functorial procedure of producing extremally disconnected compact Hausdorff spaces will be frequently used throughout the rest of this work. Most notably, given a compact Hausdorff space S we can always find an extremally disconnected compact Hausdorff space S' and a continuous surjection $\pi_S : S' \to S$. For example, this implies that a morphism of κ-condensed sets is already determined by its components on κ-small extremally disconnected compact Hausdorff spaces.

Recall that the Stone-Čech compactification (cf. Section 5.25 in [22]) of a topological space X is a continuous map $i_X : X \to \beta X$ from X to a compact Hausdorff space βX which satisfies the following universal property. If K is a compact Hausdorff space and $f : X \to K$ a continuous map, then there exists a unique continuous map $\beta f : \beta X \to K$ such that the following diagram is commutative:

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[dr, "f"'] & \beta X \arrow[d, "\beta f"] \\ & K \end{tikzcd}$$

As a direct consequence of this universal property, the Stone-Čech compactification is functorial and left-adjoint to the inclusion functor from compact Hausdorff spaces into all topological spaces. In particular, the Stone-Čech compactification commutes with all colimits. By construction, the image of i_X is dense in βX. Moreover, i_X is a homeomorphism onto an open subspace if and only if X is a locally compact Hausdorff space (cf. Lemma 5.25.2 in [22]).

5.1 Extremally disconnected sets

Definition 5.1. A topological space X is called *extremally disconnected* if the closure $\overline{U}$ of every open set U is open.

Lemma 5.2. *Let X be extremally disconnected. If U and V are open and disjoint subsets of X, then $\overline{U}$ and $\overline{V}$ are also disjoint. If X is also Hausdorff, then X is totally disconnected.*

Proof. Since U and V are disjoint, $U \subset X \backslash V$ and hence $\overline{U} \subset X \backslash V$. This means that the opens $\overline{U}$ and V are also disjoint. The same argument shows that $\overline{U}$ and $\overline{V}$ are disjoint.
Now assume X is also Hausdorff. Let $x \in X$ and let C be the unique connected component that contains x. Let $y \in X \backslash \{x\}$. We claim that $y \notin C$ which would imply that $C = \{x\}$. Since X is Hausdorff, we can find disjoint open neighborhoods U and V of x and y, respectively. By the first part of the lemma we know that $y \notin \overline{U}$, hence y is not an element of $\overline{U} \cap C$. As $\overline{U}$ is open by assumption, $\overline{U} \cap C$ is both open and closed in C and contains x. Hence, $\overline{U} \cap C = C$ and thus, $C = \{x\}$ as claimed. $\square$

For compact Hausdorff spaces Gleason gave a different description of extremally disconnected spaces (cf. Theorem 2.5 in [14]). They are exactly the projective objects in the category of compact Hausdorff spaces.

Proposition 5.3 (Gleason)**.** *A compact Hausdorff space S is extremally disconnected if and only if any surjection $S' \twoheadrightarrow S$ from a compact Hausdorff space splits, i.e. there is a continuous map $S \to S'$ such that the composition $S \to S' \twoheadrightarrow S$ is the identity.* □

Remark 5.4. It is easy to check that the condition in Proposition 5.3 is equivalent to the usual notion of projectivity. Indeed, suppose that S is projective and let $S' \xrightarrow{f} S$ be any surjection from a compact Hausdorff space. By projectivity, f factors through the identity id_S, i.e. there is some $S \xrightarrow{h} S'$ such that $f \circ h = id_S$. Conversely, let $S' \xrightarrow{g} T$ and $S \xrightarrow{f} T$ and suppose that g is surjective. Consider the fiber product $S \times_T S'$ with its projections p_1 and p_2. The projection $S \times_T S' \xrightarrow{p_1} S$ is surjective and hence admits a section $S \xrightarrow{h} S \times_T S'$ such that $p_1 \circ h = id_S$. Then $\tilde{f} := p_2 \circ h$ satisfies $f = g \circ \tilde{f}$.

Remark 5.5. For (compact) Hausdorff spaces, the condition of being extremally disconnected is stronger than the notion of being totally disconnected. Indeed, by the previous lemma, extremally disconnected Hausdorff spaces are automatically profinite, but the converse is not true. Let $X := \mathbb{N} \cup \{\infty\}$, we claim that X is a profinite set, hence totally disconnected, but not extremally disconnected. Recall that the topology on X is given by arbitrary subsets of $\mathbb{N}$ and additionally sets of the form $(\mathbb{N}\backslash F) \cup \{\infty\}$, where F is a finite subset of $\mathbb{N}$. If $X = \bigcup_j U_j$ is the union of open subsets, then there has to be one U_i of the form $(\mathbb{N}\backslash F) \cup \{\infty\}$. Since the complement of U_i is finite, finitely many of the remaining U_j and U_i must cover X already. Let $x, y \in X$ be distinct points. If both x and y lie in $\mathbb{N}$, the opens $\{x\}$ and $\{y\}$ separate x and y. Say $x = \infty$ then $\{y\}$ and $(\mathbb{N}\backslash\{y\}) \cup \{\infty\}$ separate x and y. Let $x \in \mathbb{N}$ and let C be the unique connected component containing x. As $\{x\}$ is both open and closed it follows that $C = \{x\}$. Finally, if C is the unique connected component containing ∞, then by the previous argument necessarily $C = \{\infty\}$. In particular, all connected components are one-point sets. Altogether, this shows that X is a profinite set. Let us now argue that X is not extremally disconnected, indeed, consider the open subset $2\mathbb{N}$. As $2\mathbb{N}$ is not closed in X, but $2\mathbb{N} \cup \{\infty\}$ is closed in X, it follows that $\overline{2\mathbb{N}} = 2\mathbb{N} \cup \{\infty\}$. The complement of $\overline{2\mathbb{N}}$ are thus the odd numbers, but those are not closed in X. Hence, $\overline{2\mathbb{N}}$ is not open.

From now on whenever we say extremally disconnected set we mean a compact Hausdorff space that is additionally extremally disconnected.

Proposition 5.6. *Every compact Hausdorff space S admits a surjection from the extremally disconnected set $S' := \beta S_{disc}$ where S_{disc} is the discrete space with underlying set S. Moreover, if S is κ-small compact Hausdorff space, then S' is a κ-small extremally disconnected set.*

Proof. Assume S is a compact Hausdorff space. If we consider S as the discrete space S_{disc}, we can apply the Stone-Čech compactification to S_{disc} and we obtain a compact Hausdorff space βS_{disc} together with a continuous map $i_{S_{\text{disc}}} : S_{\text{disc}} \hookrightarrow \beta S_{\text{disc}}$. Notice that the identity $S_{\text{disc}} \xrightarrow{id_S} S$ is continuous and hence, by the universal property of

$i_{S_{\text{disc}}} : S_{\text{disc}} \to \beta S_{\text{disc}}$, induces a surjection $\beta S_{\text{disc}} \to S$. Let us now show that βS_{disc} is extremally disconnected. For this we verify the condition in Proposition 5.3. If $S' \twoheadrightarrow \beta S_{\text{disc}}$ is any surjection, we may lift the map $S_{\text{disc}} \to \beta S_{\text{disc}}$ continuously to S' such that the diagram

$$\begin{tikzcd} & S' \arrow[d] \\ S_{\text{disc}} \arrow[r] \arrow[ur, dotted] & \beta S_{\text{disc}} \end{tikzcd}$$

is commutative. By the universal property of the Stone-Čech compactification, we obtain a unique continuous map $\beta S_{\text{disc}} \to S'$ such that $\beta S_{\text{disc}} \to S' \twoheadrightarrow \beta S_{\text{disc}}$ is the identity on the dense subset S_{disc} and hence on all of βS_{disc}. For the final statement let S be a κ-small compact Hausdorff space, i.e. $|S| < \kappa$. By construction of $S' := \beta S_{\text{disc}}$, we have that $|S'| \leq 2^{2^{|S|}}$ (cf. Lemma 5.25.1 in [22]). As κ is an uncountable strong limit cardinal, by definition $|S'| < \kappa$. □

Here are some results that show how useful extremally disconnected sets and their formal properties are. In some sense they behave like stalks do for sheaves on a topological space.

Proposition 5.7. *Let $f : T \to T'$ be a morphism of κ-condensed sets then:*

(i) *f is injective if and only if for all κ-small extremally disconnected sets S the map of sets $f(S) : T(S) \to T'(S)$ is injective.*

(ii) *f is surjective if and only if for all κ-small extremally disconnected sets S the map of sets $f(S) : T(S) \to T'(S)$ is surjective.*

(ii) *f is an isomorphism if and only if for all κ-small extremally disconnected sets S the map of sets $f(S) : T(S) \to T'(S)$ is bijective.*

Proof. For (i) it is enough to show that if the map of sets $f(S') : T(S') \to T'(S')$ is injective for all κ-small extremally disconnected sets S', then this assertion is also true for all κ-small profinite sets. Let S be a κ-small profinite set. Let $\pi_S : S' \to S$ be the surjection from the extremally disconnected set $S' = \beta S_{\text{disc}}$. Consider the following commutative diagram:

$$\begin{tikzcd} T(S) \arrow[r, "f(S)"] \arrow[d, "T(\pi_S)"'] & T'(S) \arrow[d, "T'(\pi_S)"] \\ T(S') \arrow[r, "f(S')"] & T'(S') \end{tikzcd}$$

The maps $T(\pi_S)$ and $T'(\pi_S)$ are injective by condition (iii) in Proposition 4.6. The map $f(S')$ is injective by assumption. By commutativity of the diagram, this implies that $f(S)$ is injective. Let us now show statement (ii). We assume first that the maps $f(S') : T(S') \to T'(S')$ are surjective for κ-small extremally disconnected sets S'. Let S be a κ-small profinite set and let $s \in T'(S)$. Consider the surjection $\pi_S : S' \to S$ from the extremally disconnected set $S' = \beta S_{\text{disc}}$. Now the map $f(S') : T(S') \to T'(S')$ is surjective and hence has $T'(\pi_S)(s)$ in its image. This shows that f is surjective in the sense of Definition 3.8. Let us now assume that f is surjective and let S be a κ-small extremally disconnected set. We need to show that $f(S) : T(S) \to T'(S)$ is surjective. Let $s \in T'(S)$, because f is surjective there is a finite jointly surjective cover $\{S_i \xrightarrow{g_i} S\}_{i\in I}$ such that $T'(g_i)(s)$ is in the image of $f(S_i) : T(S_i) \to T'(S_i)$ for all $i \in I$, i.e. there are $x_i \in T(S_i)$ such that we have

$$f(S_i)(x_i) = T'(g_i)(s).$$

Because $\{S_i \xrightarrow{g_i} S\}_{i\in I}$ is a cover, the induced map $\coprod_{i\in I} S_i \xrightarrow{g} S$ is surjective and satisfies $g \circ \phi_{S_i} = g_i$ where $\phi_{S_i} : S_i \to \coprod_{i\in I} S_i$ denote the natural inclusions. Since S is extremally disconnected, there exists $S \xrightarrow{h} \coprod_{i\in I} S_i$ such that $g \circ h = id_S$ and hence $id_{T(S)} = T(h) \circ T(g)$. Since T is a sheaf and I is finite, we have a natural bijection

$$T(\coprod_{i\in I} S_i) \to \prod_{i\in I} T(S_i), y \mapsto (T(\phi_{S_i})(y))_{i\in I}.$$

By the surjectivity there is $x \in T(\coprod_{i\in I} S_i)$ such that $(x_i)_{i\in I} = (T(\phi_{S_i})(x))_{i\in I}$. Likewise, we have a natural bijection

$$T'(\coprod_{i\in I} S_i) \to \prod_{i\in I} T'(S_i), y \mapsto (T'(\phi_{S_i})(y))_{i\in I}.$$

In particular, we have that

$$\begin{aligned}(T'(\phi_{S_i}))_{i\in I}(T'(g)(s)) &= (T'(\phi_{S_i})(T'(g)(s)))_{i\in I}\\ &= (T'(g \circ \phi_{S_i})(s))_{i\in I}\\ &= (T'(g_i)(s))_{i\in I}.\end{aligned}$$

We claim that $f(S)(T(h)(x)) = s$. Indeed, consider the following commutative diagram:

$$\begin{tikzcd}
T(S) \arrow[r, "f(S)"] & T'(S) \\
T(\coprod_{i\in I} S_i) \arrow[u, "T(h)"] \arrow[r, "f(\coprod_{i\in I} S_i)"] & T'((\coprod_{i\in I} S_i) \arrow[u, "T'(h)"'] \\
\prod_{i\in I} T(S_i) \arrow[u, "(T(\phi_{S_i}))_{i\in I}"] \arrow[r, "(f(S_i))_{i\in I}"] & \prod_{i\in I} T'(S_i) \arrow[u, "(T'(\phi_{S_i}))_{i\in I}"']
\end{tikzcd}$$

By the commutativity we have that

$$\begin{aligned}f(S)(T(h)(x)) &= (f(S) \circ T(h) \circ (T(\phi_{S_i}))^{-1}_{i\in I})((x_i)_{i\in I})\\ &= (T'(h) \circ (T'(\phi_{S_i}))^{-1}_{i\in I} \circ (f(S_i))_{i\in I})((x_i)_{i\in I})\\ &= (T'(h) \circ (T'(\phi_{S_i}))^{-1}_{i\in I})((f(S_i)(x_i))_{i\in I})\\ &= (T'(h) \circ (T'(\phi_{S_i}))^{-1}_{i\in I})((T'(g_i)(s))_{i\in I})\\ &= (T'(h) \circ T'(g))(s)\\ &= T'(g \circ h)(s)\\ &= T'(id_S)(s)\\ &= s.\end{aligned}$$

This finishes the proof because (iii) is clear. $\square$

Here is another result in the same direction. Two morphisms of κ-condensed sets are already equal if they agree on κ-small extremally disconnected sets.

Proposition 5.8. *Let $\eta, \epsilon : T \to T'$ be two morphisms of κ-condensed sets such that they agree on all κ-small extremally disconnected sets. Then $\eta = \epsilon$. Moreover, if we restrict T and T' to κ-small extremally disconnected sets and if $\tilde{\eta}$ is a natural transformation of those restrictions, then there is a necessarily unique morphism of κ-condensed sets $\eta : T \to T'$ such that $\tilde{\eta}(S) = \eta(S)$ for all κ-small extremally disconnected sets S.*

Proof. Let $\pi_S : S' \to S$ be the surjection from Proposition 5.6 with S' κ-small extremally disconnected. By naturality, we have two commutative diagrams:

$$\begin{tikzcd}
T(S) \arrow[r, shift left, "\eta(S)"] \arrow[r, shift right, "\epsilon(S)"'] \arrow[d, "T(\pi_S)"'] & T'(S) \arrow[d, "T'(\pi_S)"] \\
T(S') \arrow[r, shift left, "\eta(S')"] \arrow[r, shift right, "\epsilon(S')"'] & T'(S')
\end{tikzcd}$$

Using Proposition 4.6 we may replace the bottom rows by the two arrows

$$\begin{tikzcd}
\{x \in T(S') : T(p_1)(x) = T(p_2)(x) \in T(S' \times_S S')\} \arrow[d] \\
\{x \in T'(S') : T'(p_1)(x) = T'(p_2)(x) \in T'(S' \times_S S')\},
\end{tikzcd}$$

as the commutative squares show that the components $\eta(S')$ and $\epsilon(S')$ map the set $\{x \in T(S') : T(p_1)(x) = T(p_2)(x)\}$ to the set $\{x \in T'(S') : T'(p_1)(x) = T'(p_2)(x)\}$. By assumption $\eta(S') = \epsilon(S')$ which implies $\eta(S) = \epsilon(S)$ since the vertical arrows in the new commutative squares are bijections.

Next, assume we are given a natural transformation $\tilde{\eta}$ of the restrictions of T and T'. We need to define $\eta(S) : T(S) \to T(S)$. Consider the diagram:

$$\begin{tikzcd}
T(S) \arrow[d, "T(\pi_S)"'] & T'(S) \arrow[d, "T'(\pi_S)"] \\
T(S') \arrow[r, "\tilde{\eta}(S')"] \arrow[d, shift right, "T(p_1)"'] \arrow[d, shift left, "T(p_2)"] & T'(S') \arrow[d, shift right, "T'(p_1)"'] \arrow[d, shift left, "T'(p_2)"] \\
T(S' \times_S S') \arrow[r, "\tilde{\eta}(S' \times_S S')"] & T'(S' \times_S S')
\end{tikzcd}$$

As before, the commutative squares at the bottom show us that $\tilde{\eta}(S')$ maps the set $\{x \in T(S') : T(p_1)(x) = T(p_2)(x)\}$ to the set $\{x \in T'(S') : T'(p_1)(x) = T'(p_2)(x)\}$. This allows us to define $\eta(S)$ as the composition

$$\begin{tikzcd}
T(S) \arrow[r, "\cong"] & \{x \in T(S') : T(p_1)(x) = T(p_2)(x)\} \arrow[d] & \\
& \{x \in T'(S') : T'(p_1)(x) = T'(p_2)(x)\} \arrow[r, "\cong"] & T'(S).
\end{tikzcd}$$

It is then clear that η and $\tilde{\eta}$ agree on κ-small extremally disconnected sets because if S is a κ-small extremally disconnected set, then the upper part of the above diagram can be completed to a commutative square using $\eta(S)$. Now let X and Y be κ-small profinite sets and let $f : Y \to X$ be any continuous map. We need to show that the following diagram is commutative:

$$\begin{tikzcd} T(X) \arrow[r, "\eta(X)"] \arrow[d, "T(f)"'] & T'(X) \arrow[d, "T'(f)"] \\ T(Y) \arrow[r, "\eta(Y)"] & T'(Y) \end{tikzcd}$$

For this let $\pi_X : X' \to X$ and $\pi_Y : Y' \to Y$ as before be the surjections from the κ-small extremally disconnected sets $X' := \beta X_{disc}$ and $Y' := \beta Y_{disc}$. Consider the map $f_{\text{disc}} : Y_{disc} \to X_{disc}$, given by f. The map f_{disc} induces a map $f' := \beta f_{\text{disc}} : Y' \to X'$ such that $f' \circ i_{Y_{disc}} = i_{X_{disc}} \circ f_{\text{disc}}$. We then have that $\pi_X \circ f' = f \circ \pi_Y$ because this is true on the dense subspace Y_{disc}. Consider the following diagram:

$$\begin{tikzcd} \{x \in T(X') : T(p_1)(x) = T(p_2)(x)\} \arrow[r, "\eta(X')"] \arrow[dddd, bend right, "T(f')"'] & \{x \in T'(X') : T'(p_1)(x) = T'(p_2)(x)\} \arrow[dddd, bend left, "T'(f')"] \\ T(X) \arrow[u, "T(\pi_X)", "\cong"'] \arrow[r, "\eta(X)"] \arrow[d, "T(f)"'] & T'(X) \arrow[u, "T'(\pi_X)"', "\cong"] \arrow[d, "T'(f)"] \\ T(Y) \arrow[r, "\eta(Y)"] \arrow[d, "T(\pi_Y)"', "\cong"] & T'(Y) \arrow[d, "T'(\pi_Y)", "\cong"'] \\ \{x \in T(Y') : T(p_1)(x) = T(p_2)(x)\} \arrow[r, "\eta(Y')"] & \{x \in T'(Y') : T'(p_1)(x) = T'(p_2)(x)\} \end{tikzcd}$$

The upper and lower square commute by definition of $\eta(X)$ respectively $\eta(Y)$. The outer parts of the diagram commute because $\pi_X \circ \tilde{f} = f \circ \pi_Y$. Using that X' and Y' are κ-small extremally disconnected sets and hence $T'(\tilde{f}) \circ \eta(X') = \eta(Y') \circ T(\tilde{f})$ this implies that the square in question commutes. □

5.2 Different sites for the definition of condensed sets

In this subsection we will show that the categories of sheaves on the sites $*_{\kappa\text{-proét}}$, κ-small compact Hausdorff spaces and κ-small extremally disconnected sets are equivalent by restriction. The covers in either of the sites are given by finite jointly surjective families of maps. Working in either of those has advantages and disadvantages. For example, compact Hausdorff spaces and profinite sets are stable under fiber products while extremally disconnected sets are not. On the other hand, the latter has a particularly simple description of sheaves (cf. Corollary 5.19). This description will prove to be very useful when it comes to κ-condensed abelian groups. As fiber products of extremally disconnected spaces need not be extremally disconnected, both the definition of a site and the definition of a sheaf cannot be formulated in the usual way. Here is an alternative (cf. C2 in [16]).

Definition 5.9. Let $\mathcal{C}$ be a category. A *family of morphisms with fixed target* in $\mathcal{C}$ is given by an object $U \in \mathcal{C}$, a set I and for each $i \in I$ a morphism $U_i \to U$ of $\mathcal{C}$ with target U. We use the notation $\{U_i \to U\}_{i\in I}$ to indicate this. A *site* is given by a category $\mathcal{C}$ and a set $\mathrm{Cov}(\mathcal{C})$ of families of morphisms with fixed target $\{U_i \to U\}_{i\in I}$ called *coverings* of $\mathcal{C}$, satisfying the axiom that whenever $\{U_i \xrightarrow{f_i} U\}_{i\in I}$ is a covering and $g : V \to U$ is any morphism, then there exists a covering $\{V_j \xrightarrow{h_j} V\}_{j\in J}$ such that each composite $g \circ h_j$ factors through some f_i. By abuse of notation, we usually write $\mathcal{C}$ to indicate this site.

Remark 5.10. If $\mathcal{C}$ is a site in the sense of Definition 3.2, then axiom (iii) of Definition 3.2 already implies that $\mathcal{C}$ is a site in the sense of the previous definition. We refer the reader to the interesting discussion in C2 in [16] as to why the definition of a site here has only one axiom in contrast to Definition 3.2.

Definition 5.11. Let $\mathcal{C}$ be a site in the sense of the previous definition. A *presheaf* of sets/groups/rings/... on $\mathcal{C}$ is a functor

$$T : \mathcal{C}^{\mathrm{op}} \to \{\text{sets/groups/rings/...}\}.$$

Given a covering $\{U_i \xrightarrow{f_i} U\}_{i\in I}$, a *compatible family of sections* is a tuple

$$(x_i)_{i\in I} \in \prod_{i\in I} T(U_i),$$

such that for all $j, k \in I$ and all morphisms $g : V \to U_j$ and $h : V \to U_k$ with $f_j \circ g = f_k \circ h$ we have that $T(g)(x_j) = T(h)(x_k)$. A *sheaf* of sets on $\mathcal{C}$ is a presheaf of sets T on $\mathcal{C}$ such that for every covering family $\{U_i \xrightarrow{f_i} U\}_{i\in I}$ and for every compatible family of sections $(x_i)_{i\in I} \in \prod_{i\in I} T(U_i)$ there is a unique element $x \in T(U)$ such that $T(f_i)(x) = x_i$ for all $i \in I$. A presheaf T of groups/rings/... on $\mathcal{C}$ is a sheaf of groups/rings/... on $\mathcal{C}$ if the underlying presheaf of sets is a sheaf of sets.

Example 5.12. With the above definition, κ-small extremally disconnected sets with covers given by finite jointly surjective families $\{S_i \xrightarrow{f_i} S\}_{i\in I}$ of maps form a site. Let us see why. If $\{S_i \xrightarrow{f_i} S\}_{i\in I}$ is a cover and if $g : X \to S$ is any continuous map of κ-small extremally disconnected sets, we need to find a cover $\{X_j \xrightarrow{h_j} X\}_{j\in J}$ such that each $g \circ h_j$ factors through some f_i. Consider the fiber product $X \times_S S_i$ of g and f_i taken in the category of compact Hausdorff spaces with its projections $p_{i,1}$ and $p_{i,2}$. We have already discussed that $\{X \times_S S_i \xrightarrow{p_{i,1}} X\}_{i\in I}$ is a finite jointly surjective family in the category of compact Hausdorff spaces. Next, let $\pi_{X\times_S S_i} : (X \times_S S_i)' \to X \times_S S_i$ be the canonical surjection from the κ-small extremally disconnected space $(X \times_S S_i)' := \beta(X \times_S S_i)_{\mathrm{disc}}$. We have also discussed that the family $\{(X \times_S S_i)' \xrightarrow{p_{i,1}\circ\pi_{X\times_S S_i}} X\}_{i\in I}$ is finite jointly surjective. By construction, $g \circ p_{i,1} \circ \pi_{X\times_S S_i} = f_i \circ p_{i,2} \circ \pi_{X\times_S S_i}$. Of course, this holds even without composition with the map $\pi_{X\times_S S_i}$ and hence, this defines indeed a site.

We would like to have an analog of Proposition 4.6. The obvious problem is that (iii) in the proposition can not be formulated because it contains a fiber product. On the other hand, if T is a presheaf on the site of κ-small extremally disconnected sets, there is an analog that is actually automatically satisfied. The following lemma is inspired by a Mathoverflow post by Adam Topaz (cf. [24]).

Lemma 5.13. *Suppose T is a presheaf on the site of κ-small extremally disconnected sets. Let $f : Y \to X$ be a surjection of κ-small extremally disconnected sets and let $g : Z \to Y \times_X Y$ be any surjection from a κ-small extremally disconnected set Z onto $Y \times_X Y$. The following diagram is an equalizer diagram:*

$$T(X) \xrightarrow{T(f)} T(Y) \underset{T(p_2 \circ g)}{\overset{T(p_1 \circ g)}{\rightrightarrows}} T(Z).$$

Proof. Clearly $T(f)$ equalizes the two arrows in the diagram. As $f : Y \to X$ is surjective, there is a section $h : X \to Y$ such that $f \circ h = id_X$. By the universal property of $Y \times_X Y$ we obtain a commutative diagram:

$$\begin{tikzcd}
Y \arrow[dr, dotted, "\exists! u"] \arrow[drr, bend left, "h \circ f"] \arrow[ddr, bend right, "id_Y"'] & & \\
& Y \times_X Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "f"] \\
& Y \arrow[r, "f"] & X
\end{tikzcd}$$

Since Y is projective and $g : Z \to Y \times_X Y$ surjective, the obtained map $u : Y \to Y \times_X Y$ factors through some $t : Y \to Z$:

$$\begin{tikzcd}
& Z \arrow[d, "g"] \\
Y \arrow[ur, dotted, "\exists t"] \arrow[r, "u"] & Y \times_X Y
\end{tikzcd}$$

By construction, $p_1 \circ g \circ t = id_Y$ and $p_2 \circ g \circ t = h \circ f$. Assume we are given $e : M \to T(Y)$ such that $T(p_1 \circ g) \circ e = T(p_2 \circ g) \circ e$. As $id_{T(Y)} = T(t) \circ T(p_1 \circ g)$ we have that

$$\begin{aligned}
e &= T(t) \circ T(p_2 \circ g) \circ e \\
&= T(p_2 \circ g \circ t) \circ e \\
&= T(h \circ f) \circ e \\
&= T(f) \circ T(h) \circ e.
\end{aligned}$$

Hence, e factors uniquely through $T(f)$. Indeed, if $e = T(f) \circ v$, then

$$T(h) \circ e = T(h) \circ T(f) \circ v = v.$$

□

This observation makes it very simple to check whether a presheaf on the site of κ-small extremally disconnected sets is a sheaf.

Proposition 5.14. *A presheaf T of sets/rings/groups/... on the site of all κ-small extremally disconnected sets is a sheaf of sets/rings/groups/... if and only if $T(\emptyset) = \{*\}$ and for all κ-small extremally disconnected sets S_1, S_2, the natural map*

$$T(S_1 \coprod S_2) \to T(S_1) \times T(S_2)$$

is a bijection.

Proof. Let us first note that it is enough to treat the case of sets. Clearly, κ-small extremally disconnected spaces are closed under finite coproducts. Any sheaf T satisfies the two given conditions. Let us now show the converse. Assume we are given a cover $\{S_j \xrightarrow{f_j} S\}_{i\in I}$. The cover gives us a natural surjection $f : \coprod_{i\in I} S_i \to S$. Consider the following commutative diagram:

$$\begin{tikzcd}
T(S) \arrow[r, "T(f)"] \arrow[d, equal] & T(\coprod_{i\in I} S_i) \arrow[r, shift left, "T(p_1\circ\pi)"] \arrow[r, shift right, "T(p_2\circ\pi)"'] \arrow[d, "\cong"] & T(((\coprod_{i\in I} S_i) \times_S (\coprod_{j\in I} S_j))') \arrow[d, "\cong"] \\
T(S) \arrow[r, "(T(f_i))_{i\in I}"] & \prod_{i\in I} T(S_i) \arrow[r, shift left, "(T(q_i^{(i,j)}))_{(i,j)}"] \arrow[r, shift right, "(T(q_j^{(i,j)}))_{(i,j)}"'] & \prod_{(i,j)\in I\times I} T((S_i \times_S S_j)')
\end{tikzcd}$$

Here, $q_k^{(i,j)} = p_k^{(i,j)} \circ \pi_{S_i\times_S S_j}$ where $p_k^{(i,j)}$ denotes the projection from $S_i \times_S S_j$ onto S_k. More concretely, the maps are given by $(T(q_i^{(i,j)}))_{(i,j)}((x_k)_{k\in I}) = (T(q_i^{(i,j)})(x_i))_{(i,j)}$ and likewise, $(T(q_j^{(i,j)}))_{(i,j)}((x_k)_{k\in I}) = (T(q_i^{(i,j)})(x_j))_{(i,j)}$. The upper row is an equalizer, obtained from Lemma 5.13 applied to $f : \coprod_{i\in I} S_i \to S$ and the natural surjection from the Stone-Čech compactification of the underlying discrete set

$$\pi : ((\coprod_{i\in I} S_i) \times_S (\coprod_{j\in I} S_j))' \to (\coprod_{i\in I} S_i) \times_S (\coprod_{j\in I} S_j).$$

All vertical arrows are bijections by assumption and because

$$((\coprod_{i\in I} S_i) \times_S (\coprod_{j\in I} S_j))' = (\coprod_{i,j\in I} S_i \times_S S_j)' = \coprod_{i,j\in I} (S_i \times_S S_j)',$$

where the latter equality is due to the fact that the Stone-Čech compactification commutes with colimits. Hence, the bottom row is an equalizer diagram. Now, given a family of compatible sections $(x_k)_{k\in I}$ we have that

$$\begin{aligned}
(T(q_i^{(i,j)}))_{(i,j)}((x_k)_{k\in I}) &= (T(q_i^{(i,j)})(x_i))_{(i,j)} \\
&= (T(q_j^{(i,j)})(x_j))_{(i,j)} \\
&= (T(q_j^{(i,j)}))_{(i,j)}((x_k)_{k\in I}),
\end{aligned}$$

because $f_i \circ q_i^{(i,j)} = f_i \circ p_i^{(i,j)} \circ \pi_{S_i\times_S S_j} = f_j \circ p_j^{(i,j)} \circ \pi_{S_i\times_S S_j} = f_j \circ q_j^{(i,j)}$ and thus, by compatibility of the sections, $T(q_i^{(i,j)})(x_i) = T(q_j^{(i,j)})(x_j)$. As the bottom row is an equalizer, there is a unique $x \in T(S)$ such that $(T(f_i)(x))_{i\in I} = (x_i)_{i\in I}$. This means that T is a sheaf. □

Lemma 5.15. *Let S_1 and S_2 be κ-small profinite sets with their respective natural surjections $\pi_{S_i} : S_i' \to S_i$ from the Stone–Čech compactifications S_1' and S_2'. Then we have that*

$$(S_1' \coprod S_2') \times_{S_1 \coprod S_2} (S_1' \coprod S_2') = (S_1' \times_{S_1} S_1') \coprod (S_2' \times_{S_2} S_2').$$

Proof. By the universal property of $(S_1' \coprod S_2') \times_{S_1 \coprod S_2} (S_1' \coprod S_2')$ we obtain two commutative diagrams:

$$\begin{tikzcd}
S_i' \times_{S_i} S_i' \arrow[dr, dotted, "\exists ! u_i"] \arrow[drr, bend left, "\phi_{S_i'} \circ p_2^i"] \arrow[ddr, bend right, "\phi_{S_i'} \circ p_1^i"'] & & \\
& (S_1' \coprod S_2') \times_{S_1 \coprod S_2} (S_1' \coprod S_2') \arrow[r, "r_2"] \arrow[d, "r_1"'] & S_1' \coprod S_2' \arrow[d, "(\pi_{S_1}, \pi_{S_2})"] \\
& S_1' \coprod S_2' \arrow[r, "(\pi_{S_1}, \pi_{S_2})"] & S_1 \coprod S_2
\end{tikzcd}$$

The universal property of $(S_1' \times_{S_1} S_1') \coprod (S_2' \times_{S_2} S_2')$ induces a continuous map

$$u := (u_1, u_2) : (S_1' \times_{S_1} S_1') \coprod (S_2' \times_{S_2} S_2') \to (S_1' \coprod S_2') \times_{S_1 \coprod S_2} (S_1' \coprod S_2')$$

such that $u \circ \phi_{S_i' \times_{S_i} S_i'} = u_i$. One checks directly that u is a bijection which implies the claim since continuous bijections between compact Hausdorff spaces are isomorphisms. $\square$

Lemma 5.16. *Suppose T is a presheaf on the site $*_{\kappa\text{-proét}}$ such that for all surjections $g : Y \to X$ where Y is a κ-small extremally disconnected set and X is a κ-small profinite set we have that the following diagram is an equalizer diagram:*

$$\begin{tikzcd}
T(X) \arrow[r, "T(g)"] & T(Y) \arrow[r, shift left, "T(r_1)"] \arrow[r, shift right, "T(r_2)"'] & T(Y \times_X Y)
\end{tikzcd}$$

where $r_i : Y \times_X Y \to Y$ denote the two projections. Then the same assertion is true for any surjection of κ-small profinite sets.

Proof. Let $f : Y \to X$ be any surjection of κ-small profinite sets and let $\pi_Y : Y' \to Y$ be the natural surjection from the κ-small profinite set Y'. We then have a surjection $Y' \xrightarrow{\pi_Y} Y \xrightarrow{f} X$ and a map $\pi_Y \times \pi_Y : Y' \times_X Y' \to Y \times_X Y$. Moreover, let $Y' \times_X Y' \xrightarrow{s_i} Y'$ denote the two projections. Then we have a commutative diagram such that the lower row is an equalizer:

$$\begin{tikzcd}
T(X) \arrow[r, "T(f)"] \arrow[d, equal] & T(Y) \arrow[r, shift left, "T(r_1)"] \arrow[r, shift right, "T(r_2)"'] \arrow[d, "T(\pi_Y)"'] & T(Y \times_X Y) \arrow[d, "T(\pi_Y \times \pi_Y)"] \\
T(X) \arrow[r, "T(f \circ \pi_Y)"] & T(Y') \arrow[r, shift left, "T(s_1)"] \arrow[r, shift right, "T(s_2)"'] & T(Y' \times_X Y')
\end{tikzcd}$$

The claim follows because $T(\pi_Y)$ is injective as an equalizer (apply the assumption to $g = \pi_Y$) and because the diagram commutes. $\square$

Lemma 5.17. *The restriction functor from κ-condensed sets to the category of sheaves on the site of κ-small extremally disconnected sets with covers given by finite jointly surjective families is fully faithful.*

Proof. This follows immediately from Proposition 5.8. $\square$

Let us now come to the main result of this section. We are going to establish the equivalence between the category of sheaves on the site of κ-small extremally disconnected sets and the category of κ-condensed sets. Let us give a quick argument why this should be true. Indeed, a κ-condensed set T is already determined by its values on κ-small extremally disconnected sets. For this let S be a κ-small profinite set. As before there is a surjection $\pi_S : S' \twoheadrightarrow S$ from the κ-small extremally disconnected set $S' := \beta S_{\mathrm{disc}}$. Similarly, the κ-small profinite set $S' \times_S S'$ also admits a surjection $\pi_{S'\times_S S'} : S'' \twoheadrightarrow S' \times_S S'$ from the κ-small extremally disconnected set $S'' := (S' \times_S S')' := \beta(S' \times_S S')_{\mathrm{disc}}$. By Proposition 4.6 we obtain the two equalizer diagrams:

$$T(S) \xrightarrow{T(\pi_S)} T(S') \underset{T(p_2)}{\overset{T(p_1)}{\rightrightarrows}} T(S' \times_S S')$$

$$T(S' \times_S S') \xrightarrow{T(\pi_{S'\times_S S'})} T(S'') \underset{T(q_2)}{\overset{T(q_1)}{\rightrightarrows}} T(S'' \times_{S'\times_S S'} S'')$$

As a consequence, the following diagram is also an equalizer diagram:

$$T(S) \xrightarrow{T(\pi_S)} T(S') \underset{T(p_2 \circ \pi_{S'\times_S S'})}{\overset{T(p_1 \circ \pi_{S'\times_S S'})}{\rightrightarrows}} T(S'')$$

Hence, $T(S)$ can be expressed in terms of values of T on κ-small extremally disconnected sets.

Theorem 5.18. *Consider the site of all κ-small extremally disconnected sets, with covers given by finite families of jointly surjective maps. Its category of sheaves is equivalent to κ-condensed sets via restriction from κ-small profinite sets. Moreover, if T_1 and T_2 are κ-condensed sets such that their restrictions to κ-small extremally disconnected sets are isomorphic, then $T_1 \cong T_2$.*

Proof. Let us fix some notation. As usual, the Stone-Čech compactification of the underlying discrete set of a κ-small profinite set S will be denoted by S' and the natural surjection $S' \to S$ by π_S. Likewise, the Stone-Čech compactification of the underlying discrete set of $S' \times_S S'$ will be denoted by S''. Moreover, recall from the proof of Proposition 5.8 that a map $f : Y \to X$ of κ-small profinite sets induces a map $f' : Y' \to X'$ of κ-small extremally disconnected sets such that $\pi_X \circ f' = f \circ \pi_Y$. By Lemma 5.17 we already know that the restriction functor is fully faithful. The last statement of the theorem is true because of the same Lemma and because of Proposition 5.7. Hence, we have to show that the restriction functor is essentially surjective. For this let T be a sheaf on the site of κ-small extremally disconnected sets and let S be a κ-small profinite set. We define the value of T' on S by setting

$$T'(S) := \varprojlim_{\tilde{S} \to S} T(\tilde{S}).$$

The index category has as objects all maps $\psi : \tilde{S} \to S$ where $\tilde{S}$ is a κ-small extremally disconnected set. For such an object, we will often write $(\tilde{S}, \tilde{S} \xrightarrow{\psi} S)$ or just $(\tilde{S}, \psi)$. A morphism $(\tilde{S}_1, \tilde{S}_1 \xrightarrow{\psi_1} S) \to (\tilde{S}_2, \tilde{S}_2 \xrightarrow{\psi_2} S)$ is given by a continuous map $\beta : \tilde{S}_2 \to \tilde{S}_1$ such that $\psi_2 = \psi_1 \circ \beta$. It follows from Lemma 4.3 that the index category is essentially small and that the limit $T'(S)$ exists as a limit of sets. The natural projection map $T'(S) \to T(\tilde{S})$ corresponding to the index $(\tilde{S}, \psi)$ will be denoted by $p_{(\tilde{S},\psi)}$. Let $S' \xrightarrow{f} S$ be a continuous map of κ-small profinite sets. By the universal property of limits we obtain a unique map $T'(f) : T'(S) \to T'(S')$ that is characterized by the fact that it is the projection onto all components that factor through f. We conclude that T' is a presheaf as claimed. In fact, as a limit of sets, $T'(S)$ has an explicit description. Namely, now a morphism $(\tilde{S}_1, \psi_1) \to (\tilde{S}_2, \psi_2)$ is a continuous map $\alpha : \tilde{S}_1 \to \tilde{S}_2$ such that $\psi_1 = \psi_2 \circ \alpha$ and $T'(S)$ can be described as

$$T'(S) = \{(s_{(\tilde{S},\psi)})_{(\tilde{S},\psi)} \in \prod_{(\tilde{S},\psi)} T(\tilde{S}) \mid \forall \alpha : (\tilde{S}_1, \psi_1) \to (\tilde{S}_2, \psi_2) : \ s_{(\tilde{S}_1,\psi_1)} = T(\alpha)(s_{(\tilde{S}_2,\psi_2)})\}.$$

For more details on this construction see section 7.19 in [22]. Let us now check that T' is a κ-condensed set. We want to verify the conditions in Proposition 4.6. Clearly $T'(\emptyset) = \{*\}$ because the index category consists of just one object $(\emptyset, id)$.

Let $f : Y \to X$ be a surjection of κ-small profinite sets. We want to check that the following diagram is an equalizer diagram:

$$\begin{tikzcd} T'(X) \arrow[r, "T'(f)"] & T'(Y) \arrow[r, shift left, "T'(r_1)"] \arrow[r, shift right, "T'(r_2)"'] & T'(Y \times_X Y) \end{tikzcd}$$

By Lemma 5.16 we may assume that Y is extremally disconnected. By functoriality, $T'(f)$ equalizes the two arrows. Moreover, by definition of $T'(f)$ it is clear that $T'(f)$ is injective because any map $\tilde{S} \to X$ with $\tilde{S}$ κ-small extremally disconnected factors through the surjection f since $\tilde{S}$ is projective. As an auxiliary next step we claim that the following diagram is an equalizer diagram:

$$\begin{tikzcd} T(X') \arrow[r, "T(f')"] & T(Y') \arrow[r, shift left, "T(r_1')"] \arrow[r, shift right, "T(r_2')"'] & T((Y \times_X Y)') \end{tikzcd}$$

Indeed, as $f : Y \to X$ is surjective, the underlying map of discrete sets has a section $h : X_{disc} \to Y_{disc}$. Consequently, we obtain a map $h' : X' \to Y'$ such that $f' \circ h' = id_{X'}$. Consider the following commutative diagram:

$$\begin{tikzcd}
Y_{disc} \arrow[dr, dotted, "\exists ! u"] \arrow[drr, bend left, "id_Y"] \arrow[ddr, bend right, "h\circ f"'] & & \\
& Y_{disc} \times_{X_{disc}} Y_{disc} \arrow[r, "r_2"] \arrow[d, "r_1"'] & Y_{disc} \arrow[d, "f"] \\
& Y_{disc} \arrow[r, "f"] & X_{disc}
\end{tikzcd}$$

Hence, we obtain a map $u' : Y' \to (Y \times_X Y)'$ and the following diagram commutes:

$$\begin{tikzcd}
Y' \arrow[dr, "u'"] \arrow[drr, bend left, "id_{Y'}"] \arrow[ddr, bend right, "h'\circ f'"'] & & \\
& (Y \times_X Y)' \arrow[r, "r_2'"] \arrow[d, "r_1'"'] & Y' \arrow[d, "f'"] \\
& Y' \arrow[r, "f'"] & X'
\end{tikzcd}$$

The fact that we have $f' \circ h' = id_{X'}$ implies that $T(f')$ is injective, moreover it is clear that $T(r_1') \circ T(f') = T(r_2') \circ T(f')$. Suppose now that we are given $y \in T(Y')$ such that $T(r_1')(y) = T(r_2')(y)$. Then:

$$\begin{aligned}
(T(f') \circ T(h'))(y) &= T(h' \circ f')(y) \\
&= T(r_1' \circ u')(y) \\
&= (T(u') \circ T(r_1'))(y) \\
&= (T(u') \circ T(r_2'))(y) \\
&= T(r_2' \circ u')(y) \\
&= T(id_{Y'})(y) \\
&= y.
\end{aligned}$$

This proves that the auxiliary diagram is an equalizer diagram. Because Y is extremally disconnected, the index category corresponding to $T'(Y)$ has an initial object, namely $(Y, Y \xrightarrow{id_Y} Y)$. Consequently, we have that the projection map $p_{(Y,id_Y)} : T'(Y) \to T(Y)$ is an isomorphism. The inverse $q : T(Y) \to T'(Y)$ is characterized by the fact that

$$T(\psi) = p_{(\tilde{S}, \tilde{S} \xrightarrow{\psi} Y)} \circ q$$

as maps $T(Y) \to T(\tilde{S})$. Thus, any element $g \in T'(Y)$ is of the form

$$g = q(s) = (T(\psi)(s))_{(\tilde{S},\psi)}$$

for a unique element $s \in T(Y)$. By construction the following diagram commutes:

$$\begin{tikzcd}
T'(X) \arrow[r, "T'(f)"] \arrow[d, "p_{(X',\pi_X)}"'] & T'(Y) \arrow[r, shift left, "T'(r_1)"] \arrow[r, shift right, "T'(r_2)"'] \arrow[d, "T(\pi_Y)\circ p_{(Y,id_Y)}"'] & T'(Y \times_X Y) \arrow[d, "p_{((Y\times_X Y)', \pi_{Y\times_X Y})}"] \\
T(X') \arrow[r, "T(f')"'] & T(Y') \arrow[r, shift left, "T(r_1')"] \arrow[r, shift right, "T(r_2')"'] & T((Y \times_X Y)')
\end{tikzcd}$$

Suppose now that we are given $g \in T'(Y)$ such that $T'(r_1)(g) = T'(r_2)(g)$. Because the lower row is an equalizer and because of commutativity, there is a unique element $x \in T(X')$ such that

$$T(f')(x) = T(\pi_Y) \circ p_{(Y,id_Y)}(g).$$

If we set $s := p_{(Y,id_Y)}(g)$, we have that $T(f')(x) = T(\pi_Y)(s)$ and $q(s) = g$. Thus, by the above discussion we may write $g = (T(\psi)(s))_{(\tilde{S},\psi)}$. We define $h \in T'(X)$ by setting $h_{(\tilde{S},\tilde{S} \xrightarrow{\varphi} X)} := T(\psi)(s)$ for some factorization $\varphi = f \circ \psi$. Such a factorization always exists because f is surjective and $\tilde{S}$ projective. Let us assume for a moment that $h \in T'(X)$. Then we have that

$$T'(f)(h) = (h_{(\tilde{S}, f\circ\psi)})_{(\tilde{S},\psi)} = (T(\psi)(s))_{(\tilde{S},\psi)} = g.$$

Let us now show that h is actually an element of $T'(X)$. Assume that we are given a morphism $(\tilde{S}_1, \varphi_1) \xrightarrow{\beta} (\tilde{S}_2, \varphi_2)$ such that $\varphi_1 = \varphi_2 \circ \beta$. By projectivity of $\tilde{S}_i$ we have factorizations $\varphi_i = f \circ \psi_i$. By definition of $T'(X)$ we need to show that:

$$T(\psi_1)(s) = T(\beta) \circ T(\psi_2)(s).$$

Note that we have $f \circ \psi_1 = f \circ \psi_2 \circ \beta$. Consequently, we obtain a commutative diagram:

$$\begin{tikzcd}
\tilde{S}_1 \arrow[drr, bend left, "\psi_2 \circ \beta"] \arrow[dr, dotted, "\exists! u"] \arrow[ddr, bend right, "\psi_1"'] & & \\
& Y \times_X Y \arrow[r, "r_2"] \arrow[d, "r_1"'] & Y \arrow[d, "f"] \\
& Y \arrow[r, "f"] & X
\end{tikzcd}$$

By projectivity of $\tilde{S}_1$ we obtain a map $v : \tilde{S}_1 \to (Y \times_X Y)'$ such that $u = \pi_{Y \times_X Y} \circ v$. Recall that we have

$$\pi_Y \circ r_j' = r_j \circ \pi_{Y \times_X Y}.$$

Hence,

$$\begin{aligned}
T(\psi_1)(s) &= T(r_1 \circ u)(s) = T(r_1 \circ \pi_{Y \times_X Y} \circ v)(s) \\
&= T(\pi_Y \circ r_1' \circ v)(s) = T(r_1' \circ v) \circ T(\pi_Y)(s) \\
&= T(r_1' \circ v) \circ T(f')(x) = T(f' \circ r_1' \circ v)(x) \\
&= T(f' \circ r_2' \circ v)(x) = T(r_2' \circ v) \circ T(f')(x) \\
&= T(r_2' \circ v) \circ T(\pi_Y)(s) = T(\pi_Y \circ r_2' \circ v)(s) \\
&= T(r_2 \circ \pi_{Y \times_X Y} \circ v)(s) = T(r_2 \circ u)(s) \\
&= T(\psi_2 \circ \beta)(s) = T(\beta) \circ T(\psi_2)(s).
\end{aligned}$$

Altogether, the diagram in question is an equalizer. Let X and Y be κ-small profinite sets. We claim that $T'(X \coprod Y) = T'(X) \times T'(Y)$. For this let S be a κ-small profinite set and $\pi_S : S' \to S$ be the natural surjection. By the discussion so far, we already know that the following diagram is an equalizer:

$$T'(S) \xrightarrow{T'(\pi_S)} T'(S') \underset{T(p_2)}{\overset{T(p_1)}{\rightrightarrows}} T'(S' \times_S S')$$

Let $\pi : S'' \to S' \times_S S'$ be the natural surjection from the κ-small extremally disconnected set $S'' = (S' \times_S S')'$. It is then clear that the following diagram is an equalizer because the map $T'(S' \times_S S') \to T'(S'')$ is injective as shown earlier:

$$T'(S) \xrightarrow{T'(\pi_S)} T'(S') \underset{T'(p_2 \circ \pi)}{\overset{T'(p_1 \circ \pi)}{\rightrightarrows}} T'(S'').$$

In fact, because $T(Y) = T'(Y)$ for any κ-small extremally disconnected set, this means that, with the induced arrows, we have a natural identification

$$T'(S) = \mathrm{eq}(T(S') \rightrightarrows T(S'')).$$

Using Lemma 5.15 and that Stone-Čech compactification commutes with colimits we know that:

$$\begin{aligned}
(X \coprod Y)'' &= ((X \coprod Y)' \times_{X \coprod Y} (X \coprod Y)')' \\
&= ((X' \times_X X') \coprod (Y' \times_Y Y'))' \\
&= (X' \times_X X')' \coprod (Y' \times_Y Y')' \\
&= X'' \coprod Y''.
\end{aligned}$$

Thus:

$$\begin{aligned}
T'(X \coprod Y) &= \mathrm{eq}(T((X \coprod Y)') \rightrightarrows T((X \coprod Y)'')) \\
&= \mathrm{eq}(T(X' \coprod Y') \rightrightarrows T(X'' \coprod Y'') \\
&= \mathrm{eq}(T(X') \times T(Y') \rightrightarrows T(X'') \times T((Y'')) \\
&= \mathrm{eq}(T(X') \rightrightarrows T(X'')) \times \mathrm{eq}(T(Y') \rightrightarrows T(Y'')) \\
&= T'(X) \times T'(Y).
\end{aligned}$$

Let us now show that the restriction of T' to κ-small extremally disconnected sets is naturally isomorphic to T. In fact, we have discussed everything necessary already. If S is a κ-small extremally disconnected set, we already know that $p_{(S.id_S)} : T'(S) \to T(S)$ is bijective. Moreover, by definition we have that $T(f) \circ p_{(S,id_S)} = p_{(S',id_{S'})} \circ T'(f)$ for any map $f : S' \to S$. □

Corollary 5.19. *The category of κ-condensed sets/rings/groups/... is equivalent to the category of functors*

$$T : \{\kappa\text{-small extremally disconnected sets}\}^{\mathrm{op}} \to \{\text{sets/rings/groups/...}\}$$

such that $T(\emptyset) = \{*\}$ and for all κ-small extremally disconnected sets S_1, S_2, the natural map $T(S_1 \coprod S_2) \to T(S_1) \times T(S_2)$ is a bijection.

Proof. This is Theorem 5.18 and Proposition 5.14. Although we have proved Theorem 5.18 for sheaves of sets the proof goes through mutatis mutandis in the other cases. For example, one has to check that the defined sheave T' is actually a sheave of rings/groups/modules/... . □

As promised, there is a similar result for the category of sheaves on the site of compact Hausdorff spaces and κ-condensed sets which we will state but not prove although one could prove it as in Theorem 5.18. Like in the case of κ-small extremally disconnected sets, a sheaf T on the site of all κ-small compact Hausdorff spaces is already determined by its values on κ-small profinite sets. Let S be a κ-small compact Hausdorff space. By Proposition 5.6 we can find a surjection $\pi_S : S' \twoheadrightarrow S$ from the κ-small extremally disconnected set $S' := \beta S_{\mathrm{disc}}$. As established in Lemma 5.2, S is also profinite. By Remark 4.7 and Proposition 4.6 the value $T(S)$ is already determined by the values $T(S')$ and $T(S' \times_S S')$ on the κ-small profinite sets S' and $S' \times_S S'$ (with its projections p_1 and p_2). Indeed, the following diagram is an equalizer diagram:

$$T(S) \xrightarrow{\pi_S} T(S') \overset{T(p_1)}{\underset{T(p_2)}{\rightrightarrows}} T(S' \times_S S').$$

Proposition 5.20. *Consider the site of all κ-small compact Hausdorff spaces, with covers given by finite families of jointly surjective maps. Its category of sheaves is equivalent to κ-condensed sets via restriction to κ-small profinite sets.* □

Remark 5.21. Proposition 5.20 follows formally from Lemma 7.29.1 in section 7.29 in [22]. Indeed, the inclusion functor from the category of profinite sets into the category of compact Hausdorff spaces satisfies all of the conditions in the lemma and induces thereby an equivalence of the topoi in question. Given a κ-condensed set T, its extension is constructed similarly as in the proof of Theorem 5.18. In contrast, the above mentioned lemma in [22] cannot be applied to prove Theorem 5.18 because κ-small extremally disconnected sets lack limits and the inclusion functor to κ-small profinite sets is therefore not continuous.

5.3 The Stone–Čech compactification revisited

In the proof of Theorem 5.18 two functors came up. Namely, given a κ-condensed set T, the value on the κ-small profinite set S was already determined by the values on the κ-small extremally disconnected sets S' and S''. More precisely, the following diagram is an equalizer diagram:

$$T(S) \xrightarrow{T(\pi_S)} T(S') \underset{T(p_2 \circ \pi_{S' \times_S S'})}{\overset{T(p_1 \circ \pi_{S' \times_S S'})}{\rightrightarrows}} T(S'').$$

One key ingredient in the proof was the fact that the functor $S \mapsto S'$ preserves epimorphisms. In fact, this is not true for the functor $S \mapsto S''$ as opposed to an erroneous statement in Proposition 2.3 in version 1 of [2]. Although by Lemma 5.15, we can at least say that the functor preserves finite coproducts. In this section we will have a closer look at the construction of the Stone–Čech compactification S' as the set of ultrafilters on the underlying discrete set S_{disc}, as well as the construction of the maps $f' : Y' \to X'$ for a given map $f : Y \to X$. We will use this description to show that the functor $S \mapsto S''$ does not preserve epimorphisms in general. The basic facts about ultrafilters and the Stone–Čech compactification of a discrete space are taken from the book *Algebra in the Stone–Čech compactification* by Neil Hindman and Dona Strauss in [15].

Definition 5.22. Let X be a set. A *filter* on X is a non-empty set $\mathcal{F}$ of subsets of X such that:

(i) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

(ii) If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$.

(iii) $\emptyset \notin \mathcal{F}$.

A filter $\mathcal{F}$ is called an *ultrafilter* if for all $A \subseteq X$ either $A \in \mathcal{F}$ or $X \backslash A \in \mathcal{F}$.

Lemma 5.23. *Let X be a set. A filter $\mathcal{F}$ on X is an ultrafilter if and only if $\mathcal{F}$ is not properly contained in any other filter.*

Proof. This is Theorem 3.6 in [15]. □

Lemma 5.24. *Let X be a set and let $\mathcal{U}$ be a set of subsets of X with the finite intersection property, i.e. for any finite non-empty subset $\mathcal{V} \subseteq \mathcal{U}$ we have that $\cap_{A \in \mathcal{V}} A \neq \emptyset$. Then there exists an ultrafilter $\mathcal{F}$ on X such that $\mathcal{U} \subseteq \mathcal{F}$.*

Proof. This is Theorem 3.8 in [15]. □

Lemma 5.25. *Let X be a set and let $x \in X$. Consider $i_X(x) := \{A \subseteq X : x \in A\}$. Then $i_X(x)$ is an ultrafilter on X.*

Proof. Clearly $i_X(x)$ is a filter on X. Suppose $A \subseteq X$, then either $x \in A$ or $x \in X \backslash A$ and hence, either $A \in i_X(x)$ or $X \backslash A \in i_X(x)$. $\square$

One says that $i_X(x)$ is the principal ultrafilter associated to $x \in X$. Now let X be a discrete space and set

$$\beta X := \{\mathcal{F} : \mathcal{F} \text{ is an ultrafilter on } X\}$$

and for $A \subset X$ we define

$$\hat{A} := \{\mathcal{F} \in \beta X : A \in \mathcal{F}\}.$$

The assignment $x \mapsto i_X(x)$ defines an injective map $X \to \beta X$. If we equip βX with the topology for which $\{\hat{A} : A \subseteq X\}$ forms a basis, then the pair $(\beta X, i_X)$ is a Stone–Čech compactification of X (cf. Theorem 3.27 in [15]). Of course, we are interested in the case where X is a compact Hausdorff space. As usual we set $X' := \beta X_{\mathrm{disc}}$. Let us first have a look at how the natural surjection $\pi_X : X' \to X$ is actually constructed. Recall that we applied the universal property of X' to the identity $id_X : X_{disc} \to X$ to obtain the unique map $\pi_X : X' \to X$ such that $\pi_X \circ i_X = id_X$. Let $\mathcal{F}$ be an ultrafilter on X_{disc}. The proof of Theorem 3.27 in [15] shows that $\pi_X(\mathcal{F})$ is the unique point in the intersection $\cap_{B\in\mathcal{F}}\overline{B}$ where $\overline{B}$ denotes the closure of B in the compact Hausdorff space X. Likewise, given a map $f : Y \to X$ of compact Hausdorff spaces X and Y, the map $f' : Y' \to X'$ is the unique map such that $f' \circ i_Y = i_X \circ f$. Hence, $f'(\mathcal{F})$ is the unique point in the intersection $\cap_{B\in\mathcal{F}}\overline{(i_X \circ f)(B)}$. If f is surjective, f' has a particularly nice description. This is our next goal.

Lemma 5.26. *If $f : Y \to X$ is a surjective map and $\mathcal{F}$ is an ultrafilter on Y, then $f(\mathcal{F}) := \{f(B) : B \in \mathcal{F}\}$ is an ultrafilter on X.*

Proof. Assume $A_1, A_2 \in f(\mathcal{F})$. We may write $A_j = f(B_j)$ for some $B_j \in \mathcal{F}$. Then

$$B_j \subseteq f^{-1}(f(B_j)) = f^{-1}(A_j)$$

and hence, $f^{-1}(A_j) \in \mathcal{F}$. This implies that

$$f^{-1}(A_1) \cap f^{-1}(A_2) = f^{-1}(A_1 \cap A_2) \in \mathcal{F}.$$

Because f is surjective,

$$A_1 \cap A_2 = f(f^{-1}(A_1 \cap A_2)) \in f(\mathcal{F}).$$

Now let $A \in \mathcal{F}$ and $A' \subseteq X$ such that $A \subseteq A'$. Write $A = f(B)$ for some $B \in \mathcal{F}$. Then

$$B \subseteq f^{-1}(f(B)) = f^{-1}(A) \subseteq f^{-1}(A'),$$

consequently, $f^{-1}(A') \in \mathcal{F}$. Thus, $A' = f(f^{-1}(A')) \in f(\mathcal{F})$ by the surjectivity of f. Since $\emptyset \notin \mathcal{F}$, we also have that $\emptyset \notin f(\mathcal{F})$. This means that $f(\mathcal{F})$ is a filter on X. Let us now show that it is also an ultrafilter. For this let $A \subseteq X$. Since $\mathcal{F}$ is an ultrafilter on Y, either $f^{-1}(A) \in \mathcal{F}$ or $Y \backslash f^{-1}(A) \in \mathcal{F}$. Then by the surjectivity of f either

$$A = f(f^{-1}(A)) \in f(\mathcal{F})$$

or

$$X \backslash A = f(f^{-1}(X \backslash A)) = f(Y \backslash f^{-1}(A)) \in f(\mathcal{F}).$$

$\square$

Lemma 5.27. *Let $f : Y \to X$ be a surjection of compact Hausdorff spaces and let $\mathcal{G}$ be an ultrafilter on X_{disc}. Then there is an ultrafilter $\mathcal{F}$ on Y_{disc} such that the family $f^{-1}(\mathcal{G}) := \{f^{-1}(B) : B \in \mathcal{G}\}$ is contained in $\mathcal{F}$.*

Proof. Because f is surjective, $f^{-1}(\mathcal{G})$ has the finite intersection property. Hence, by Lemma 5.24 the existence of $\mathcal{F}$ is ensured. □

Proposition 5.28. *Let $f : Y \to X$ be a surjection of compact Hausdorff spaces and let $\mathcal{F}$, $\mathcal{G}$ be ultrafilters on Y_{disc} and X_{disc}, respectively. Then the following statements are true:*

(i) *$f' : Y' \to X'$ is given by $f'(\mathcal{F}) = f(\mathcal{F})$.*

(ii) *$f'(\mathcal{F}) = \mathcal{G}$ if and only if $f^{-1}(\mathcal{G}) \subseteq \mathcal{F}$.*

(iii) *f' is surjective.*

Proof. (i) By Lemma 5.26 we may define a map $g : Y' \to X'$ by $g(\mathcal{F}) := f(\mathcal{F})$. We claim that g is continuous. For this let $\hat{A} = \{\mathcal{G} \in X' : A \in \mathcal{G}\}$ be a basic open set in X'. Set $B := f^{-1}(A)$. Then $g^{-1}(\hat{A}) = \hat{B}$. Indeed, let $\mathcal{F} \in g^{-1}(\hat{A})$, then we have $f(\mathcal{F}) \in \hat{A}$. Hence, $A = f(C)$ for some $C \in \mathcal{F}$. Thus

$$C \subseteq f^{-1}(f(C)) = f^{-1}(A) = B.$$

This implies that $B \in \mathcal{F}$ which means that $\mathcal{F} \in \hat{B}$. Conversely, let $\mathcal{F} \in \hat{B}$. By definition of $\hat{B}$ this means that $f^{-1}(A) \in \mathcal{F}$. As f is surjective we have

$$A = f(f^{-1}(A)) \in f(\mathcal{F})$$

which implies that $f(\mathcal{F}) \in \hat{A}$ and consequently, $\mathcal{F} \in g^{-1}(\hat{A})$. Hence, g is continuous as claimed. Let us now show that g satisfies $g \circ i_Y = i_X \circ f$ which would imply $g = f'$ by uniqueness of f'. We have

$$\begin{aligned} g \circ i_Y(y) &= g(i_Y(y)) \\ &= g(\{A \subseteq Y_{\text{disc}} : y \in A\}) \\ &= \{f(A) : y \in A\} \\ &= \{B \subseteq X_{disc} : f(y) \in B\} \\ &= i_X(f(y)) \\ &= i_X \circ f(y). \end{aligned}$$

(ii) Suppose that $f^{-1}(\mathcal{G}) \subseteq \mathcal{F}$. By Lemma 5.23 and part (i) it is enough to show that $\mathcal{G} \subseteq f(\mathcal{F})$. Let $B \in \mathcal{G}$, then by the surjectivity of f we have that

$$B = f(f^{-1}(B)) \in f(\mathcal{F}).$$

Hence, $\mathcal{G} = f'(\mathcal{F})$. Assume now that $f'(\mathcal{F}) = \mathcal{G}$. Let $B \in \mathcal{G}$. By assumption and part (i) we may write $B = f(A)$ for some $A \in \mathcal{F}$. But then

$$A \subseteq f^{-1}(f(A)) = f^{-1}(B)$$

which implies that $f^{-1}(B) \in \mathcal{F}$. Thus, we may conclude that $f^{-1}(\mathcal{G}) \subseteq \mathcal{F}$.
(iii) This follows from Lemma 5.27 and part (ii).

□

We are now ready to discuss that the functor $S \mapsto S'' = (S' \times_{S' \times_S S'})'$ does not preserve epimorphisms. Given a map $f : Y \to X$, consider the induced map $f' : Y' \to X'$. Let us quickly see how $f'' : Y'' \to X''$ is constructed. First of all, we have a commutative diagram

$$\begin{tikzcd}
Y' \times_Y Y' \arrow[dr, dotted, "\exists! f' \times f'"] \arrow[drr, bend left, "f' \circ q_2"] \arrow[ddr, bend right, "f' \circ q_1"'] & & \\
& X' \times_X X' \arrow[r, "p_2"] \arrow[d, "p_1"] & X' \arrow[d, "\pi_X"] \\
& X' \arrow[r, "\pi_X"] & X
\end{tikzcd}$$

We then define $f'' := (f' \times f')' : Y'' \to X''$. Notice that a map $f : Y \to X$ is surjective if and only if $f' : Y' \to X'$ is surjective. Indeed, we already know that the only if part is true (cf. Proposition 5.28). On the other hand, if f' is surjective, then f is surjective because of the equality $\pi_X \circ f' = f \circ \pi_Y$. Let us first discuss some positive results, i.e. instances where the map f'' is surjective. Suppose $f : Y \to X$ is surjective, by the above, we can focus on $f' \times f'$. We have the following commutative diagram:

$$\begin{tikzcd}
Y' \arrow[dr, dotted, "\exists! f' \times \pi_Y"] \arrow[drr, bend left, "\pi_Y"] \arrow[ddr, bend right, "f'"'] & & \\
& X' \times_X Y \arrow[r, "p_2"] \arrow[d, "p_1"] & X' \arrow[d, "f"] \\
& X' \arrow[r, "\pi_X"] & X
\end{tikzcd}$$

Let us assume that $f' \times \pi_Y$ is surjective. We claim that in this case $f' \times f'$ is surjective as well. Let $(x_1, x_2) \in X' \times_X X'$. By the surjectivity of f' we may write $x_2 = f'(y_2)$ for some $y_2 \in Y'$. Then

$$\pi_X(x_1) = \pi_X(x_2) = \pi_X \circ f'(y_2) = f \circ \pi_Y(y_2)$$

and consequently, $(x_1, \pi_Y(y_2)) \in X' \times_X Y$. By the surjectivity of $f' \times \pi_Y$ there is some $y_1 \in Y'$ such that

$$(x_1, \pi_Y(y_2)) = (f' \times \pi_Y)(y_1) = (f'(y_1), \pi_Y(y_1)).$$

Then $(y_1, y_2) \in Y' \times_Y Y'$ such that $(f' \times f')((y_1, y_2)) = (x_1, x_2)$. The following lemma shows that $f' \times \pi_Y$ is surjective if f is surjective and open.

Lemma 5.29. *If $f : Y \to X$ is surjective and open, then $f' \times \pi_Y : Y' \to X' \times_X Y$ is surjective. Consequently, $f'' : Y'' \to X''$ is surjective.*

Proof. Since $(f' \times \pi_Y)(Y')$ is closed in $X' \times_X Y$, it is enough to show that $(f' \times \pi_Y)(Y')$ is dense in $X' \times_X Y$. Let $U \times_X V := (U \times V) \cap (X' \times_X Y)$ be a non-empty basic open set in $X' \times_X Y$, i.e. $U \subseteq X'$ and $V \subseteq Y$ are open. Consider the open $W := f(V) \subseteq X$. Since $U \times_X V$ is not empty, we have that $U' := U \cap \pi_X^{-1}(W) \subseteq X'$ is non-empty and open. As $i_X(X_{\mathrm{disc}})$ is dense in X', there is $x \in X$ such that $i_X(x) \in U'$. By construction,

$x = \pi_X \circ i_X(x) \in W$ and hence, there is some $y \in V$ such that $f(y) = x$. But then $i_Y(y) \in Y'$ such that

$$(f' \times \pi_Y)(i_Y(y)) = (f' \circ i_Y(y), \pi_Y \circ i_Y(y)) = (i_X \circ f(y), y) = (i_X(x), y) \in U \times_X V.$$

This shows the claim. $\square$

Unfortunately this is not always the case, as surjections of compact Hausdorff spaces are always closed, but they are not necessarily open. The following counterexamples originated in discussion with Xavier Xarles and Niklas Müller.

Example 5.30. We start with compact Hausdorff spaces and will later see that these examples give rise to examples of profinite sets. Let $Y := [0,1] \subset \mathbb{R}$ be the unit interval and $X := [0,1]/\{0 \sim 1\}$, the quotient of Y where we identify the endpoints of the interval. We identify X with the unit circle S^1 under the homeomorphism induced by the surjection

$$f : Y \to S^1, y \mapsto (\cos(2\pi y), \sin(2\pi y)).$$

Let us fix some notation. We denote by $\mathcal{H}^+$ the open upper half plane and by $\mathcal{H}^-$ the open lower half plane in $\mathbb{R}^2$ and for $0 < \epsilon < 1$ we set $B_\epsilon := B_\epsilon((1,0)) \cap S^1$ as well as $U_\epsilon := B_\epsilon \cap \mathcal{H}^+$ and $V_\epsilon := B_\epsilon \cap \mathcal{H}^-$. Let $\mathcal{G}$ be an ultrafilter on X_{disc} such that $\mathcal{G}$ contains the sets V_ϵ for $0 < \epsilon < 1$. By Lemma 5.24 such an ultrafilter exists. In particular, $(\mathcal{G}, 0) \in X' \times_X Y$ because by construction we have for all $0 < \epsilon < 1$ that

$$\{\pi_X(\mathcal{G})\} = \cap_{B \in \mathcal{G}} \overline{B} \subset \overline{V_\epsilon},$$

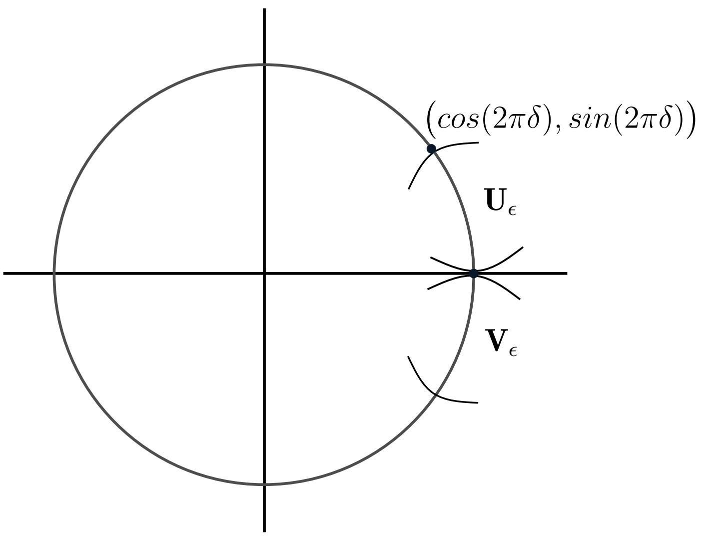
which already implies that $\pi_X(\mathcal{G}) = (1,0)$ and hence

$$\pi_X(\mathcal{G}) = f(0) = (1,0).$$

Suppose now that there is $\mathcal{F} \in Y'$ such that $(f' \times \pi_Y)(\mathcal{F}) = (\mathcal{G}, 0)$. The equality

$$\{0\} = \{\pi_Y(\mathcal{F})\} = \cap_{B \in \mathcal{F}} \overline{B}$$

implies that $0 \in \overline{B}$ for all $B \in \mathcal{F}$. We have that for all $0 < \epsilon < 1$ the set $[0, \epsilon)$ is contained in $\mathcal{F}$ because otherwise there is some ϵ such that $[\epsilon, 1] \in \mathcal{F}$ which would imply that $0 \in [\epsilon, 1]$. By construction we may choose $0 < \delta, \epsilon < 1$ such that $[0, \delta) \in \mathcal{F}$ and $f([0, \delta)) = U_\epsilon \cup \{(1,0)\}$. By Proposition 5.28 we may conclude that $f([0, \delta))$ is contained in $f(\mathcal{F}) = \mathcal{G}$. This is of course absurd because then $f([0, \delta)) \cap V_\epsilon = \emptyset \in \mathcal{G}$. This establishes that $f' \times \pi_Y$ is not surjective in general. Here is a picture of the situation:



Let us now see why $f' \times f'$ is not surjective in general. For this choose ultrafilters $\mathcal{G}_1$ and $\mathcal{G}_2$ on X_{disc} as above such that for all $0 < \epsilon < 1$ the sets U_ϵ and V_ϵ are contained in $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. We then have that

$$\pi_X(\mathcal{G}_1) = (1, 0) = \pi_X(\mathcal{G}_2)$$

and hence, $(\mathcal{G}_1, \mathcal{G}_2) \in X' \times_X X'$. Suppose that there is $(\mathcal{F}_1, \mathcal{F}_2) \in Y' \times_Y Y'$ such that

$$(f' \times f')((\mathcal{F}_1, \mathcal{F}_2)) = (\mathcal{G}_1, \mathcal{G}_2).$$

By Proposition 5.28 we know that $f^{-1}(\mathcal{G}_i) \subseteq \mathcal{F}_i$ and hence, for all $0 < \epsilon < 1$ we have that $[0, \epsilon) \in \mathcal{F}_1$ and $(1 - \epsilon, 1] \in \mathcal{F}_2$. This implies that

$$\pi_Y(\mathcal{F}_1) = 0 \neq 1 = \pi_Y(\mathcal{F}_2),$$

which is clearly a contradiction because this means that $(\mathcal{F}_1, \mathcal{F}_2) \notin Y' \times_Y Y'$. To get a counterexample in the profinite setting, we replace $[0, 1]$ with the cantor set $C \subset [0, 1]$. Then the exact same arguments work because we can identify $C/\{0 \sim 1\}$ with a subspace of $[0, 1]/\{0 \sim 1\}$. It is well known that C is a profinite set, we just have to argue that $C/\{0 \sim 1\}$ is profinite as well. For this it is enough to show that $C/\{0 \sim 1\}$ is totally disconnected. We will do this by showing that any two distinct points have disjoint clopen neighborhoods such that their union is $C/\{0 \sim 1\}$. In a first step we separate $\{x\}$ and $\{0, 1\}$. As C is totally disconnected we may write $C = U \cup V$ where $x \in U$ and $0 \in V$ with $U \cap V = \emptyset$ as well as $C = U' \cap V'$ where $x \in U'$ and $1 \in V'$ with $U' \cap V' = \emptyset$. In particular, U, U' and V, V' are clopen. Hence, $W := U \cap U'$ is clopen and contains x. The complement $C \backslash W$ contains $\{0, 1\}$ and is also clopen. Let $\pi : C \to C/\{0 \sim 1\}$ be the natural surjection, we then have that $C/\{0 \sim 1\} = \pi(W) \cup \pi(C \backslash W)$. Moreover, both $\pi(W)$ and $\pi(C \backslash W)$ are clopen and they are disjoint, $\pi(W)$ contains $\{x\}$ and $\pi(C \backslash W)$ contains $\{0, 1\}$. Let us now separate $\{x\}$ and $\{y\}$ for distinct $x, y \in C \backslash \{0, 1\}$. By the first step we may write $C = U_y \cup V_y$ such that $y \in U_y$ and $\{0, 1\} \subset V_y$ with U_y and V_y clopen and disjoint. If $x \in V_y$, we are done because we can take the image of U_y and $C \backslash U_y$ under π. Let us then assume that $x \in U_y$. As $x \neq y$ we may decompose $C = W_x \cup W_y$ such that $x \in W_x$, $y \in W_y$ and W_x, W_y are clopen and disjoint. Now $W := U_y \cap W_x$ is clopen and contains x but neither y nor $\{0, 1\}$. Hence, the images of W and $C \backslash W$ under π work.

6 Compactly generated topological spaces

Let us now look at another very important class of topological spaces, namely *compactly generated* topological spaces (cf. Definition 6.1 below). The main goal of this section is to show that this large class of topological spaces embeds fully faithfully into κ-condensed sets (cf. Proposition 6.5). As usual, the material presented here is taken from the lecture notes [7] of Clausen and Scholze.

Definition 6.1. A topological space X is *compactly generated* if a map $f : X \to Y$ to another topological space Y is continuous as soon as the composite $S \to X \to Y$ is continuous for all compact Hausdorff spaces S mapping continuously to X.

In other words, X is compactly generated if and only if $V \subset X$ is closed as soon as the preimage of V is closed in S for all compact Hausdorff spaces S mapping to X. This topology is also called *final topology* with respect to the collection of all maps from compact Hausdorff spaces mapping continuously to X. Note that this collection is not a set but a proper class. We can avoid talking about classes here. In particular this will allow us, by abuse of notation, to define this topology as the quotient topology for the natural map $\coprod_{S \to X} S \to X$ where S runs through compact Hausdorff spaces mapping to X. The reason is that we can always choose a *set* $\mathcal{C}_X$ of compact Hausdorff spaces mapping to X such that the induced final topology is the same (thus allowing us to take the coproduct over a set). The surprisingly simple arguments are taken from [4] (cf. Result 5.9.1). Let us go over the construction of the set $\mathcal{C}_X$. If $A \subset X$ is not closed, there is a compact Hausdorff space S_A and a map $t : S_A \to X$ such that the preimage $t^{-1}(A)$ is not closed in S_A. For each such non-closed A we choose one S_A and one $t : S_A \to X$ as above. Let $\mathcal{C}_X$ be the set consisting of those t. We claim that the final topology induced by $\mathcal{C}_X$ coincides with the final topology of all compact Hausdorff spaces mapping to X. Indeed, let $A \subset X$ be closed for the topology induced by $\mathcal{C}_X$. The existence of $s : S \to X$ such that the preimage $s^{-1}(A)$ is not closed in S would, by construction of the set $\mathcal{C}_X$, imply that there is some $t : S_A \to X$ in the set $\mathcal{C}_X$ such that $t^{-1}(A)$ is not closed in S_A. But this is in contradiction to the fact that $h^{-1}(A)$ is closed for all $h \in \mathcal{C}_X$. We will use this result without mentioning it again. In particular, we can achieve that the natural map $\coprod_{S \to X} S \to X$ is actually a quotient map. The benefit of this approach is the fact that we can use the universal property of the coproduct to make continuity arguments.

There is a straightforward functorial way to produce compactly generated topological spaces. For this let X be a topological space. On the underlying set of X, we define the topological space X^{cg} where the topology is given by the quotient topology for the natural map $q : \coprod_{S \to X} S \to X$, here S runs through compact Hausdorff spaces that map continuously to X. In particular, X^{cg} and X have the same class of continuous maps from compact Hausdorff spaces, i.e. a map $S \to X^{\mathrm{cg}}$ is continuous if and only if $S \to X$ is continuous. The canonical inclusions $S \to \coprod_{T \to X} T$ will typically be denoted by ϕ_S. Let us quickly see why X^{cg} is compactly generated. Assume Y is a topological space and assume $f : X^{\mathrm{cg}} \to Y$ is a map such that for all compact Hausdorff spaces $S \to X^{\mathrm{cg}}$ the composite $S \to X^{\mathrm{cg}} \to Y$ is continuous. By the universal property of $\coprod_{S \to X^{\mathrm{cg}}} S$ there is a unique continuous map $g : \coprod_{S \to X^{\mathrm{cg}}} S \to Y$ such that the following diagram is commutative for all S:

$$\begin{tikzcd} \coprod_{T\to X^{\mathrm{cg}}} T \arrow[drr, dotted, "\exists ! g"] & & \\ S \arrow[u, "\phi_S"] \arrow[r] & X \arrow[r, "f"'] & Y \end{tikzcd}$$

On the other hand, $f \circ q$ is another continuous map such that the diagram commutes and thus, by uniqueness, $g = f \circ q$. In particular, this implies that f is continuous for the topology induced by q. Hence, X^{cg} is compactly generated.

Notice that by definition of X^{cg} the natural identity map $X^{\mathrm{cg}} \to X$ is continuous. Indeed, if $U \subset X$ is open, then the preimage of U under the natural map $q : \coprod_{S\to X} S \to X$ is open, i.e. $U \subset X^{\mathrm{cg}}$ is open. In particular, the topology on X^{cg} is finer than the given topology and X is compactly generated if and only if the identity $X^{\mathrm{cg}} \to X$ is a homeomorphism.

Let us now check that if $f : X \to Y$ is continuous and if $f^{\mathrm{cg}} : X^{\mathrm{cg}} \to Y^{\mathrm{cg}}$ is given by f, then f^{cg} is continuous. Let $\psi_S : S \to X^{\mathrm{cg}}$ be a compact Hausdorff space mapping continuously into X^{cg}. We need to check that the composition

$$S \xrightarrow{\psi_S} X^{\mathrm{cg}} \xrightarrow{f^{\mathrm{cg}}} Y^{\mathrm{cg}}$$

is continuous. This is the case if and only if the composition

$$S \xrightarrow{\psi_S} X^{\mathrm{cg}} \xrightarrow{f^{\mathrm{cg}}} Y^{\mathrm{cg}} \xrightarrow{id_Y} Y$$

is continuous. The latter is the same as the composition

$$S \xrightarrow{\psi_S} X^{\mathrm{cg}} \xrightarrow{id_X} X \xrightarrow{f} Y$$

and hence continuous as composition of continuous maps.

Proposition 6.2. *The constructed functor $X \mapsto X^{cg}$ is right adjoint to the inclusion functor from the full subcategory of compactly generated topological spaces to the category of all topological spaces.*

Proof. Let X be a compactly generated topological space and let Y be an arbitrary topological space. We claim that the map $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Y^{\mathrm{cg}}), f \mapsto f$, is bijective and natural in X and Y. The first statement reduces to the claim that $f : X \to Y$ is continuous if and only if $f : X \to Y^{\mathrm{cg}}$ is continuous. For this let $f : X \to Y$ be continuous. We need to check that $f : X \to Y^{\mathrm{cg}}$ is still continuous. Since X is compactly generated, we can test the continuity on compact Hausdorff spaces $S \to X$ mapping continuously to X. Hence, we need to show that the composition

$$S \to X \xrightarrow{f} Y^{\mathrm{cg}}$$

is continuous. This is the case if and only if the composition

$$S \to X \xrightarrow{f} Y^{\mathrm{cg}} \xrightarrow{id_Y} Y$$

is continuous. The latter is continuous because it agrees with the composition

$$S \to X \xrightarrow{f} Y.$$

Finally, let us assume that $f : X \to Y^{\mathrm{cg}}$ is continuous. Then $f : X \to Y$ can be written as composition $X \xrightarrow{f} Y^{\mathrm{cg}} \xrightarrow{id_Y} Y$, and thus, $f : X \to Y$ is continuous because $Y^{\mathrm{cg}} \xrightarrow{id_Y} Y$ is continuous. □

Remark 6.3. Let S be a compact Hausdorff space. By Proposition 5.6 we know that the extremally disconnected set βS_{disc} surjects onto S and by Lemma 5.2 βS_{disc} is also profinite. This observation is the key to the fact that a topological space X is already compactly generated if $X \to Y$ is continuous as soon as $S \to X \to Y$ is continuous for all profinite sets S. In particular, in the previous discussion, we can replace compact Hausdorff spaces with profinite sets without altering the class of compactly generated spaces or the functor $X \mapsto X^{\mathrm{cg}}$.

To avoid set theoretic issues we need the following variant (cf. Remark 7.20). We choose an uncountable strong limit cardinal κ as above, and we say that a (not necessarily κ-small) topological space X is *κ-compactly generated* if it is equipped with the quotient topology from the natural map $\coprod_{S \to X} S \to X$ where S runs over compact Hausdorff spaces with $|S| < \kappa$. We denote by $X \mapsto X^{\kappa-\mathrm{cg}}$ the right adjoint of the inclusion functor from the full subcategory of κ-compactly generated topological spaces into the category of all topological spaces; the construction is essentially the same as before. Namely, $X^{\kappa-\mathrm{cg}}$ has the underlying set of X equipped with the quotient topology of the natural map $\coprod_{S \to X} S \to X$ where S runs over all compact Hausdorff spaces S with $|S| < \kappa$. By the above argument, we can again work with κ-small profinite sets instead of with κ-small compact Hausdorff spaces because the profinite set βS_{disc} is κ-small.

Example 6.4. If X is a first-countable topological space, for example a metrizable topological space, then X is compactly generated; in fact, it is even κ-compactly generated for any uncountable strong limit cardinal κ. To verify the claim let $V \subset X$ be such that for all κ-small compact Hausdorff spaces S mapping to X, the preimage of V in S is closed. We need to see that V is closed in X. Let $x \in \overline{V}$ arbitrary. Since X is first-countable, x has a countable open neighborhood basis $\{B_j\}_{j \in \mathbb{N}}$. Set $U_n := \bigcap_{i=1}^n B_i$ such that for all $n \in \mathbb{N}$ we have that $U_{n+1} \subset U_n$. For all $n \in \mathbb{N}$ choose $x_n \in V \cap U_n$ which is possible since $x \in \overline{V}$. The sequence $\{x_n\}_{n \in \mathbb{N}}$ converges to x and thus defines a continuous map $\mathbb{N} \cup \{\infty\} \to X$ by sending n to x_n and ∞ to x. The preimage of V is by assumption closed in $\mathbb{N} \cup \{\infty\}$ and contains $\mathbb{N}$, hence also ∞ (closed sets either contain ∞ or are finite). But then $x \in V$, which means that V is closed.

If T is a κ-condensed set, we equip the underlying set $T(*)$ with a topology in the following way. Let S be a κ-small profinite set. We want maps coming from elements of $T(S) = \mathrm{Hom}(\underline{S}, T)$ (Yoneda) to be continuous. Hence we equip $T(*)$ with the final topology for the maps $\underline{S}(*) \to T(*)$ where $\underline{S}(*) \cong S$ is endowed with the topology of S. We obtain a topological space which we denote by $T(*)_{\mathrm{top}}$.

The next proposition tells us that κ-compactly generated topological spaces embed fully faithfully into κ-condensed sets. By the previous example this already covers a large class of topological spaces that typically come up in practice and the transition comes with no loss.

Proposition 6.5. *The functor $X \mapsto \underline{X}$ from topological spaces/rings/groups/... to the category of κ-condensed sets/rings/groups/... is faithful, and fully faithful when restricted to the full subcategory of all X that are κ-compactly generated as a topological space.*
The functor $X \mapsto \underline{X}$ from topological spaces to κ-condensed sets admits a left adjoint $T \mapsto T(*)_{top}$ sending any κ-condensed set T to the topological space $T(*)_{top}$.
The counit $\underline{X}(*)_{top} \to X$ of the adjunction agrees with the counit $X^{\kappa-cg} \to X$ of the adjunction between κ-compactly generated spaces and all topological spaces; in particular, $\underline{X}(*)_{top} \cong X^{\kappa-cg}$.

Proof. Let us first see why the functor $X \mapsto \underline{X}$ is faithful. A morphism $f : X \to Y$ is mapped to a natural transformation $\underline{f} : \underline{X} \to \underline{Y}$. Given a κ-small profinite set S, the morphism $\underline{f}(S) : \underline{X}(S) \to \underline{Y}(S)$ is given by composition with f. Now assume we have $f, g : X \to Y$ such that $\underline{f} = \underline{g} : \underline{X} \to \underline{Y}$. In particular, $\underline{f}(*) = \underline{g}(*) : \underline{X}(*) \to \underline{Y}(*)$. But this means that $f = g$ as we can identify $\underline{X}(*) = \mathrm{Hom}_{\mathrm{Cont}}(*, X)$ with the underlying set of X.
Next, let us restrict to κ-compactly generated topological spaces and show that $X \mapsto \underline{X}$ is also full. Assume $f : \underline{X} \to \underline{Y}$ is a natural transformation, then $f(*) : \underline{X}(*) \to \underline{Y}(*)$ gives rise to a map $g : X \to Y$ as follows. If $x \in X$, there is a unique $\gamma_x : \{*\} \to X$ such that $\gamma_x(*) = x$. Likewise, $f(*)(\gamma_x) : \{*\} \to Y$ corresponds to an element $y \in Y$, the map g is then given by $g(x) := (f(*)(\gamma_x))(*)$. Now let S be a κ-small profinite set and $\gamma_s : \{*\} \to S$ corresponding to an element $s \in S$. As f is a natural transformation we have a commutative diagram:

$$\begin{tikzcd}
\underline{X}(S) \arrow[r, "f(S)"] \arrow[d, "\underline{X}(\gamma_s)"'] & \underline{Y}(S) \arrow[d, "\underline{Y}(\gamma_s)"] \\
\underline{X}(*) \arrow[r, "f(*)"] & \underline{Y}(*)
\end{tikzcd}$$

As $s \in S$ was arbitrary, the diagram shows that $f(S)$ is given by composition with g. Indeed, if $\alpha \in \underline{X}(S)$, then

$$\begin{aligned}
(f(S)(\alpha))(s) &= (f(S)(\alpha))(\gamma_s(*)) \\
&= ((f(S)(\alpha)) \circ \gamma_s)(*) \\
&= (\underline{Y}(\gamma_s)(f(S)(\alpha)))(*) \\
&= ((\underline{Y}(\gamma_s) \circ f(S))(\alpha))(*) \\
&= ((f(*) \circ \underline{X}(\gamma_s))(\alpha))(*) \\
&= (f(*)(\underline{X}(\gamma_s)(\alpha)))(*) \\
&= (f(*)(\alpha \circ \gamma_s))(*) \\
&= g(\alpha(s)) \\
&= (g \circ \alpha)(s).
\end{aligned}$$

This means two things. On the one hand this shows that g is continuous since X is κ-compactly generated and because $S \to X \xrightarrow{g} Y$ is continuous for all κ-small profinite sets S mapping continuously to X as $f(S)(\alpha) = g \circ \alpha \in \underline{Y}(S)$ for all $\alpha \in \underline{X}(S)$. On the other hand, this means that $f = \underline{g}$ by definition of $\underline{g}$.

Now let X be a topological space and T be a κ-condensed set. We want to show that there is a bijection $\mathrm{Hom}(T, \underline{X}) \to C(T(*)_{\mathrm{top}}, X)$ that is natural in X and T.
We do this by constructing maps

$$\begin{aligned} \Phi &: C(T(*)_{\mathrm{top}}, X) \to \mathrm{Hom}(T, \underline{X}) \\ \Psi &: \mathrm{Hom}(T, \underline{X}) \to C(T(*)_{\mathrm{top}}, X), \end{aligned}$$

that are inverse to each other. Given a continuous map $T(*)_{\mathrm{top}} \to X$, we want to give a morphism $T \to \underline{X}$. For this let $S \in *_{\kappa\text{-proét}}$. By the Yoneda lemma, $T(S) = \mathrm{Hom}(\underline{S}, T)$. Hence any $\alpha \in T(S)$ induces a map $\alpha(*) : \underline{S}(*) \to T(*)_{\mathrm{top}}$ that is continuous by definition of the topology on $T(*)_{\mathrm{top}}$. In particular, the composite $S \cong \underline{S}(*) \to T(*)_{\mathrm{top}} \to X$ is continuous and hence an element of $\underline{X}(S)$. This assignment defines a morphism $T \to \underline{X}$. Indeed, for any $\beta : S' \to S$ we have a commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}(\underline{S}, T) \arrow[r] \arrow[d] & \underline{X}(S) \arrow[d] \\ \mathrm{Hom}(\underline{S'}, T) \arrow[r] & \underline{X}(S') \end{tikzcd}$$

Now suppose we have a morphism $f : T \to \underline{X}$ and let S be κ-small profinite set with a morphism $g : \underline{S} \to T$. We would like to know that the induced map $T(*) \to \underline{X}(*) = X$ is continuous for the topology introduced on $T(*)_{\mathrm{top}}$. By naturality, for any $s \in S$ we get a corresponding map $\gamma_s : \{*\} \to S$ and thus a commutative diagram:

$$\begin{tikzcd} \underline{S}(S) \arrow[r, "g(S)"] \arrow[d, "\underline{S}(\gamma_s)"'] & T(S) \arrow[r, "f(S)"] \arrow[d, "T(\gamma_s)"] & \underline{X}(S) \arrow[d, "\underline{X}(\gamma_s)"] \\ \underline{S}(*) \arrow[r, "g(*)"] & T(*) \arrow[r, "f(*)"] & \underline{X}(*) \end{tikzcd}$$

As above, this shows that $f(S) \circ g(S)$ is given by composition with $f(*) \circ g(*)$. Hence the continuous map $(f(S) \circ g(S))(id_S)$ is equal to the composition $\underline{S}(*) \to T(*) \to X$, which means that the map $T(*)_{\mathrm{top}} \to X$ is continuous by definition of the topology on $T(*)_{\mathrm{top}}$.
Moreover, the constructions involved are inverse to each other. Indeed, given a map $f : T(*)_{\mathrm{top}} \to X$ we are given another map $(\Psi \circ \Phi)(f) : T(*)_{\mathrm{top}} \to X$. By construction, the latter sends $t \in T(*)$ to the composition $\{*\} \xrightarrow{\gamma_t} T(*)_{\mathrm{top}} \xrightarrow{f} X$ that corresponds to the element $f(t) \in X$. Hence, $f = (\Psi \circ \Phi)(f)$. Likewise, given a morphism $f : T \to \underline{X}$ we are given another morphism $(\Phi \circ \Psi)(f) : T \to \underline{X}$. The map $f(*) : T(*) \to \underline{X}(*)$ obtained by evaluation at a point induces a map $T(*) \to X$ which we still denote $f(*)$. Let S be a κ-small profinite set and let $\gamma_s : \{*\} \to S$ correspond to an element $s \in S$. We then have a commutative diagram:

$$\begin{tikzcd} T(S) \arrow[r, "f(S)"] \arrow[d, "T(\gamma_s)"'] & \underline{Y}(S) \arrow[d, "\underline{X}(\gamma_s)"] \\ T(*) \arrow[r, "f(*)"] & \underline{X}(*) \end{tikzcd}$$

As before the diagram shows that $f(S)$ maps an element $\alpha \in T(S)$ to the composition $\underline{S}(*) \xrightarrow{\alpha(*)} T(*) \xrightarrow{f(*)} X$ which is how $(\Psi \circ \Phi)(f)(S)$ is defined. So $(\Phi \circ \Psi)(f) = f$. The statement about the counits is immediate because both are given by identities on the components. In particular, by definition, we have a natural identification $\underline{X}(*)_{\text{top}} \xrightarrow{\sim} X^{\kappa-\text{cg}}$ because elements of $\underline{X}(S)$ are exactly the continuous maps $S \to X$ and because the topology on $X^{\kappa-\text{cg}}$ is already determined by κ-small profinite sets.

$\square$

Remark 6.6. We could have formally deduced the first statement from the remaining proposition. Indeed,

$$\operatorname{Hom}(\underline{X}, \underline{Y}) = C(\underline{X}(*)_{\text{top}}, Y) = C(X^{\kappa-\text{cg}}, Y) \hookrightarrow C(X, Y).$$

The last arrow is a bijection if X is κ-compactly generated.

7 Condensed sets

In this section we focus on the appendix to the second lecture in [7]. The main goal of this section is to give a definition of *condensed sets* which is independent of the cut off cardinal κ.

Definition 7.1. Let κ be an infinite cardinal. The *cofinality* of κ is the least cardinal λ such that there exists an index set I of cardinality λ, cardinals $\lambda_i < \kappa$ such that the λ_i sum up to κ.

If I is an index set and if S_i for $i \in I$ are pairwise disjoint sets of cardinality κ_i, then $\sum_{i\in I} \kappa_i$ is defined to be the cardinality of the disjoint union of the S_i. Hence, because any cardinal κ is the disjoint union of κ singletons, it is clear that $\lambda \leq \kappa$. Moreover, by Theorem 9S in [11] the cofinality λ of an infinite cardinal κ is always *regular*, i.e. the cofinality of λ is equal to λ.

Definition 7.2. Let λ be a regular cardinal. A λ-filtered category is a category $\mathcal{C}$ such that any diagram $I \to \mathcal{C}$ has a cocone when I has less than λ arrows.

Definition 7.3. If M is any set of cardinals, the *supremum* of M is $\sup_{\lambda\in M} \lambda := \bigcup_{\lambda \in M} \lambda$.

Remark 7.4. The supremum is a cardinal itself (cf. Section 3.6 in [22]).

In the proof of the following lemma we need some facts about cardinal arithmetic. They can be found in [11]. Namely, Theorem 6I and Theorem 6L.

Lemma 7.5. *Let κ be an uncountable strong limit cardinal and let λ be its cofinality. Let I be a set and for $i \in I$ let S_i be a κ-small profinite set. Suppose that $\eta := |I| < \lambda$, then $\prod_{i\in I} S_i$ is κ-small.*

Proof. We already know that the claim is true if I is finite. Thus, in the following we assume that I is infinite. Let μ be the supremum of all $|S_i|$, i.e. if $\lambda_i := |S_i|$, then $\mu = \bigcup_{i\in I} \lambda_i$. It is then clear that $\mu < \kappa$ because we have that $\eta < \lambda$ so that the λ_i can not sum up to κ by definition of cofinality. Hence

$$|\prod_{i\in I} S_i| \leq \mu^\eta \leq (2^\mu)^\eta = 2^{\mu\otimes\eta} < \kappa$$

because $\mu \otimes \eta = \max\{\mu, \eta\} < \kappa$ and because κ is a strong limit cardinal. □

Let κ be an uncountable strong limit cardinal. Let $\tilde{S}$ be a profinite set. Consider the category $I_{\tilde{S}}$ whose objects are all continuous maps $\tilde{S} \to S$ with S being a κ-small profinite set. A morphism $(\tilde{S} \xrightarrow{q_1} S_1) \to (\tilde{S} \xrightarrow{q_2} S_2)$ is given by a continuous map $S_2 \xrightarrow{\phi} S_1$ such that $q_1 = \phi \circ q_2$. It is straightforward to check that this category is always filtered. If we consider extremally disconnected sets, one can do even better.

Lemma 7.6. *Let κ be an uncountable strong limit cardinal and let $\tilde{S}$ be an extremally disconnected set. Then the category $I_{\tilde{S}}$ is λ-filtered where λ denotes the cofinality of κ.*

Proof. Let $I \to I_{\tilde{S}}$ be a diagram of continuous maps $\tilde{S} \xrightarrow{q_i} S_i$ with $|I| < \lambda$ (by abuse of notation). We need to check that the diagram has a cocone. Since the category of profinite sets is λ-filtered, we may form the limit $T := \varprojlim_{i\in I^{\mathrm{op}}} S_i$ in the category of profinite sets. Denote by p_i the canonical projections $T \to S_i$. By Lemma 7.5 we know that T is a κ-small profinite set. Hence, $S := T'$ is a κ-small extremally disconnected set and we may consider the canonical surjection $\pi : S \to T$. The maps $q_i : \tilde{S} \to S_i$ induce a unique continuous map $q : \tilde{S} \to T$ such that $q_i = p_i \circ q$. By projectivity of $\tilde{S}$ we obtain a continuous map $\psi : \tilde{S} \to S$ such that $q = \pi \circ \psi$. For $i \in I$ observe that $p_i \circ \pi : S \to S_i$ defines a morphism $(\tilde{S} \xrightarrow{q_i} S_i) \to (\tilde{S} \xrightarrow{\psi} S)$. Let $(\tilde{S} \xrightarrow{q_i} S_i) \to (\tilde{S} \xrightarrow{q_j} S_j)$ be a morphism given by $\phi : S_j \to S_i$ that comes from a morphism $i \to j$ in I. Then by definition of the limit $p_i = \phi \circ p_j$ which implies that $\phi \circ p_j \circ \pi = p_i \circ \pi$. Altogether, all $\tilde{S} \xrightarrow{q_i} S_i$ factor compatibly over $\tilde{S} \xrightarrow{\psi} S$ and thus, the diagram has a cocone. $\square$

Proposition 7.7. *Let $\kappa' > \kappa$ be uncountable strong limit cardinals. There is a natural functor from κ-condensed sets to κ'-condensed sets, given by sending a κ-condensed set T to the κ'-condensed set $T_{\kappa'}$ given by the sheafification of*

$$\tilde{S} \mapsto \varinjlim_{\tilde{S} \to S} T(S)$$

where the filtered colimit is taken over all κ-small profinite sets S with a map $\tilde{S} \to S$. The functor $T \mapsto T_{\kappa'}$ is fully faithful and commutes with all colimits and all λ-small limits where λ is the cofinality of κ.

Proof. By Corollary 5.19 we may identify the category of κ-condensed sets with functors

$$\{\kappa\text{-small extremally disconnected sets}\}^{\mathrm{op}} \to \{\text{sets}\}$$

sending finite disjoint unions to finite products and likewise, we may do the same for the category κ'-condensed sets. We claim that no sheafification is required in the definition of $T_{\kappa'}$. Indeed, define for a κ'-small extremally disconnected set $\tilde{S}$ the value on $\tilde{S}$ as $T'(\tilde{S}) := \varinjlim_{\tilde{S} \to S} T(S)$ (cf. Proposition 4.3). It is straightforward to check that this defines a presheaf. Let us now check the conditions from Proposition 5.14. Because there is a map $\emptyset \to \emptyset$ (hence an initial object), we have that $T'(\emptyset) = \{*\}$. Now let $\tilde{S}_1$ and $\tilde{S}_2$ be κ'-small extremally disconnected sets. We want to show that the natural map

$$T'(\tilde{S}_1 \coprod \tilde{S}_2) \to T'(\tilde{S}_1) \times T'(\tilde{S}_2)$$

is bijective. An element on the right hand side corresponds to elements $x_1 \in T(S_1)$ and $x_2 \in T(S_2)$ for κ-small extremally disconnected sets S_1 and S_2 with continuous maps $\tilde{S}_i \xrightarrow{\psi_i} S_i$ for $i = 1, 2$. We know that the natural map

$$T(S_1 \coprod S_2) \to T(S_1) \times T(S_2)$$

is bijective. Hence, we obtain an element $x \in T(S_1 \coprod S_2)$ such that $T(\phi_{S_i})(x) = x_i$ for $i = 1, 2$ where ϕ_{S_i} denotes the natural inclusion $S_i \to S_1 \coprod S_2$. We have a natural map $\tilde{S}_1 \coprod \tilde{S}_2 \to S_1 \coprod S_2$ such that x gives rise to an element of $T'(\tilde{S}_1 \coprod \tilde{S}_2)$. In fact, $T(\phi_{S_i})(x) = x_i = T(id_{S_i})(x_i)$ for $i = 1, 2$ implies that the class of x gets mapped to the class of x_i under the natural map $T'(\tilde{S}_1 \coprod \tilde{S}_2) \to T'(S_i)$ for $i = 1, 2$. This shows that the map in question is surjective. Let us now verify that it is also injective. Suppose that S_1 and S_2 are κ-small extremally disconnected sets with continuous maps $\tilde{S}_1 \coprod \tilde{S}_2 \xrightarrow{\psi_j} S_j$ and let $x_j \in T(S_j)$ for $j = 1, 2$. Assume that the classes of x_1 and x_2

get mapped to the same element in $T'(\tilde{S}_1) \times T'(\tilde{S}_2)$. This implies that the classes of x_1 and x_2 agree in $T'(S_1)$ as well as in $T'(S_2)$. Hence, we obtain morphisms

$$(\tilde{S}_1 \xrightarrow{\psi_i \circ \phi_{\tilde{S}_1}} S_i) \xrightarrow{f_i} (\tilde{S}_1 \xrightarrow{\varphi_1} Y_1)$$

such that $T(f_1)(x_1) = T(f_2)(x_2)$ and

$$(\tilde{S}_2 \xrightarrow{\psi_i \circ \phi_{\tilde{S}_2}} S_i) \xrightarrow{g_i} (\tilde{S}_2 \xrightarrow{\varphi_2} Y_2)$$

such that $T(g_1)(x_1) = T(g_2)(x_2)$. The compositions $\tilde{S}_i \to Y_i \to Y_1 \coprod Y_2$ give rise to a continuous map

$$\tilde{S}_1 \coprod \tilde{S}_2 \to Y_1 \coprod Y_2.$$

We obtain morphisms

$$(\tilde{S}_1 \coprod \tilde{S}_2 \to S_i) \xrightarrow{h_i} (\tilde{S}_1 \coprod \tilde{S}_2 \to Y_1 \coprod Y_2)$$

given by $h_i := (f_i, g_i) : Y_1 \coprod Y_2 \to S_i$ such that $T(h_1)(x_1) = T(h_2)(x_2)$. This implies that x_1 and x_2 define the same element in $T'(\tilde{S}_1 \coprod \tilde{S}_2)$. Hence, T' is already a sheaf. Moreover, the functor $T \mapsto T_{\kappa'}$ is left adjoint to the forgetful functor from the category of sheaves of sets on the site of κ'-small extremally disconnected sets to the category of sheaves of sets on the site of κ-small extremally disconnected sets (cf. Lemma 7.5.4 in [22]). The unit of the adjunction can be identified with the identity transformation and is therefore an isomorphism. By Lemma 4.24.4 in [22] this means that the functor $T \mapsto T_{\kappa'}$ is fully faithful. Since left adjoints commute with colimits, we also see that the functor $T \mapsto T_{\kappa'}$ commutes with colimits. Using Lemma 7.6 we conclude that the functor $T \mapsto T_{\kappa'}$ also commutes with λ-small limits because λ is regular (cf. Satz 5.2 in [13]). □

Remark 7.8. The last proposition applies to all categories of interest for us such as condensed rings/groups/modules/... (cf. Remark 2.10 in [7]).

Definition 7.9. Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be categories. Suppose $T : \mathcal{A} \to \mathcal{C}$ and $\iota : \mathcal{A} \to \mathcal{B}$ are functors. A *left Kan extension* of T along ι consists of a functor $L : \mathcal{B} \to \mathcal{C}$ and a natural transformation $\eta : T \to L \circ \iota$ satisfying the universal property that whenever $X : \mathcal{B} \to \mathcal{C}$ is a functor and $\alpha : T \to X \circ \iota$ is a natural transformation there exists a unique natural transformation $\sigma : L \to X$ such that $\alpha(A) = \sigma(\iota(A)) \circ \eta(A)$ for all objects A of $\mathcal{A}$.

Remark 7.10. In our setting one can describe the left Kan extension in terms of colimits. This is due to Theorem 3.7.2 in [3]. For example, let $\kappa' > \kappa$ be uncountable strong limit cardinals. If T is a κ-condensed set and ι_κ the inclusion from the opposite category of κ-small extremally disconnected sets into the opposite category of κ'-small extremally disconnected sets, then T' defined as in the proof of Proposition 7.7 is the left Kan extension of T along ι_κ. The natural transformation $\eta : T \to T' \circ \iota_\kappa$ is given by the natural choice. In the same way one can define left Kan extensions along the inclusion from the opposite category of κ-small extremally disconnected sets into the opposite category of extremally disconnected sets. This means that the functor $T \mapsto T_{\kappa'}$ corresponds to left Kan extension along the inclusion of κ-small extremally disconnected sets into κ'-small extremally disconnected sets.

We are now ready to give the definition of the category of *condensed sets/rings/groups/...*.

Definition 7.11. The category of condensed sets/rings/groups/... is given by the filtered colimit of the category of κ-condensed sets along the filtered poset of all uncountable strong limit cardinals.

Equivalently, the category of condensed sets/rings/groups/... is the category of functors

$$T : \{\text{extremally disconnected sets}\}^{\mathrm{op}} \to \{\text{sets/rings/groups/...}\}$$

such that $T(\emptyset) = \{*\}$ and for all extremally disconnected sets S_1, S_2, the natural map $T(S_1 \coprod S_2) \to T(S_1) \times T(S_2)$ is a bijection, and such that for some uncountable strong limit cardinal κ it is the left Kan extension of its restriction to κ-small extremally disconnected sets (cf. Definition 2.11 in [7]).

Remark 7.12. The category of condensed sets/rings/groups/modules/... is not a topos, i.e. not the category of sheaves of sets/rings/groups/modules/... on any site but it shares many of its features (cf. Remark 2.12 in [7]). The category of condensed sets/rings/groups/modules/... has all colimits and all limits. Limits and filtered colimits can even be computed pointwise.
Moreover, the category of condensed sets/rings/groups/modules/... is locally small. Fore more details see the discussion on page 15 of [7].

Let $f : \tilde{S} \to X$ be a morphism in a category $\mathcal{C}$. There is an associated category I of factorizations of f. An object of I is a pair (h, g) of morphisms $h : \tilde{S} \to S$ and $g : S \to X$ in $\mathcal{C}$ such that $f = g \circ h$. A morphism $h : (h_1, g_1) \to (h_2, g_2)$ is a morphism $h : S_2 \to S_1$ in $\mathcal{C}$ such that the following diagram is commutative:

$$\begin{tikzcd} & S_1 \arrow[dr, "g_1"] & \\ \tilde{S} \arrow[ur, "h_1"] \arrow[r, "h_2"] & S_2 \arrow[u, "h"] \arrow[r, "g_2"] & X \end{tikzcd}$$

For more details on this construction see Definition 3.1 in [20].

Lemma 7.13. *Let X be a topological space. Assume that there is some uncountable strong limit cardinal κ such that for all extremally disconnected sets $\tilde{S}$ any continuous map $\tilde{S} \to X$ factors over a continuous map $S \to X$ with S κ-small extremally disconnected. Moreover, suppose that for all extremally disconnected sets $\tilde{S}$ the category of factorizations $\tilde{S} \to S \to X$ with S being κ-small is filtered. Then the functor $S \mapsto \underline{X}(S) = C(S, X)$ is a condensed set, namely, $\underline{X}$ is the left Kan extension of its restriction $\underline{X}_\kappa$ to κ-small extremally disconnected sets.*

Proof. It is clear that $\underline{X}(\emptyset) = \{*\}$ and that $\underline{X}(S_1 \coprod S_2) = \underline{X}(S_1) \times \underline{X}(S_2)$ for extremally disconnected sets S_1 and S_2. Denote by ι_κ the inclusion from κ-small extremally disconnected sets to extremally disconnected sets. Let $\tilde{S}$ be an extremally disconnected set. We claim that the natural map

$$\varinjlim_{\tilde{S} \to S} \underline{X}_\kappa(S) \xrightarrow{t} \underline{X}(\tilde{S})$$

is bijective. The map t sends the class of a continuous map $S \to X$ corresponding to the index $\tilde{S} \to S$ to the composition $\tilde{S} \to S \to X$. By assumption, any continuous map

$\tilde{S} \to X$ factors over some κ-small extremally disconnected set S. Hence, t is surjective. Let us now check that t is injective. Let $S_1 \xrightarrow{g_1} X$ and $S_2 \xrightarrow{g_2} X$ be continuous maps corresponding to the indices $\tilde{S} \xrightarrow{\phi_1} S_1$ and $\tilde{S} \xrightarrow{\phi_2} S_2$, respectively. Assume that their classes get mapped to the same element in $\underline{X}(\tilde{S})$, i.e. $g_1 \circ \phi_1 = g_2 \circ \phi_2$. By assumption on the category of factorizations we are given a third factorization $\tilde{S} \xrightarrow{\phi} S \xrightarrow{g} X$ and two continuous maps $S \xrightarrow{f_i} S_i$ such that the diagram

$$\begin{tikzcd}
& S_1 \arrow[dr, "g_1"] & \\
\tilde{S} \arrow[ur, "\phi_1"] \arrow[r, "\phi"] \arrow[dr, "\phi_2"'] & S \arrow[u, "f_1"'] \arrow[r, "g"] \arrow[d, "f_2"] & X \\
& S_2 \arrow[ur, "g_2"'] &
\end{tikzcd}$$

is commutative. For $i = 1, 2$ this diagram defines morphisms $(\tilde{S} \xrightarrow{\phi_i} S_i) \to (\tilde{S} \xrightarrow{\phi} S)$ given by the f_i. In particular, the commutativity implies that

$$\underline{X}_\kappa(f_1)(g_1) = g_1 \circ f_1 = g = g_2 \circ f_2 = \underline{X}_\kappa(f_2)(g_2).$$

Hence, the classes of g_1 and g_2 agree and t is injective as claimed. $\square$

Proposition 7.14. *If X is a topological space all of whose points are closed, then there exists some uncountable strong limit cardinal κ of cofinality λ such that for all extremally disconnected sets $\tilde{S}$ any continuous map $\tilde{S} \to X$ factors over a continuous map $S \to X$ with S κ-small extremally disconnected. Moreover, for all extremally disconnected sets $\tilde{S}$ the category of factorizations $\tilde{S} \to S \to X$ with S κ-small is λ-filtered.*

Proof. As a first step we show that if $\tilde{S}$ is an extremally disconnected set with a continuous map $\tilde{S} \xrightarrow{g} X$ such that there exists a κ-small extremally disconnected set S with a continuous map $\tilde{S} \xrightarrow{f} S$ such that the two continuous maps

$$\begin{tikzcd}
\tilde{S} \times_S \tilde{S} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \tilde{S} \arrow[r, "g"] & X
\end{tikzcd}$$

agree, then there is a κ-small extremally disconnected set T and continuous maps $\tilde{S} \xrightarrow{k} T$ and $T \xrightarrow{h} X$ such that $g = h \circ k$. Consider the profinite space $f(\tilde{S})$. We define the map $h' : f(\tilde{S}) \to X$ by setting $h'(f(s)) = g(s)$. By assumption this map is well-defined. Let $U \subseteq X$ be open. Since g is continuous, we know that $g^{-1}(U) \subseteq \tilde{S}$ is also open. Hence, $A := \tilde{S} \backslash g^{-1}(U)$ is closed. This implies that $f(A) \subseteq f(\tilde{S})$ is closed. Moreover, we have that $f(A) = f(\tilde{S}) \backslash h'^{-1}(U)$ and thus $h'^{-1}(U)$ is open, i.e. h' is continuous. Set $T := f(\tilde{S})'$ and let $\pi_{f(\tilde{S})} : T \to f(\tilde{S})$ be the natural surjection. By projectivity, there is a continuous map $\tilde{S} \xrightarrow{k} T$ such that $f = \pi_{f(\tilde{S})} \circ k$. Let $h := h' \circ \pi_{f(\tilde{S})} : T \to X$. Then $h \circ k = h' \circ \pi_{f(\tilde{S})} \circ k = h' \circ f = g$ as required.

Now let κ be an uncountable strong limit cardinal with cofinality $\lambda > |X|$. Let $\tilde{S}$ be an extremally disconnected set and let $\tilde{S} \xrightarrow{g} X$ be a continuous map. We claim that there exists a κ-small extremally disconnected set T and a continuous map $f : \tilde{S} \to T$ such that the two continuous maps

$$\tilde{S} \times_T \tilde{S} \overset{p_1}{\underset{p_2}{\rightrightarrows}} \tilde{S} \xrightarrow{g} X$$

agree. Let $x, y \in X$ be distinct points. Let S be a κ-small extremally disconnected set with a continuous map $\tilde{S} \to S$. Note that by assumption $(x, y) \in X \times X$ is closed. Thus, the preimage $T_{x,y,S}$ of (x, y) under the induced map

$$\tilde{S} \times_S \tilde{S} \to X \times X$$

is closed in $\tilde{S} \times_S \tilde{S}$ and hence profinite itself. Let us construct an inverse system of these profinite sets indexed by $I_{\tilde{S}}$ (cf. 4.3 and 7.6). We say that $(\tilde{S} \to S_i) \leq (\tilde{S} \to S_j)$ if and only if there is a morphism $(\tilde{S} \to S_i) \to (\tilde{S} \to S_j)$. If $(\tilde{S} \xrightarrow{\phi_i} S_i) \leq (\tilde{S} \xrightarrow{\phi_j} S_j)$ we need a continuous map $f_{ij} : T_{x,y,S_j} \to T_{x,y,S_j}$. Consider the following commutative diagram

$$\begin{tikzcd}
\tilde{S} \times_{S_j} \tilde{S} \arrow[dr, dotted, "\exists! f_{ij}"] \arrow[drr, bend left, "p_2^j"] \arrow[ddr, bend right, "p_1^j"'] & & \\
& \tilde{S} \times_{S_i} \tilde{S} \arrow[r, "p_2^i"] \arrow[d, "p_1^i"'] & \tilde{S} \arrow[d, "\phi_i"] \\
& \tilde{S} \arrow[r, "\phi_i"] & S_i
\end{tikzcd}$$

where the p_k^l are the projections of the fiber product $\tilde{S} \times_{S_l} \tilde{S}$. One checks that f_{ij} is just the inclusion $T_{x,y,S_j} \to T_{x,y,S_i}$ and that the f_{ij} satisfy the compatibility conditions of an inverse system. This allows us to consider the inverse limit

$$\varprojlim_{\tilde{S} \to S} T_{x,y,S} = \{(t_{\tilde{S} \to S})_{\tilde{S} \to S} \in \prod_{\tilde{S} \to S} T_{x,y,S} : t_{\tilde{S} \to S} = t_{\tilde{S} \to T} \text{ for all } (\tilde{S} \to S) \leq (\tilde{S} \to T)\}.$$

Since the essentially small indexing category is directed, any sequence $(t_{\tilde{S} \to S})_{\tilde{S} \to S} \in \varprojlim_{\tilde{S} \to S} T_{x,y,S}$ must be constant using that the transition maps are inclusions. For some index $\tilde{S} \xrightarrow{\phi} S$ write $t_{\tilde{S} \to S} = (a, b)$. As $x = g(a)$ and $y = g(b)$ are distinct, so are a and b. Since $\tilde{S}$ is totally disconnected, we may write $\tilde{S} = U_a \coprod U_b$ where U_a and U_b are clopen neighborhoods of a and b, respectively. If S is a κ-small extremally disconnected set with at least two distinct elements, then there is a continuous map $\tilde{S} \xrightarrow{\psi} S$ with $\psi(a) \neq \psi(b)$. This implies that $(a, b) \notin T_{x,y,S}$ which is a contradiction, i.e. the inverse limit must be empty. We claim that one of the $T_{x,y,S_{(x,y)}}$ has to be empty. Indeed, otherwise the inverse limit $\varprojlim_{\tilde{S} \to S} T_{x,y,S}$ would be the intersection of the nonempty closed subspaces

$$\{(t_{\tilde{S} \to S})_{\tilde{S} \to S} \in \prod_{\tilde{S} \to S} T_{x,y,S} : t_{\tilde{S} \to T} = t_{\tilde{S} \to T'}\}$$

ranging over all $(\tilde{S} \to T) \leq (\tilde{S} \to T')$. As these closed subsets have the finite intersection property, the limit would be nonempty. Hence, there must be some κ-small extremally disconnected set $S_{(x,y)}$ with a continuous map $\tilde{S} \to S_{(x,y)}$ such that $T_{x,y,S_{(x,y)}} = \emptyset$. Note that if X is infinite, we have $\lambda > |X| = |X \times X|$ and if X is finite, so is $X \times X$ and thus $\lambda > |X \times X|$ because λ is infinite. As in the proof of Lemma 7.6, we may form the limit $\varprojlim_{\tilde{S} \to S_{(x,y)}} S_{(x,y)}$ where the limit is taken over all pairs of distinct x and y. The proof of the same lemma shows that $T := (\varprojlim_{\tilde{S}\to S_{(x,y)}} S_{(x,y)})'$ and the continuous map $\tilde{S} \xrightarrow{\psi} T$ are what we are looking for. Indeed, the two continuous maps

$$\tilde{S} \times_T \tilde{S} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \tilde{S} \xrightarrow{\ g\ } X$$

agree by construction because $T_{x,y,S} \subseteq T_{x,y,S_{(x,y)}} = \emptyset$ for all distinct $x, y \in X$ as all maps $\tilde{S} \to S_{(x,y)}$ are a composition $\tilde{S} \xrightarrow{\psi} T \to S_{(x,y)}$.

Finally, let us show that the category I of factorizations $\tilde{S} \to S \to X$ with S being κ-small is λ-filtered. Let $J \to I$ be a diagram of κ-small extremally disconnected sets S_j of compatible factorizations $\tilde{S} \xrightarrow{q_j} S_j \xrightarrow{g_j} X$ with $|J| < \lambda$. The diagram $J \to I$ gives rise to a diagram $J \to I_{\tilde{S}}$ and we can proceed as in the proof of Lemma 7.6 to obtain a cocone $\tilde{S} \xrightarrow{\psi} S \to X$. One checks directly that the required diagrams commute. □

Corollary 7.15. *If X is a topological space all of whose points are closed, then $\underline{X}$ is a condensed set.* □

Remark 7.16. Proposition 5.7 still holds because all the arguments in the proof go through in the new setting. This allows us for simplicity to still refer to Proposition 5.7.

Definition 7.17. We say that a condensed set T is *quasicompact* if there exists a profinite set S and an epimorphism $\underline{S} \to T$. A morphism $X \to T$ of condensed sets is *quasicompact* if for all profinite sets S with a morphism $\underline{S} \to T$ the fiber product $\underline{S} \times_T X$ is quasicompact as a condensed set.

Let T be a condensed set. As before, we make the underlying set of T into a topological space $T(*)_{\mathrm{top}}$ by declaring maps $S \cong \underline{S}(*) \to T(*)$ coming from elements of the set $T(S) = \mathrm{Hom}(\underline{S}, T)$ to be continuous, i.e. we equip $T(*)$ with the final topology for the maps $\underline{S}(*) \to T(*)$ where $\underline{S}(*) \cong S$ is endowed with the topology of S. Here S runs through κ-small profinite sets for any κ such that T is determined by its values on κ-small profinite sets. Let us briefly discuss why these topologies agree for different choices of κ. Suppose $\kappa' > \kappa$. Clearly, if a map $T(*)_{\mathrm{top},\kappa'} \to Y$ is continuous, then it is continuous as map $T(*)_{\mathrm{top},\kappa} \to Y$. Here, $T(*)_{\mathrm{top},\kappa'}$ and $T(*)_{\mathrm{top},\kappa}$ denote the topological spaces corresponding to κ' and κ, respectively. Suppose now that $T(*)_{\mathrm{top},\kappa} \to Y$ is continuous. We claim that it is continuous as map $T(*)_{\mathrm{top},\kappa'} \to Y$ as well. Let $\tilde{S}$ be a κ'-small profinite set with a morphism $\alpha : \underline{\tilde{S}} \to T$. We need to show that the induced composition $\tilde{S} \cong \underline{\tilde{S}}(*) \to T(*)_{\mathrm{top},\kappa'} \to Y$ is continuous. As T is determined by its values on κ-small profinite sets, we have that $T(\tilde{S}) = \varinjlim_{\tilde{S}\to S} T(S)$. Consequently, the morphism $\alpha : \underline{\tilde{S}} \to T$ factors over $\underline{S}$ for some κ-small profinite set S. This implies that $\tilde{S} \to T(*)_{\mathrm{top},\kappa'}$ factors over S. Hence, the composition $\tilde{S} \to T(*)_{\mathrm{top},\kappa'} \to Y$ is continuous.

Proposition 7.18. *If X is a topological space all of whose points are closed, then $\underline{X}$ is a condensed set for which all maps from points are quasicompact. Conversely, if T is a condensed set such that all maps from points are quasicompact, then $T(*)_{top}$ is a compactly generated space all of whose points are closed.*

Proof. By Corollary 7.15 we know that $\underline{X}$ is a condensed set. Let $g : \underline{\{x\}} \to \underline{X}$ and let S be a profinite set with a morphism $f : \underline{S} \to \underline{X}$, we need to show that the fiber product $\underline{S} \times_{\underline{X}} \underline{\{x\}}$ is quasicompact. By Example 4.14 we know that $\underline{S} \times_{\underline{X}} \underline{\{x\}} = \underline{S \times_X \{x\}}$. Notice that $S \times_X \{x\}$ identifies with the preimage of x under the induced map $S \to X$. Because singletons are closed in X, the preimage is closed as well. As closed subsets of profinite sets are profinite, the fiber product $S \times_X \{x\}$ is profinite. Hence, $\underline{S} \times_{\underline{X}} \underline{\{x\}}$ is quasicompact. Suppose now that T is a condensed set such that all maps from points are quasicompact. Note that $T(*)_{\text{top}}$ is compactly generated by definition of the topology on $T(*)_{\text{top}}$. Let S be a profinite set with a morphism $\underline{S} \to T$. By definition of the topology on $T(*)_{\text{top}}$, we need to show that the preimage of $x \in T(*)_{\text{top}}$ under the induced map $S \to T(*)_{\text{top}}$ is closed. Since $\underline{\{x\}} \to T$ is quasicompact, we know that the fiber product $\underline{S} \times_T \underline{\{x\}}$ is quasicompact. This implies that there is a profinite set S' and a surjection $\underline{S'} \to \underline{S} \times_T \underline{\{x\}}$. By Proposition 5.7 the induced map $S' \to S \times_{T(*)_{\text{top}}} \{x\}$ is then surjective because $\{*\}$ is extremally disconnected. Hence, S' surjects onto the preimage of $\{x\}$ in S which implies that the preimage of $\{x\}$ is compact and hence closed in S. This means that $\{x\}$ is closed. □

Corollary 7.19. *The functor $X \mapsto \underline{X}$ that sends a topological space X all of whose points are closed to the condensed set $\underline{X}$ admits a left adjoint $T \mapsto T(*)_{top}$ sending any condensed set T for which all maps from points are quasicompact to the topological space $T(*)_{top}$.*

Proof. This follows immediately from Proposition 7.18 and Proposition 6.5. □

Remark 7.20. $\underline{X}$ is never a condensed set if not all points of X are closed. Indeed, suppose X is a topological space such that $\underline{X}$ is a condensed set. Let $Y \subseteq X$ be a subspace. We claim that $\underline{Y}$ is a condensed set as well. Let $\tilde{S}$ be an extremally disconnected set. We know that the natural map

$$\varinjlim_{\tilde{S} \to S} \underline{X}_{\kappa}(S) \xrightarrow{t} \underline{X}(\tilde{S})$$

is a bijection because $\underline{X}$ is the left Kan extension of its restriction $\underline{X}_{\kappa}$ for some uncountable strong limit cardinal κ. We claim that that $\underline{Y}$ is the left Kan extension of its restriction $\underline{Y}_{\kappa}$. For this, we need to check that the natural map

$$\varinjlim_{\tilde{S} \to S} \underline{Y}_{\kappa}(S) \xrightarrow{t'} \underline{Y}(\tilde{S})$$

is bijective as well. We start with the surjectivity. Let $\tilde{S} \xrightarrow{f} Y$ be a continuous map. Then f gives rise to a continuous map $\tilde{S} \xrightarrow{f} Y \xrightarrow{\iota} X$ by composition with the inclusion $Y \xrightarrow{\iota} X$.
By the surjectivity of t we know that there is an index $\tilde{S} \xrightarrow{\phi} S$ and a continuous map $S \xrightarrow{g} X$ such that $g \circ \phi = \iota \circ f$. Consider the κ-small profinite set $T := \phi(\tilde{S})$. The continuous map $\tilde{S} \xrightarrow{\phi} T$ factors by projectivity of $\tilde{S}$ over the natural surjection $\pi_T : T' \to T$ from the κ-small extremally disconnected set T'. Hence, we obtain a continuous map $\tilde{S} \xrightarrow{\phi'} T'$ such that $\pi_T \circ \phi' = \phi$. We define $T' \xrightarrow{g'} Y$ as the composition $T' \xrightarrow{\pi_T} T \xrightarrow{g} Y$. By construction, the class of g' gets mapped to f. The injectivity of t' directly follows from the injectivity of t. Altogether, $\underline{Y}$ is a condensed set.

One can show that the Sierpinski space $X = \{s, \eta\}$ where $\{\eta\}$ is closed but $\{s\}$ is not does not give rise to a condensed set. By the above this implies that any topological space, not all of whose points are closed, also does not give rise to a condensed set either because it contains a Sierpinski space which would necessarily give rise to a condensed set (cf. Warning 2.14 in [7]).

Definition 7.21. We say that a condensed set T is *quasiseparated* if for any two profinite sets S_1 and S_2 with morphisms to T, the fiber product $\underline{S_1} \times_T \underline{S_2}$ is quasicompact. A morphism $X \to T$ of condensed sets is *quasiseparated* if the diagonal map $X \to X \times_T X$ is quasicompact.

Lemma 7.22. *The following statements are true:*

(i) *A condensed set T is quasiseparated if and only if for all morphisms $T_1 \to T$ and $T_2 \to T$ where T_1 and T_2 are quasicompact the fiber product $T_1 \times_{T'} T_2$ is quasicompact.*

(ii) *If T is a condensed set that is both quasicompact and quasiseparated, then $T(*)_{top}$ is a compact Hausdorff space.*

(iii) *If $T \to T'$ is a morphism of condensed set where both T and T' are quasicompact and quasiseparated, then $T \times_{T'} T$ is quasicompact and quasiseparated.*

Proof. (i) The if part is true because profinite sets give rise to quasicompact condensed sets. For the only if part let S_1 and S_2 be profinite sets with epimorphisms $\underline{S_1} \to T_1$ and $\underline{S_2} \to T_2$. Because T' is quasiseparated, we know that there is a profinite set S and an epimorphism $\underline{S} \to \underline{S_1} \times_{T'} \underline{S_2}$. Thus, it suffices to check that the morphism $\underline{S_1} \times_{T'} \underline{S_2} \to T_1 \times_{T'} T_2$ is an epimorphism. By Proposition 5.7 we may check this on extremally disconnected sets where it follows from the fact that both $\underline{S_1} \to T_1$ and $\underline{S_2} \to T_2$ are epimorphisms. Hence, $T_1 \times_{T'} T_2$ is quasicompact.

(ii) First note that we have an epimorphism $\underline{S} \to T$ where S is a profinite set. Because T is quasiseparated, $\underline{S} \times_T \underline{S}$ is quasicompact. Thus, there is a profinite set S' and an epimorphism $\underline{S'} \to \underline{S} \times_T \underline{S}$. By Proposition 5.7 this means that we have a surjection $S' \to S \times_{T(*)_{\mathrm{top}}} S \subset S \times S$. This implies that $S \times_{T(*)_{\mathrm{top}}} S \subset S \times S$ is closed. Because $\underline{S} \to T$ is an epimorphism, the diagram

$$\underline{S} \times_T \underline{S} \rightrightarrows \underline{S} \to T$$

is a coequalizer diagram (cf. the arguments for Lemma 7.11.3 in [22]). Hence, so is

$$\underline{S}(*) \times_{T(*)_{\mathrm{top}}} \underline{S}(*) \rightrightarrows \underline{S}(*) \to T(*)_{\mathrm{top}}.$$

Thus, $T(*)_{\mathrm{top}}$ is the quotient of $\underline{S}(*)$ by the closed equivalence relation $\underline{S}(*) \times_{T(*)_{\mathrm{top}}} \underline{S}(*)$ and therefore a compact Hausdorff space.

(iii) By part (i) it is enough to show that $T \times_{T'} T$ is quasiseparated. Let S_1 and S_2 be profinite sets with morphisms $\underline{S_1} \to T \times_{T'} T$ and $\underline{S_2} \to T \times_{T'} T$. We need to show that $\underline{S_1} \times_{T \times_{T'} T} \underline{S_2}$ is quasicompact. Note that we have

$$\begin{aligned} \underline{S_1} \times_{T \times_{T'} T} \underline{S_2} &= \underline{S_1} \times_{T \times T} \underline{S_2} \\ &= (\underline{S_1} \times_T \underline{S_2}) \times_{\underline{S_1} \times \underline{S_2}} (\underline{S_1} \times_T \underline{S_2}), \end{aligned}$$

which can be checked pointwise on extremally disconnected sets by Proposition 5.7. The latter is quasicompact because both fiber products $\underline{S_1} \times_T \underline{S_2}$ are and because $\underline{S_1} \times \underline{S_2}$ is quasiseparated. $\square$

The arguments in the following lemma are taken from Proposition IV.2.4 in [6].

Lemma 7.23. *Let $h : T \to T'$ be a morphism of condensed sets where T is quasicompact and where T' is both quasicompact and quasiseparated. Suppose that the induced map $h(*) : T(*)_{top} \to T'(*)_{top}$ is surjective. Then $h : T \to T'$ is an epimorphism.*

Proof. As T' is quasicompact there exists an epimorphism $\underline{S} \to T'$ where S is a profinite set. By Lemma 7.22 the fiber product $\underline{S} \times_{T'} T$ is quasicompact. Hence, there is a profinite set $\tilde{S}$ and an epimorphism $\underline{\tilde{S}} \to \underline{S} \times_{T'} T$. Because $\underline{S} \to T'$ is an epimorphism, Proposition 5.7 implies that the induced map $\underline{S}(*) \to T'(*)_{\mathrm{top}}$ is surjective. The projection $\underline{S}(*) \times_{T'(*)_{\mathrm{top}}} T(*)_{\mathrm{top}} \to \underline{S}(*)$ is surjective because $T(*)_{\mathrm{top}} \to T'(*)_{\mathrm{top}}$ is surjective. Hence, the composition

$$\underline{\tilde{S}}(*) \to \underline{S}(*) \times_{T'(*)_{\mathrm{top}}} T(*)_{\mathrm{top}} \to \underline{S}(*)$$

is surjective. By Example 4.14 we know that $\underline{\tilde{S}} \to \underline{S}$ is an epimorphism. This implies that the composition $\underline{\tilde{S}} \to \underline{S} \to T'$ is an epimorphism. By construction we have a commutative diagram:

$$\begin{tikzcd}
\underline{\tilde{S}} \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \underline{S} \times_{T'} T \arrow[r] \arrow[d] & T \arrow[d] \\
& \underline{S} \arrow[r] & T'
\end{tikzcd}$$

Hence, the composition $\underline{\tilde{S}} \to \underline{S} \to T'$ agrees with the composition $\underline{\tilde{S}} \to T \xrightarrow{h} T'$. This implies that the latter is an epimorphism. But this implies that h is an epimorphism. $\square$

Corollary 7.24. *Let $h : T \to T'$ be a morphism of condensed sets where both T and T' are quasicompact and quasiseparated. Suppose that $h(*) : T(*)_{top} \to T'(*)_{top}$ is an isomorphism. Then $h : T \to T'$ is an isomorphism.*

Proof. Lemma 7.23 implies that h is an epimorphism. By Proposition 5.7 it is enough to show that h is also a monomorphism. Since a morphism is a monomorphism if and only if the diagonal is an isomorphism, we can focus on the diagonal. The diagonal is always a monomorphism. Hence, it is enough to show that the diagonal $T \to T \times_{T'} T$ is an epimorphism. But because $h(*) : T(*)_{\mathrm{top}} \to T'(*)_{\mathrm{top}}$ is injective, we know that the diagonal $T(*)_{\mathrm{top}} \to T(*)_{\mathrm{top}} \times_{T'(*)_{\mathrm{top}}} T(*)_{\mathrm{top}}$ is an isomorphism. By Lemma 7.22 (iii) and Lemma 7.23 we conclude that the diagonal is an epimorphism and hence an isomorphism. $\square$

Definition 7.25. We say that a topological space X is *weak Hausdorff* if for any compact Hausdorff space S mapping to X, the image is compact Hausdorff (in the subspace topology).

The adjunction in Corollary 7.19 lets us translate some of the introduced properties a condensed set can have.

Theorem 7.26. *The following statements are true:*

(i) *The functor $X \mapsto \underline{X}$ induces an equivalence of categories between compact Hausdorff spaces X and condensed sets T that are quasicompact and quasiseparated.*

(ii) *A compactly generated space X is weak Hausdorff if and only if $\underline{X}$ is quasiseparated. For any quasiseparated condensed set T, the topological space $T(*)_{top}$ is compactly generated weak Hausdorff.*

Proof. (i) We start by claiming that the functor $X \mapsto \underline{X}$ maps any compact Hausdorff space to a quasicompact and quasiseparated condensed set. If X is a compact Hausdorff space, let $\pi_X : X' \to X$ be the natural surjection. We claim that the induced morphism $\underline{X'} \to \underline{X}$ is an epimorphism. By Proposition 5.7 we need to show that for any extremally disconnected set S the map $\underline{X'}(S) \to \underline{X}(S)$ is surjective. But this is true because S is extremally disconnected and hence any continuous map $S \to X$ factors over the surjection $X' \to X$. Hence, $\underline{X}$ is quasicompact. To see that $\underline{X}$ is quasiseparated let S_1 and S_2 be profinite sets with morphisms to $\underline{X}$. Because $S_1 \times_X S_2$ is a compact Hausdorff space, we know by the above argument that $\underline{S_1} \times_{\underline{X}} \underline{S_2}$ is quasicompact. Hence, $\underline{X}$ is both quasicompact and quasiseparated. Let us now show that any quasicompact and quasiseparated condensed set T is isomorphic to $\underline{X}$ for some compact Hausdorff space X. Indeed, by Lemma 7.22 (ii) we know that $T(*)_{\text{top}}$ is a compact Hausdorff space. By Corollary 7.19 the identity $T(*)_{\text{top}} \to T(*)_{\text{top}}$ gives rise to a morphism $T \to \underline{T(*)_{\text{top}}}$ whose evaluation on $*$ is the identity. Since the identity is an isomorphism, Lemma 7.24 implies that $T \to \underline{T(*)_{\text{top}}}$ is an isomorphism.

(ii) Suppose that X is weak Hausdorff. Any point $x \in X$ is the image of the continuous map $\{*\} \to X, * \mapsto x$ and hence is closed. By Corollary 7.15 we know that $\underline{X}$ is a condensed set. Let us now show that $\underline{X}$ is quasiseparated. For this let S_1 and S_2 be profinite sets with morphisms $\underline{S_1} \to \underline{X}$ and $\underline{S_2} \to \underline{X}$. Note that $\underline{S_1} \times_{\underline{X}} \underline{S_2} = \underline{S_1 \times_X S_2}$ (cf. Example 4.14). By (i) it is enough to show that $S_1 \times_X S_2$ is a compact Hausdorff space. Consider the continuous map $S_1 \coprod S_2 \to X$ and denote its image by Y. We then have $S_1 \times_X S_2 = S_1 \times_Y S_2$ and since Y is a compact Hausdorff space by assumption, so is $S_1 \times_Y S_2$. This implies that $\underline{X}$ is quasiseparated. The converse is shown in the proof of Theorem 2.16 in [7]. □

8 κ-Condensed abelian groups

By definition, a κ-condensed abelian group is a sheaf of abelian groups on the site $*_{\kappa\text{-proét}}$. By using Corollary 5.19 we identify the category of κ-condensed abelian groups with the category of sheaves of abelian groups on the site of κ-small extremally disconnected sets. Using the simple description of sheaves in Corollary 5.19, we will see that κ-condensed abelian groups form a particularly nice abelian category (cf. Theorem 8.32). They satisfy the same of Grothendieck's axioms as the category of abelian groups and are even a Grothendieck abelian category. However the generator comes from κ-small extremally disconnected sets. In particular, the discrete abelian group $\underline{\mathbb{Z}}$ is not a generator, in contrast to the category of abelian groups (cf. Example 8.31). The existence of *compact projective generators* is the key to many of the good properties of the category (cf. Remark 8.33). We present two different proofs for their existence. One is taken from the original lecture notes on condensed mathematics [7], the other one makes use of the characterization of families of projective generators in Corollary 2.43. This section is mostly based on the lecture notes of Clausen and Scholze in [7].

Our first goal is to show that the category of κ-condensed abelian groups is in fact an abelian category. Although this is true for abelian sheaves on any site (cf. Section 18.3 in [22]), we do this in a very concrete manner to get a good feeling for κ-condensed abelian groups. Here, as indicated, we make use of Corollary 5.19 throughout this section, i.e. we use the description of κ-condensed abelian groups as sheaves on the site of κ-small extremally disconnected sets. In particular, we often do not need sheafification which otherwise we would in the profinite setting, that is, arguments are simpler. The usual instances where this might happen are listed in Section 18.3 in [22]. Nevertheless, to make things easier, we will still speak of κ-condensed abelian groups. Finally, we note that both approaches give rise to the same abelian category, i.e. the structure of an abelian category on the category of shaves of abelian groups on the site of κ-small extremally disconnected sets gives rise to the same structure of an abelian category on the category of κ-condensed abelian groups described in Section 18.3 in [22] via the equivalence in Corollary 5.19. This is true because the pre-additive structure on a category is necessarily unique (cf. Lemma 8.2.14 in [17]).

Suppose we are given two morphisms of κ-condensed abelian groups $\eta, \epsilon : M \to M'$. We define the sum of η and ϵ as one would expect. Namely, if S is a κ-small extremally disconnected set, then $(\eta + \epsilon)(S) := \eta(S) + \epsilon(S) : M(S) \to M'(S)$. This addition makes $\mathrm{Hom}(M, M')$ into an abelian group and the composition law is clearly bilinear. By Proposition 4.12, the category of κ-condensed sets admits products. With the same arguments we see that the category of κ-condensed abelian groups also admits products. As a consequence, we obtain the following statement.

Corollary 8.1. *The category of κ-condensed abelian groups is an additive category.* $\square$

Not only do we have arbitrary products, we also have arbitrary direct sums. Note that finite direct sums agree with products and thereby do not need sheafification even in the profinite setting.

Proposition 8.2. *Let $\{M_i\}_{i\in I}$ be κ-condensed abelian groups indexed by a set I. The direct sum $\bigoplus_{i\in I} M_i$ exists in the sense of a categorical coproduct.*

Proof. Consider the assignment $S \mapsto (\bigoplus_{i\in I} M_i)(S) := \bigoplus_{i\in I} M_i(S)$ where the latter denotes the usual direct sum of abelian groups. Given $f : S' \to S$, we define component wise a group homomorphism

$$(\bigoplus_{i\in I} M_i)(f) : \bigoplus_{i\in I} M_i(S) \to \bigoplus_{i\in I} M_i(S').$$

Thus, we obtain a presheaf $\bigoplus_{i\in I} M_i$ of abelian groups. Moreover, with this definition it is clear, that the canonical group homomorphisms $\phi_j(S) : M_j(S) \to \bigoplus_{i\in I} M_i(S)$ define a natural transformation $\phi_j : M_j \to \bigoplus_{i\in I} M_i$. We clearly have $\bigoplus_{i\in I} M_i(\emptyset) = 0$. Since finite direct sums of abelian groups are the same as finite products, we have for κ-small extremally disconnected sets S_1 and S_2 that

$$\bigoplus_{i\in I} M_i(S_1 \coprod S_2) = \bigoplus_{i\in I}(M_i(S_1) \oplus M_i(S_2)) = (\bigoplus_{i\in I} M_i(S_1)) \oplus (\bigoplus_{i\in I} M_i(S_2)).$$

Hence, $\bigoplus_{i\in I} M_i$ is a κ-condensed abelian group. The universal property directly follows from the corresponding universal property of the components $\bigoplus_{i\in I} M_i(S)$. □

Definition 8.3. Let M be a κ-condensed abelian group. A *condensed abelian subgroup* of M is a κ-condensed abelian group N such that $N \subseteq M$, i.e. for all $f : S' \to S$ there is a commutative diagram:

$$\begin{tikzcd} N(S) \arrow[r, "\subseteq"] \arrow[d, "N(f)"'] & M(S) \arrow[d, "M(f)"] \\ N(S') \arrow[r, "\subseteq"] & M(S') \end{tikzcd}$$

Example 8.4. Let I be a set and let $\{M_i\}_{i\in I}$ be κ-condensed abelian groups. The direct sum $\bigoplus_{i\in I} M_i$ is a condensed abelian subgroup of $\prod_{i\in I} M_i$. They are equal if I is finite; there is only a difference between $\bigoplus_{i\in I} M_i$ and $\prod_{i\in I} M_i$ if I is infinite and $M_i \neq 0$ for infinitely many $i \in I$.

Definition 8.5. Let $\eta : M \to M'$ be a morphism of κ-condensed abelian groups. We define the *kernel* of η as the sheaf of abelian groups

$$S \mapsto \ker(\eta)(S) := \ker(\eta(S))$$

and the *image* of η as the sheaf of abelian groups

$$S \mapsto \mathrm{im}(\eta)(S) := \mathrm{im}(\eta(S)).$$

Remark 8.6. The kernel of η is a condensed abelian subgroup of M and the image of η is condensed abelian subgroup of M'. We will soon see that they are a categorical kernel and image of η. That both the kernel and the image are κ-condensed abelian groups follows from the following commutative diagram for κ-small extremally disconnected sets S_1 and S_2:

$$\begin{tikzcd} M(S_1 \coprod S_2) \arrow[r, "\eta(S_1 \coprod S_2)"] \arrow[d, "\cong"] & M'(S_1 \coprod S_2) \arrow[d, "\cong"] \\ M(S_1) \times M(S_2) \arrow[r, "(\eta(S_1), \eta(S_2))"] & M'(S_1) \times M'(S_2) \end{tikzcd}$$

Definition 8.7. Let M be a κ-condensed abelian group and let N be a condensed abelian subgroup. The *condensed quotient group* M/N of M and N is the sheaf of abelian groups $(S \mapsto M(S)/N(S))$.

Remark 8.8. Let us quickly unpack the definition. It is straightforward to check that M/N satisfies the conditions from Corollary 5.19. We just need to argue that M/N is a presheaf in the first place. Let S and S' be κ-small extremally disconnected sets and let $f : S' \to S$. $(M/N)(f) : M(S)/N(S) \to M(S')/N(S')$ is the unique group homomorphism that makes the following diagram commutative:

$$\begin{tikzcd} M(S) \arrow[r, "\pi(S)"] \arrow[d, "M(f)"'] & M(S)/N(S) \arrow[d, dotted, "\exists!(M/N)(f)"] \\ M(S') \arrow[r, "\pi(S')"] & M(S')/N(S') \end{tikzcd}$$

Here, $\pi(S)$ denotes the usual projection. On the one hand this shows that M/N is a κ-condensed abelian group and on the other hand this shows that we have a natural projection $\pi : M \to M/N$. By Proposition 5.7, π is an epimorphism.

Definition 8.9. Let $\eta : M \to M'$ be a morphism of κ-condensed abelian groups. We define the cokernel of η as the sheaf of abelian groups

$$S \mapsto \operatorname{coker}(\eta)(S) := \operatorname{coker}(\eta(S)) = M'(S)/\operatorname{im}(\eta(S))$$

and the coimage of η as the sheaf of abelian groups

$$S \mapsto \operatorname{coim}(\eta)(S) := \operatorname{coim}(\eta(S)) = M(S)/\ker(\eta(S)).$$

Remark 8.10. As the image of η is a condensed abelian subgroup of M' and the kernel of η is condensed abelian subgroup of M, both quotients are well defined. We will see below that they are a categorical cokernel and coimage of η.

Proposition 8.11. *Let M be a κ-condensed abelian group and let N be a condensed abelian subgroup of M. If $\eta : M \to M'$ is a morphism of κ-condensed abelian groups such that $N \subset \ker(\eta)$, then there exists a unique morphism $\overline{\eta} : M/N \to M'$ of κ-condensed abelian groups such that $\eta = \pi \circ \overline{\eta}$. Moreover, we have that* $\operatorname{im}(\tilde{\eta}) = \operatorname{im}(\eta)$.

Proof. It is clear that for any κ-small extremally disconnected set S we have a unique group homomorphism $\overline{\eta}(S) : M(S)/N(S) \to M'(S)$ such that $\eta(S) = \overline{\eta}(S) \circ \pi(S)$. This defines a morphism $\overline{\eta}$ of κ-condensed abelian groups as required. Indeed, the naturality of $\overline{\eta}$ follows from the naturality of η. Moreover, it is clear that $\eta = \overline{\eta} \circ \pi$ and that $\overline{\eta}$ is unique. The final statement follows immediately from the corresponding statement for abelian groups. $\square$

Proposition 8.12. *Let $\eta : M \to M'$ be a morphism of κ-condensed abelian groups. The following statements are true:*

(i) *The inclusion of the condensed subgroup* $\ker(\eta) \hookrightarrow M$ *is a categorical kernel of* η.

(ii) *The inclusion of the condensed subgroup* $\operatorname{im}(\eta) \hookrightarrow M'$ *is a categorical image of* η.

(iii) *The quotient map* $M' \to M'/\operatorname{im}(\eta)$ *is a categorical cokernel of* η.

(iv) *The quotient map $M \to M/\ker(\eta)$ is a categorical coimage of η.*

Proof. The statements (iii) and (iv) follow immediately from Proposition 8.11. Let us now show (i) and (ii). By definition, the image of η is the kernel of the cokernel $\pi : M' \to M'/\operatorname{im}(\eta)$. Hence, it is enough to show (i). Denote by ι the inclusion $\ker(\eta) \hookrightarrow M$. Suppose that $h : K \to M$ is a morphism of κ-condensed abelian groups such that $\eta \circ h = 0$. This means that $\operatorname{im}(h)$ is a condensed abelian subgroup of $\ker(\eta)$. Clearly, h factors uniquely as $h = \iota \circ h$. $\square$

Corollary 8.13. *Let $\eta : M \to M'$ be a morphism of κ-condensed abelian groups.*
(a) *The following statements are equivalent:*

(i) *η is a monomorphism.*

(ii) *For all κ-small extremally disconnected sets S the morphism of abelian groups $\eta(S) : M(S) \to M'(S)$ is injective.*

(iii) $\ker(\eta) = 0$.

(b) *The following statements are equivalent:*

(i) *η is an epimorphism.*

(ii) *For all κ-small extremally disconnected sets S the morphism of abelian groups $\eta(S) : M(S) \to M'(S)$ is surjective.*

(iii) $\operatorname{im}(\eta) \to M'$ *is an isomorphism.*

(c) *The following statements are equivalent:*

(i) *η is an isomorphism.*

(ii) *For all κ-small extremally disconnected sets S the morphism of abelian groups $\eta(S) : M(S) \to M'(S)$ is bijective.*

Proof. We can calculate kernels and cokernels componentwise on κ-small extremally disconnected sets. Hence, the statements follow immediately from Lemma 2.11 and Proposition 5.7. $\square$

The exact same corollary is true even if we consider κ-condensed abelian groups as sheaves of abelian groups on the site of κ-small profinite sets (cf. Proposition 5.7).

Corollary 8.14. *Let M be a κ-condensed abelian group and let $\eta : M \to N$ be a morphism of κ-condensed abelian groups, then $\overline{\eta} : M/\ker(\eta) \to \operatorname{im}(\eta)$ is an isomorphism of κ-condensed abelian groups.*

Proof. This follows immediately from Corollary 8.13. $\square$

Corollary 8.15. *The category of κ-condensed abelian groups is an abelian category.* $\square$

Remark 8.16. A sequence of κ-condensed abelian groups

$$0 \to M' \to M \to M'' \to 0$$

is exact if and only if for all κ-small extremally disconnected sets the sequence

$$0 \to M'(S) \to M(S) \to M''(S) \to 0$$

of abelian groups is exact.

Example 8.17. We can now come back to the example $id : \mathbb{R}_{\text{disc}} \to \mathbb{R}$. Consider the induced map $\underline{id} : \underline{\mathbb{R}_{\text{disc}}} \to \underline{\mathbb{R}}$ of condensed abelian groups (cf. Example 4.10). Proposition 8.12 allows us to compute the kernel and the cokernel of $\underline{id}$. The kernel is zero and $\underline{id}$ thereby a monomorphism. For a κ-small extremally disconnected set S the cokernel Q is given by

$$Q(S) = C(S, \mathbb{R})/\{\text{locally constant maps } S \to \mathbb{R}\}.$$

The "underlying" abelian group $Q(*)$ is still zero but in general there are S such that $Q(S) \neq 0$. For example, the map $\mathbb{N} \to [0,1], n \mapsto \frac{1}{n}$ extends to a continuous map $S := \beta\mathbb{N} \to [0,1] \subset \mathbb{R}$ that takes the constant value zero on $\beta\mathbb{N}\backslash\mathbb{N}$ which is not open. In particular, $S \to \mathbb{R}$ is not locally constant. While the above formula for $Q(S)$ is a priori only true for κ-small extremally disconnected sets it is more generally true for κ-small profinite sets (cf. Introduction in [9]). A simple example of such an S is then given by $S := \mathbb{N} \cup \{\infty\}$. $Q(S)$ can then be identified with the quotient of all convergent sequences in $\mathbb{R}$ by eventually constant sequences in $\mathbb{R}$.

In the category of κ-condensed abelian groups taking direct sums and products is exact. In general it is only true that taking direct sums is right exact and taking products is left exact. However, in our case the exactness follows directly from the corresponding statement in the category of abelian groups.

Proposition 8.18. *Let I be a set and let $\{M_i \xrightarrow{\eta_i} M_i'\}_{i\in I}$ be a family of monomorphisms (epimorphisms) of κ-condensed abelian groups. Then the induced morphism*

$$\bigoplus_{i\in I} M_i \xrightarrow{(\eta_i)_{i\in I}} \bigoplus_{i\in I} M_i',$$

is a monomorphism (epimorphism). In particular, taking direct sums is exact.

Proof. For any κ-small extremally disconnected set S, the morphism in question is given by the group homomorphism

$$\bigoplus_{i\in I} M_i(S) \xrightarrow{(\eta_i(S))_{i\in I}} \bigoplus_{i\in I} M_i'(S).$$

The statement now follows from the corresponding statement in the category of abelian groups and from Corollary 8.13. □

Proposition 8.19. *Let I be a set and let $\{M_i \xrightarrow{\eta_i} M_i'\}_{i\in I}$ be a family of (monomorphisms) epimorphisms of κ-condensed abelian groups. Then the induced morphism*

$$\prod_{i\in I} M_i \xrightarrow{(\eta_i)_{i\in I}} \prod_{i\in I} M_i',$$

is a monomorphism (epimorphism). In particular, taking products is exact.

Proof. For any κ-small extremally disconnected set S, the morphism in question is given by the group homomorphism

$$\prod_{i\in I} M_i(S) \xrightarrow{(\eta_i(S))_{i\in I}} \prod_{i\in I} M_i'(S).$$

The statement now follows from the corresponding statement in the category of abelian groups and from Corollary 8.13. □

Lemma 8.20. *The forgetful functor from κ-condensed abelian groups to κ-condensed sets admits a left adjoint $T \mapsto \mathbb{Z}[T]$ which is given as the sheafification of the functor $\mathbb{Z}[T]_{pre}$ that sends a κ-small extremally disconnected set S to the free abelian group $\mathbb{Z}[T(S)]$ on the set $T(S)$.*

Proof. Recall that sheafification is left adjoint to the inclusion functor from the category of sheaves of sets (abelian groups) to the category of presheaves of sets (abelian groups) on the site of κ-small extremally disconnected sets (cf. Proposition 2.26 in [16]). As the composition of left adjoint functors yields a left adjoint again it is enough to show that the inclusion functor from the category of presheaves of abelian groups to the category of presheaves of sets has a left adjoint. Namely, this is the functor sending a presheaf of sets T to the presheaf of abelian groups $\mathbb{Z}[T]_{\text{pre}}$ sending a κ-small extremally disconnected set S to the free abelian group $\mathbb{Z}[T(S)]$. As $\mathbb{Z}[T]_{\text{pre}}$ is the composition of the functor T and the functor $\mathbb{Z}[-]$ that sends a set M to the free abelian group $\mathbb{Z}[M]$ it is clear that $\mathbb{Z}[T]_{\text{pre}}$ is indeed a presheaf. Since the functor $\mathbb{Z}[-]$ is left adjoint to the forgetful functor from the category of abelian groups to the category of sets, the claim follows. Indeed, if T is a presheaf of sets and if T' is a presheaf of abelian groups, the natural bijections $\mathrm{Hom}(\mathbb{Z}[T(S)], T'(S)) = \mathrm{Hom}(T(S), T'(S))$ give rise to a bijection $\mathrm{Hom}(\mathbb{Z}[T]_{\text{pre}}, T') = \mathrm{Hom}(T, T')$ that is natural in T and T'. □

Remark 8.21. Note that sheafification is really required in the above lemma. Let S_1 and S_2 be κ-small extremally disconnected sets and let T be a κ-condensed set. Then we have that $\mathbb{Z}[T(S_1 \coprod S_2)] = \mathbb{Z}[T(S_1) \times T(S_2)] = \mathbb{Z}[T(S_1)] \otimes_{\mathbb{Z}} \mathbb{Z}[T(S_2)]$ which is in general not isomorphic to $\mathbb{Z}[T(S_1)] \times \mathbb{Z}[T(S_2)]$.

Proposition 8.22. *The category of κ-condensed abelian groups admits all limits and all colimits indexed by small categories.*

Proof. The key observation is that all limits and colimits exist in the category of presheaves on the site of all κ-small extremally disconnected sets and are calculated pointwise on extremally disconnected sets in the category of abelian groups (cf. Section 7.4 in [22]). That is, for any small category I and any functor $i \mapsto M_i$ to κ-condensed abelian groups, we have:

$$(\varprojlim_i M_i)(S) = \varprojlim_i M_i(S)$$
$$(\varinjlim_i M_i)(S) = \varinjlim_i M_i(S).$$

Using this and noting that in the category of abelian groups limits and colimits commute with finite products, we can calculate

$$\begin{aligned}(\varprojlim_i M_i)(S_1 \coprod S_2) &= \varprojlim_i M_i(S_1 \coprod S_2) \\ &= \varprojlim_i (M_i(S_1) \times M_i(S_2)) \\ &= (\varprojlim_i M_i(S_1)) \times (\varprojlim_i M_i(S_2)) \\ &= (\varprojlim_i M_i)(S_1) \times (\varprojlim_i M_i)(S_2),\end{aligned}$$

and similarly

$$(\varinjlim_i M_i)(S_1 \coprod S_2) = (\varinjlim_i M_i)(S_1) \times (\varinjlim_i M_i)(S_2).$$

Moreover, by the above, it is clear that

$$(\varinjlim_i M_i)(\emptyset) = 0,$$

and

$$(\varprojlim_i M_i)(\emptyset) = 0.$$

Hence, all limits and all colimits exist in the category of κ-condensed abelian groups and are calculated pointwise. $\square$

Remark 8.23. The last proposition is true more generally in abelian categories that admit all products and all coproducts. See for example Lemma 4.14.11 and Lemma 4.14.12 in [22].

Corollary 8.24. *Let S be a κ-small extremally disconnected set. The evaluation functor $M \mapsto M(S)$ from κ-condensed abelian groups to abelian groups commutes with all limits and all colimits.* $\square$

Corollary 8.25. *The following statements are true:*

(i) *For any index set J and filtered categories $I_j, j \in J$, with functors $i \mapsto M_i$ from I_j to κ-condensed abelian groups, the natural map*

$$\varinjlim_{(i_j \in I_j)_{j \in J}} \prod_{j \in J} M_{i_j} \to \prod_{j \in J} \varinjlim_{i_j \in I_j} M_{i_j}$$

is an isomorphism.

(ii) *Filtered colimits are exact.*

Proof. Both statements are true in the category of abelian groups. By Proposition 8.22 and Corollary 8.13 we can reduce both claims to abelian groups. $\square$

Recall the notion of compactness.

Definition 8.26. Let $\mathcal{C}$ be an abelian category that admits filtered colimits. An object X of $\mathcal{C}$ is called *compact* if the functor $Y \mapsto \mathrm{Hom}(X, Y)$ from $\mathcal{C}$ to the category of sets preserves filtered colimits.

Proposition 8.27. *The category of κ-condensed abelian groups is generated by compact projective objects.*

Proof. We claim that for a κ-small extremally disconnected set S the κ-condensed abelian group $\mathbb{Z}[\underline{S}]$ is compact and projective. Moreover, together they generate the category of κ-small condensed abelian groups. Let us first show that $\mathbb{Z}[\underline{S}]$ is compact and projective. Indeed, if M is a κ-condensed abelian group, then

$$\mathrm{Hom}(\mathbb{Z}[\underline{S}], M) = \mathrm{Hom}(\underline{S}, M) = M(S),$$

by Lemma 8.20 and by the Yoneda Lemma. Corollary 8.24 says that the evaluation functor commutes with all limits and all colimits. Hence, so does the hom-functor $M \mapsto \mathrm{Hom}(\mathbb{Z}[\underline{S}], M)$. In particular, the hom-functor is exact and commutes with filtered colimits. This means that $\mathbb{Z}[\underline{S}]$ is projective and compact. Let us now show that the $\mathbb{Z}[\underline{S}]$ generate the category of κ-condensed abelian groups. By Proposition 2.41 it is enough to show that any κ-condensed abelian group M admits a surjection from a direct sum $\bigoplus_{i \in I} \mathbb{Z}[\underline{S_i}]$ for some set I. Given M consider the set X of all condensed abelian subgroups $M' \subseteq M$ such that there exists a surjection $\bigoplus_{i \in I} \mathbb{Z}[\underline{S_i}] \to M'$. Note that X is really a set because a condensed abelian subgroup corresponds to an element of the set $\prod_S \mathcal{P}(M(S))$ satisfying certain conditions where $\mathcal{P}$ denotes the power set and where the product runs over the small skeleton of the category of κ-small extremally disconnected sets (cf. Proposition 4.3). The set X is non-empty and partially ordered by inclusion. Assume that we are given a chain $(M_j)_{j \in J}$ in X. Consider the assignment $S \mapsto M'(S) := \bigcup_{j \in J} M_j(S)$. We claim that M' is a sheaf of abelian groups. Indeed, as the M_j are a chain, $M'(S)$ is an abelian group. If $M_k \subseteq M_l$, we are given a natural transformation via inclusion. This means that for $f : S' \to S$ the restriction of $M_l(f)$ to $M_k(S)$ is equal to $M_k(f)$. Hence, we can define $M'(f) : M'(S) \to M'(S')$ in the obvious way. It is clear that $M'(\emptyset) = \{0\}$. Moreover, because the M_j are a chain, we have that $M'(S_1 \coprod S_2) = M'(S_1) \times M'(S_2)$. By Corollary 5.19 we see that M' is a κ-condensed abelian group. By construction M' is an upper bound for the chain $(M_j)_{j \in J}$. As we have a surjection onto all the M_j, we have one onto M'. Thus, $M' \in X$. By Zorn's lemma there is a maximal condensed abelian subgroup M' admitting a surjection $\bigoplus_{i \in I} \mathbb{Z}[\underline{S_i}] \to M'$. We claim that $M' = M$. Assume this is not the case. Then there is some κ-small extremally disconnected set S such that $0 \neq (M/M')(S) = \mathrm{Hom}(\mathbb{Z}[\underline{S}], M/M')$. Hence, we can pick some non-zero $g : \mathbb{Z}[\underline{S}] \to M/M'$ and by the projectivity of $\mathbb{Z}[\underline{S}]$ we obtain a commutative diagram:

$$\begin{tikzcd} & M \arrow[d, "\mathrm{can}"] \\ \mathbb{Z}[\underline{S}] \arrow[ur, dotted, "\exists h"] \arrow[r, "g"] & M/M' \end{tikzcd}$$

As g is non-zero the image of h is not contained in M'. The surjection $\bigoplus_{i \in I} \mathbb{Z}[\underline{S_i}] \to M'$ and h induce a map $(\bigoplus_{i \in I} \mathbb{Z}[\underline{S_i}]) \bigoplus \mathbb{Z}[\underline{S}] \to M$ such that the image strictly contains M'. This is clearly in contradiction to the maximality of M' and thus, $M = M'$. Hence, the $\mathbb{Z}[\underline{S}]$ generate the category of κ-condensed abelian groups. $\square$

Remark 8.28. Here is an alternative proof for the statement that the $\mathbb{Z}[\underline{S}]$ generate the category of κ-condensed abelian groups. If M is a κ-condensed abelian group, then

$$\mathrm{Hom}(\mathbb{Z}[\underline{S}], M) = \mathrm{Hom}(\underline{S}, M) = M(S),$$

by Lemma 8.20 and by the Yoneda Lemma. In particular, if M is a nonzero κ-condensed abelian group, there is some κ-small extremally disconnected set S such that $M(S) \neq 0$ and hence, $\mathrm{Hom}(\mathbb{Z}[\underline{S}], M) \neq 0$. By Corollary 2.43 this already means that the $\mathbb{Z}[\underline{S}]$ generate the category of κ-condensed abelian groups.

The last proposition allows a reformulation of Corollary 8.13.

Corollary 8.29. *Let $\eta : M \to M'$ be a morphism of κ-condensed abelian groups.*
(a) *The following statements are equivalent:*

(i) *η is a monomorphism.*

(ii) *For all κ-small extremally disconnected sets S the induced group homomorphism $\eta_* : \mathrm{Hom}(\mathbb{Z}[\underline{S}], M) \to \mathrm{Hom}(\mathbb{Z}[\underline{S}], M')$ is injective.*

(b) *The following statements are equivalent:*

(i) *η is an epimorphism.*

(ii) *For all κ-small extremally disconnected sets S the induced group homomorphism $\eta_* : \mathrm{Hom}(\mathbb{Z}[\underline{S}], M) \to \mathrm{Hom}(\mathbb{Z}[\underline{S}], M')$ is surjective.*

(c) *The following statements are equivalent:*

(i) *η is an isomorphism.*

(ii) *For all κ-small extremally disconnected sets S the induced group homomorphism $\eta_* : \mathrm{Hom}(\mathbb{Z}[\underline{S}], M) \to \mathrm{Hom}(\mathbb{Z}[\underline{S}], M')$ is bijective.*

Proof. Let S be a κ-small extremally disconnected set. By the Yoneda Lemma and Lemma 8.20 we have that $M(S) = \mathrm{Hom}(\mathbb{Z}[\underline{S}], M)$. Using Corollary 8.13, the statement follows now immediately. $\square$

Corollary 8.30. *The category of κ-condensed abelian groups admits a projective generator, given by $\bigoplus_S \mathbb{Z}[\underline{S}]$, where S ranges over a set of representatives of the isomorphism classes of all κ-small extremally disconnected sets. In particular, the category of κ-condensed abelian groups has enough projectives and is a Grothendieck abelian category.*

Proof. Recall that the category of κ-small extremally disconnected sets is essentially small (cf. Proposition 4.3). By Proposition 8.27 the $\mathbb{Z}[\underline{S}]$ form a family of generators that are all projective. With this information all but the last assertion follow immediately from Corollary 2.42 and Corollary 2.37. The last assertion follows from Proposition 8.22 and Corollary 8.25. $\square$

Example 8.31. We claim that $\underline{\mathbb{Z}} = \mathbb{Z}[\underline{*}]$ and that this is a compact, projective object which does not generate the category of κ-condensed abelian groups. Indeed, the underlying presheaf $\mathbb{Z}[\underline{*}]_{\mathrm{pre}}$ of the sheaf $\mathbb{Z}[\underline{*}]$ is the constant presheaf with value $\mathbb{Z}$. Hence, $\mathbb{Z}[\underline{*}]_{\mathrm{pre}}$ embeds into $\underline{\mathbb{Z}}$ by sending $n \in \mathbb{Z}[\underline{*}]_{\mathrm{pre}}(S)$ to the constant function with value n in $\underline{\mathbb{Z}}(S) = C(S, \mathbb{Z})$. The naturality is clear. If we call this injection η^{pre}, using the universal property of sheafification, we obtain an injection $\eta : \mathbb{Z}[\underline{*}] \to \underline{\mathbb{Z}}$ and a commutative diagram (cf. Lemma 7.10.14 in [22]):

$$\begin{tikzcd} \mathbb{Z}[\underline{*}]_{\mathrm{pre}} \arrow[r, "\eta^{\mathrm{pre}}"] \arrow[d, "\iota"'] & \underline{\mathbb{Z}} \\ \mathbb{Z}[\underline{*}] \arrow[ur, dotted, "\exists! \eta"'] & \end{tikzcd}$$

We need to check that η is surjective. By Corollary 8.13 we need to show that the group homomorphism $\eta(S) : \mathbb{Z}[\underline{*}](S) \to \underline{\mathbb{Z}}(S)$ is surjective for all κ-small extremally disconnected sets S. For this let $f \in \underline{\mathbb{Z}}(S) = C(S, \mathbb{Z})$. f then induces a covering of S as follows. Since S is compact, the image $f(S) \subset \mathbb{Z}$ is compact and hence finite because $\mathbb{Z}$ is discrete. The covering is then given by $S_i := f^{-1}(\{i\})$ for $i \in f(S)$. Note that $S = \coprod_{i \in f(S)} S_i$. Consider the following commutative diagram where ϕ_i denotes the inclusion of S_i into S:

$$\begin{tikzcd}
\prod_{i\in f(S)} \mathbb{Z}[\underline{*}](S_i) \arrow[r, "(\eta(S_i))_i"] & \prod_{i\in f(S)} \underline{\mathbb{Z}}(S_i) \\
\mathbb{Z}[\underline{*}](S) \arrow[u, "(\mathbb{Z}[\underline{*}](\phi_i))_i"] \arrow[r, "\eta(S)"] & \underline{\mathbb{Z}}(S) \arrow[u, "(\underline{\mathbb{Z}}(\phi_i))_i"']
\end{tikzcd}$$

By definition of S_i we have that $\underline{\mathbb{Z}}(\phi_i)(f)$ is the constant function with value i. Moreover, since the vertical arrows in the diagram are bijections, it is enough to find a preimage for $\underline{\mathbb{Z}}(\phi_i)(f)$ under $\eta(S_i)$. By the discussion so far, this is not a problem, because $\eta(S_i)(\iota(S_i)(i)) = \eta^{\text{pre}}(S_i)(i) = \underline{\mathbb{Z}}(\phi_i)(f)$. Hence, $\underline{\mathbb{Z}} = \mathbb{Z}[\underline{*}]$. By the last proposition this shows that $\underline{\mathbb{Z}}$ is compact and projective. Suppose now that $\underline{\mathbb{Z}}$ is a generator. Proposition 2.37 (iii) tells us that for all κ-condensed abelian groups T, $\text{Hom}(\underline{\mathbb{Z}}, T) \neq 0$. Let Q be the nonzero cokernel of $\underline{id} : \underline{\mathbb{R}_{\text{disc}}} \to \underline{\mathbb{R}}$ with $Q(*) = 0$ (cf. Example 8.17). Using Lemma 8.20 and the Yoneda Lemma we see that

$$\begin{aligned}
\text{Hom}(\underline{\mathbb{Z}}, Q) &= \text{Hom}(\mathbb{Z}[\underline{*}], Q) \\
&= \text{Hom}(\underline{*}, Q) \\
&= Q(*) \\
&= 0,
\end{aligned}$$

which is clearly a contradiction. Hence, $\underline{\mathbb{Z}}$ is not a generator.

If we collect the statements we have proven, we obtain the main result of this section.

Theorem 8.32. *The category of κ-condensed abelian groups is an abelian category which satisfies Grothendieck's axioms (AB3), (AB4), (AB5), (AB6), (AB3*) and (AB4*), to wit: all limits (AB3*) and colimits (AB3) exist, arbitrary products (AB4*), arbitrary direct sums (AB4) and filtered colimits (AB5) are exact, and (AB6) for any index set J and filtered categories $I_j, j \in J$, with functors $i \mapsto M_i$ from I_j to κ-condensed abelian groups, the natural map*

$$\varinjlim_{(i_j \in I_j)_{j\in J}} \prod_{j \in J} M_{i_j} \to \prod_{j\in J} \varinjlim_{i_j \in I_j} M_{i_j}$$

is an isomorphism. Moreover, the category of κ-condensed abelian groups is generated by compact projective objects.

Remark 8.33. Throughout this section we have exploited the description of κ-condensed abelian groups as certain functors on the site of κ-small extremally disconnected sets. The importance of the compact projective generators is hidden in this approach. As Corollary 8.29 still holds for κ-condensed abelian groups on the site $*_{\kappa\text{-proét}}$, statements like Proposition 8.18 can be proved using the corollary. Indeed, for example suppose that we are given a family $\{M_i \xrightarrow{\eta_i} M_i'\}_{i\in I}$ of monomorphisms of κ-condensed abelian groups. The induced morphism

$$\bigoplus_{i\in I} M_i \to \bigoplus_{i\in I} M_i'$$

is a monomorphism if and only if for all κ-small extremally disconnected sets S the induced group homomorphism

$$\mathrm{Hom}(\mathbb{Z}[\underline{S}], \bigoplus_{i\in I} M_i) \to \mathrm{Hom}(\mathbb{Z}[\underline{S}], \bigoplus_{i\in I} M_i')$$

is an isomorphism in the category of abelian groups. By the compactness of $\mathbb{Z}[\underline{S}]$, this reduces to the claim that for all $i \in I$ the group homomorphism

$$\mathrm{Hom}(\mathbb{Z}[\underline{S}], M_i) \to \mathrm{Hom}(\mathbb{Z}[\underline{S}], M_i')$$

is injective, but this is true by the projectivity of $\mathbb{Z}[\underline{S}]$.

Thus far we have seen that the category of κ-condensed abelian groups is an abelian category that is generated by compact projective objects; it even has a single generator. Here are some further categorical properties of the category of κ-condensed abelian groups.

Let M and N be κ-condensed abelian groups viewed (as before) as sheaves on the site of κ-small extremally disconnected sets. We want to define a tensor product $M \otimes N$. For this let $M \otimes N$ be the sheafification of the presheaf of abelian groups $(M \otimes N)_{\mathrm{pre}}$ given by $S \mapsto M(S) \otimes_{\mathbb{Z}} N(S)$. Given a κ-small extremally disconnected set S we have a canonical bilinear map $M(S) \times N(S) \to M(S) \otimes_{\mathbb{Z}} N(S)$. These maps induce a canonical (component wise) bilinear map $M \times N \to (M \otimes N)_{\mathrm{pre}}$. Let ι be the composition

$$M \times N \to (M \otimes N)_{\mathrm{pre}} \to M \otimes N.$$

The next proposition shows that $\iota : M \times N \to M \otimes N$ satisfies the usual universal property one would expect of a tensor product.

Proposition 8.34. *Let M and N be κ-condensed abelian groups. Then $M \times N \xrightarrow{\iota} M \otimes N$ satisfies the universal property of a tensor product, i.e. if P is a κ-condensed abelian group and if $h : M \times N \to P$ is a componentwise bilinear map, then there exists a unique morphism of κ-condensed abelian groups $\tilde{h} : M \otimes N \to P$ such that $h = \tilde{h} \circ \iota$.*

Proof. Assume we are given a componentwise bilinear map $h : M \times N \to P$. Then h induces a unique map $\overline{h} : (M \otimes N)_{\mathrm{pre}} \to P$. By the universal property of the sheafification we obtain a unique morphism of κ-condensed abelian groups $M \otimes N \xrightarrow{\tilde{h}} P$. It is straightforward to check that $\tilde{h}$ is the desired morphism. □

One can describe the tensor product $M \otimes N$ of κ-condensed abelian groups M and N more explicitly. If M is a κ-condensed abelian group, there is a natural action $\underline{\mathbb{Z}}[*]_{\mathrm{pre}} \times M \to M$ which gives rise to a natural action $\underline{\mathbb{Z}} \times M \to M$ by sheafification (cf. Section 18.11 in [22]). For example, if G is a topological abelian group, then for a κ-small extremally disconnected set S the action $\underline{\mathbb{Z}}(S) \times \underline{G}(S) \to \underline{G}(S)$ is given by pointwise multiplication. The action $\underline{\mathbb{Z}} \times M \to M$ gives $M(S)$ the structure of an $\underline{\mathbb{Z}}(S)$-module. Hence, we can define the presheaf of abelian groups $(S \mapsto M(S) \otimes_{\underline{\mathbb{Z}}(S)} N(S))$. This is actually already a sheaf, which follows from the isomorphisms

$$\begin{aligned}
& M(S_1 \coprod S_2) \otimes_{\underline{\mathbb{Z}}(S_1 \coprod S_2)} N(S_1 \coprod S_2) \\
= \ & (M(S_1) \times M(S_2)) \otimes_{\underline{\mathbb{Z}}(S_1) \times \underline{\mathbb{Z}}(S_2)} (N(S_1) \times N(S_2)) \\
= \ & (M(S_1) \otimes_{\underline{\mathbb{Z}}(S_1)} N(S_1)) \times (M(S_2) \otimes_{\underline{\mathbb{Z}}(S_2)} N(S_2)).
\end{aligned}$$

It is now completely formal that $M \times N \to (S \mapsto M(S) \otimes_{\underline{\mathbb{Z}}(S)} N(S))$ has the universal property of a tensor product of M and N. With this at hand, it is straightforward to check that $\underline{\mathbb{Z}} \otimes M = M$ for any κ-condensed abelian group M.

Remark 8.35. As a direct consequence of the universal property, the tensor product makes the category of κ-condensed abelian groups a symmetric monoidal category with respect to the tensor product, where the unit object is given by $\underline{\mathbb{Z}}$. Moreover, the functor $T \to \mathbb{Z}[T]$ from the category of κ-condensed sets to the category of κ-condensed abelian groups is symmetric monoidal with respect to the tensor product, i.e. it takes products to the tensor product:

$$\mathbb{Z}[T_1 \times T_2] = \mathbb{Z}[T_1] \otimes \mathbb{Z}[T_2].$$

This follows directly from the corresponding statement for the underlying presheaves (cf. Section 18.26 in [22]). Moreover, we claim that $\mathbb{Z}[T]$ is flat for any κ-condensed set T in the sense that tensoring with $\mathbb{Z}[T]$ is an exact functor. To see this note that tensoring with the presheaf $S \mapsto \mathbb{Z}[T(S)]$ is exact as exactness is tested componentwise and because $\mathbb{Z}[T(S)]$ is free and hence flat for any κ-small extremally disconnected set S. Thus, we see that $\mathbb{Z}[T]$ is flat because sheafification is exact (cf. Section 18.3 in [22]).

Recall that in the category of abelian groups we have the tensor-hom adjunction. More precisely, for a fixed abelian group G, the functor $H \mapsto H \otimes_{\mathbb{Z}} G$ is left adjoint to the functor $H \mapsto \mathrm{Hom}(G, H)$. Let M be a fixed κ-condensed abelian group. As left adjoint functors commute with colimits and because colimits are calculated componentwise in the category of presheaves, the functor $N \mapsto (N \otimes M)_{\mathrm{pre}}$ commutes with colimits. Finally, sheafification commutes with colimits because sheafification is a left adjoint functor and hence, the functor $N \mapsto N \otimes M$ commutes with colimits. By the adjoint functor theorem (cf. Remark 2.46) the functor $N \mapsto N \otimes M$ has a right-adjoint functor $N \mapsto \underline{\mathrm{Hom}}(M, N)$, usually called internal hom. We obtain:

Proposition 8.36. *The category of κ-condensed abelian groups admits an internal hom object $\underline{\mathrm{Hom}}(M, N)$ for any κ-condensed abelian groups M and N. Moreover, if S is a κ-small extremally disconnected set S, we have $\underline{\mathrm{Hom}}(M, N)(S) = \mathrm{Hom}(\mathbb{Z}[\underline{S}] \otimes M, N)$.*

Proof. We have done most of the work already. By the adjunction formula we have:

$$\mathrm{Hom}(P, \underline{\mathrm{Hom}}(M, N)) = \mathrm{Hom}(P \otimes M, N).$$

If we let $P = \mathbb{Z}[\underline{S}]$, we obtain

$$\begin{aligned}
\mathrm{Hom}(\mathbb{Z}[\underline{S}] \otimes M, N) &= \mathrm{Hom}(\mathbb{Z}[\underline{S}], \underline{\mathrm{Hom}}(M, N)) \\
&= \mathrm{Hom}(\underline{S}, \underline{\mathrm{Hom}}(M, N)) \\
&= \underline{\mathrm{Hom}}(M, N)(S),
\end{aligned}$$

where we have used the adjunction from Lemma 8.20 and the Yoneda Lemma. $\square$

Remark 8.37. All relevant statements in this section such as Theorem 8.32 are still true if we pass to the category of condensed abelian groups that is independent of κ as introduced in the last section. However, the category of condensed abelian groups is no longer a Grothendieck category because the objects $\mathbb{Z}[\underline{S}]$, where S is extremally disconnected, no longer form a set (cf. Corollary 8.30).

Our initial motivation was to embed the category of topological abelian groups into an abelian category. Theorem 7.26 and Corollary 7.19 imply that compactly generated weak Hausdorff topological abelian groups embed fully faithfully into the abelian category of condensed abelian groups. By now, the theory of condensed mathematics was already applied successfully. For example in [8] Scholze and Clausen have proved finiteness of coherent cohomology, Serre duality, GAGA in the algebraic case and the (Grothendieck–)Hirzebruch–Riemann–Roch theorem. Notably, these '*proofs are proofs by "formal nonsense" and in particular analysis-free*'. More recently, in [10] Scholze and Clausen presented an approach to unify different geometric theories in the form of analytic stacks, the starting point of the theory are so called light condensed sets. In [5], Juan Camargo presents the theory in written form.

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