Babyseminar Proposal: Deformation Theory

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The goal of this seminar would be to study deformation theory, which is an important tool in algebraic geometry, providing insights on the study of the local structure of moduli spaces, and in number theory, being part of the proof of the modularity theorem. Deformation theory is the study of a family in the neighbourhood of a given element. A typical situation would be a family of schemes, i.e. a flat morphism of schemes $f: X \to T$ of which we want to study the fibers X_t for varying $t \in T$. Deformation theory is the infinitesimal study of the family in the neighborhood of a special fiber X_0. This is closely related to the study of existence of moduli spaces, deformation theory helps understanding the local structure of the moduli space. A deformation of a structure S over a field k, is an extension of S to a structure over a k-algebra A.

The idea of the seminar would be to give an introduction to deformation theory and study some applications. The main references for the seminar will be [Har10] and [Ser06].

The main examples of deformation problems that we can look at are:

- The Hilbert scheme: deforming subschemes of a certain scheme while keeping the ambient space fixed;
- The Picard scheme: deforming line bundles on a scheme;
- Deformations of vector bundles/coherent sheaves on a fixed scheme;
- Deformations of abstract schemes.

First we would start by looking at first order deformations, which are deformations over the ring of dual numbers $k[\epsilon]/(\epsilon^2)$. In any of our situations, when a good moduli space exists, the deformations over the dual numbers will allow us to compute the Zariski tangent space to the moduli space. Then, we can talk about higher order deformations, i.e. deformations over $k[\epsilon]/(\epsilon^n)$. Given a deformation over $k[\epsilon]/(\epsilon^n)$, can one extend it to $k[\epsilon]/(\epsilon^{n+1})$? In general this is not always possible, and leads to an obstruction theory.

Then we pass to the study of formal deformation theory. Deformation functors can be formalized as functors of Artin rings. We want to study when they are pro-representable. In this case we get a universal family. However, it turns out that they are not always pro-representable, so the more relaxed notions of versal and miniversal family were introduced. A powerful tool in this theory is

Schlessinger's theorem, which gives necessary and sufficient conditions for the existence of miniversal families and pro-representability.

Once we have a formal family of deformations of some object (either versal, miniversal or universal), the question is, can we extend it to an actual family of deformations? The main result in this sense is Artin's algebraization theorem.

Since deformation theory is a tool used in many applications, it would be nice to have some final talks about these, depending on the taste of people. For example we could look at deformations of Galois representations. For this, possible references could be [Ber+13] and [Gee22].

References

[Ber+13] Laurent Berger et al. *Elliptic curves, Hilbert modular forms and Galois deformations*. Springer, 2013.

[Gee22] Toby Gee. "Modularity lifting theorems". In: *arXiv preprint arXiv:2202.05818* (2022).

[Har10] Robin Hartshorne. *Deformation theory*. Vol. 257. Springer, 2010.

[Ser06] Edoardo Sernesi. *Deformations of algebraic schemes*. Springer, 2006.