

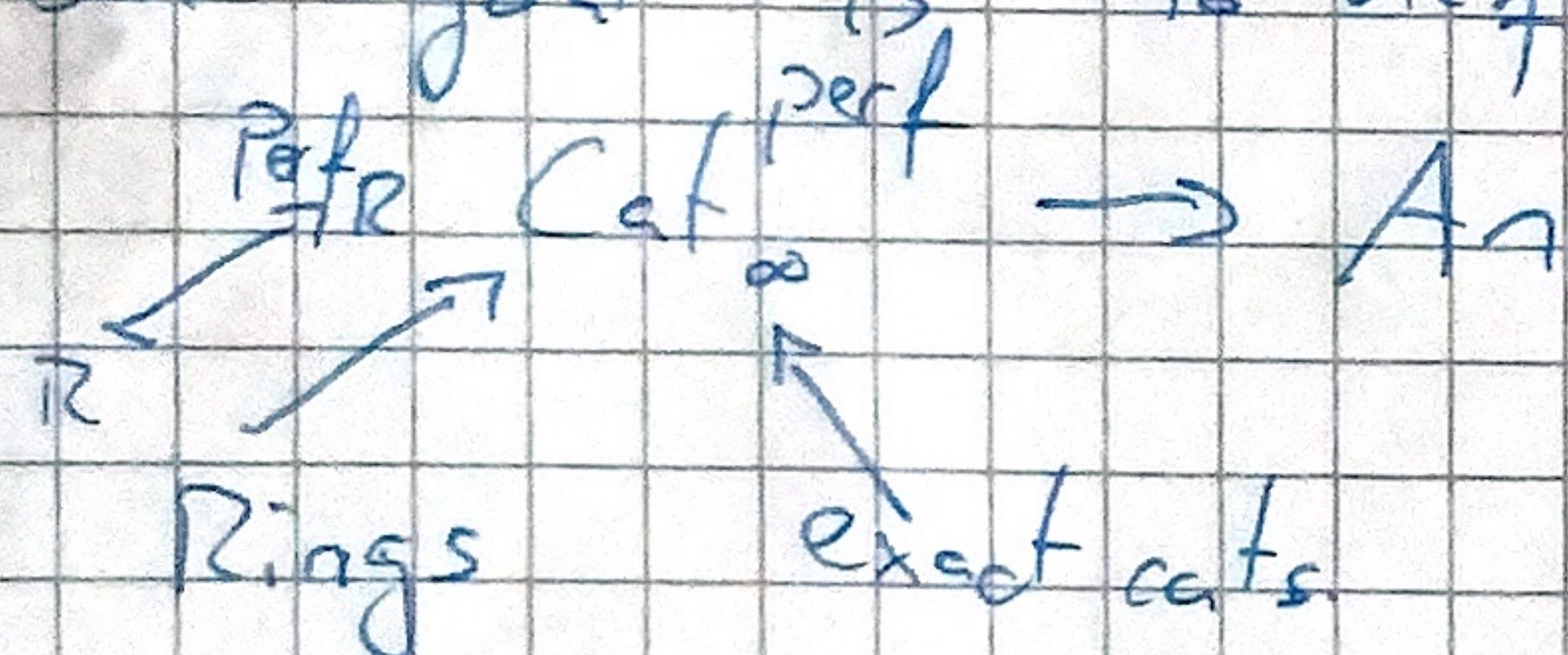
{ Categorical preliminaries

We will not seriously need the following

Def A category C is idempotent complete if every idempotent morphism $X \xrightarrow{f} X$ determines a retract $X \xrightarrow{i} Y \xrightarrow{p} X$

Def Let $\text{Cat}_a^{\text{perf}}$ be the category of small idempotent-complete categories
($\simeq$ triangulated cat. with set of generators & idem-complete)

Our goal is to define K-theory as a functor



As we saw last time, K-theory satisfies certain excision properties for rings eg. $(K(\text{Perf}_R^{\text{f-tors}})) \rightarrow K(\mathbb{Z}) \rightarrow K(\mathbb{Z}/p\mathbb{Z})$

Def A seq. of st. in $\text{Cat}_a^{\text{perf}}$

$$C \rightarrow D \xrightarrow{p} E$$

is a verdier seq. if it is both a fiber & cofiber seq. eg.

$$\begin{array}{ccc} C & \rightarrow & D \\ \downarrow & & \downarrow \\ * & \rightarrow & E \end{array}$$

is a pushout & pullback square

Examples - if $R \rightarrow R[\frac{1}{p}]$, then

$$D(R)^{\text{f-tors}} \rightarrow D(R) \rightarrow D(R[\frac{1}{p}])$$

- if $m \in \mathcal{O}_{\mathbb{A}^1_p}$ then

$$D(\mathcal{O}_{\mathbb{A}^1_p}) \rightarrow D(\mathcal{O}_{\mathbb{A}^1_p}) \rightarrow D^s(\mathcal{O}_{\mathbb{A}^1_p})$$

almost 0

- if $D_c(X, \mathcal{I})$ denotes constructible sheaves on $X/\mathbb{C}$
 $\exists Z \subset X \supset U$ is a closed-open decomp.

$$D_c(Z) \rightarrow D_c(X) \rightarrow D_c(U)$$

- semi-orthogonal decomp. $D(\mathbb{A}^1) = \langle \mathcal{O}_X, \mathcal{O}_X(1), \dots, \mathcal{O}_X(n-1) \rangle$

We can always construct Verdier seq.

Def Let $i: \mathcal{C} \rightarrow \mathcal{D}$ be a functor in $\text{Cat}_{\infty}^{\text{perf}}$

Define

$$D/\mathcal{C} = D[\mathcal{C}\text{-equiv.}^{-1}], \quad f: A \rightarrow B \text{ is a } \mathcal{C}\text{-eq.}$$

This is also the cofiber of i in $\text{Cat}_{\infty}^{\text{perf}}$ if $\text{fib}(f) = 0$

lemma The fiber of $D \rightarrow D/\mathcal{C}$ is the full subcat. of $\mathcal{D}$ spanned by retracts of $\mathcal{C}$ in D .

Def A verdier sequence $\mathcal{C} \xrightarrow{i} D \xrightarrow{p} \mathcal{E}$ is split

if p admits a left & right adjoint.

In that case, i also admits 2 adjoints.

Example s.o.d. & constructible

$$D_c(Z) \xrightleftharpoons[\text{res.}]{\text{incl.}} D_c(X) \xrightleftharpoons[\text{incl.}]{\text{res.}} D_c(U)$$

If $R \rightarrow S$ is a morphism of rings, then

$$\begin{array}{ccc} \text{Mod}_R & \xrightleftharpoons[\text{Hom}_R(S, -)]{\text{res.}} & \text{Mod}_S \end{array}$$

If $\mathcal{C}$ & D are stable then $\mathcal{C} \hookrightarrow \mathcal{C} \times D \rightarrow D$ is split verdier

§ Towards localizing invariants

3

Def. A functor $F: \text{Cat}_{\infty}^{\text{st}} \rightarrow \text{Sp}^d$ with $F(0) = 0$ is

- additive if F carries split Verdier seq. to fiber seq.
- Verdier localizing if F carries Verdier seq. to fiber seq.

Note that, if $F: \text{Cat}^{\text{st}} \rightarrow \text{An}$ is additive, then the direct sum $A \times A \rightarrow A$ gives $F(A)$ the structure of a commutative monoid, in particular, F factors as

$$\begin{array}{ccc} \text{Cat}_{\infty}^{\text{perf}} & \rightarrow & \text{Mon}(\text{An}) \\ & \searrow F & \downarrow \\ & & \text{An} \end{array}$$

We say F is grouplike if F factors through $\text{Grp}(\text{An})$

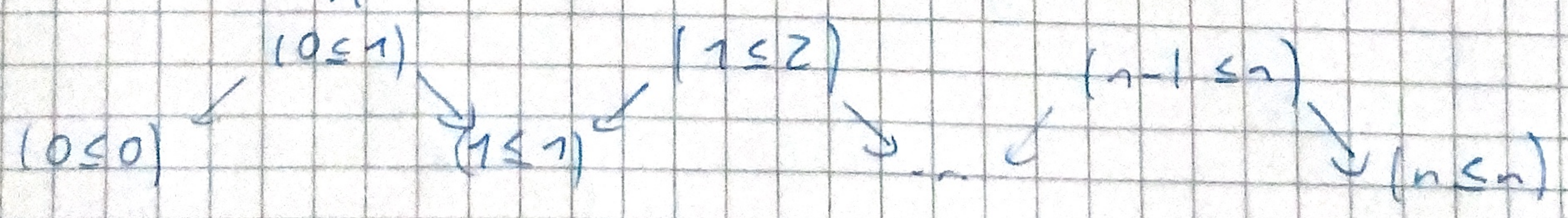
We denote $\text{Fun}^{\text{gp}}(\text{Cat}_{\infty}^{\text{perf}}, \text{An}) \subseteq \text{Fun}^{\text{add}}(\text{Cat}_{\infty}^{\text{perf}}, \text{An})$

Important example: Since $(-)^{\text{op}}$ is right-adjoint it preserves pullbacks & hence it is additive.

It is not grouplike.

§ K-theory of Stable Categories

Defn. Let $\tilde{\mathcal{I}}_n$ denote the 1-cat.



We define, for $\mathcal{C}$ a category,

$$Q_n(\mathcal{C}) = \text{Fun}(\tilde{\mathcal{I}}_n, \mathcal{C})$$

This becomes a simplicial object

$$Q: \Delta^{\text{op}} \rightarrow \text{Cat} \\ [n] \mapsto Q_n(\mathcal{C})$$

Def Let $e \in \text{Cat}_\infty^{\text{perf}}$, we define

$$K(e) = \coprod | (Q(e))^{\sim} |$$

Theorem Let $F: \text{Cat}_\infty^{\text{st}} \rightarrow \text{An}$ be additive, then

$|F(Q(e))|: \text{Cat}_\infty^{\text{st}} \rightarrow \text{An}$ is additive & grouplike.

Taking $F = (-)^{\sim}$, we see that $K(-)$ factors through $\text{CGrp}(\text{An}) \simeq \text{Sp}_{\geq 0}$

Very rough proof sketch

Let $e \rightarrow D$ be ^{split} ~~ar~~ ^{ar} ~~vedier~~ ^{square} seq.

$$\begin{array}{ccc} e & \rightarrow & D \\ \downarrow \cong & & \downarrow \\ \bullet & \rightarrow & \bullet \end{array}$$

Applying Q_n & F preserves this, hence

$$FQ_n(e) \rightarrow FQ_n(D) \rightarrow FQ_n(E)$$

is a fiber seq. in $s\text{An} \subseteq \text{Fun}(\Delta^{\text{op}}, \text{An})$

In fact, all of the $s\text{An}$ above are Segal anima
& there is an eq. $\{\text{Segal anima}\} \xrightarrow{\sim} \text{Cat}_\infty$
which commutes with $| - |$.

$\Rightarrow$ It suffices to show that

$$\text{ac } FQ_n(e) \rightarrow \text{ac } FQ_n(D) \xrightarrow{p} \text{ac } FQ_n(E)$$

gets sent to a fiber seq. by $| - |$.

For this, one shows that p is a

"realisation" fibration.

$\nwarrow$ HARD!

§ Universal Property of K-theory

5

Theorem The inclusion

$$\text{Fun}^{\text{gp}}(\text{Cat}^{\text{st}}, \text{An}) \hookrightarrow \text{Fun}^{\text{add}}(\text{Cat}^{\text{st}}, \text{An})$$

admits a left-adjoint

$$F \mapsto \Omega | \text{Span}^F(-) |$$

$\Rightarrow$ The unit $(-)^{\Omega} \rightarrow K(-)$ is the initial map from $(-)^{\Omega}$ to grouplike additive invariants.

In fact, K-theory is also Verdier localizing

Theorem $K: \text{Cat}_0^{\text{st}} \rightarrow \text{CGrp}(\text{An}) \xrightarrow{\sim} \text{Sp}_{\geq 0} \hookrightarrow \text{Sp}$
sends Verdier seq. to fiber seq.

However, this is not fully satisfying yet as there are certain sequences which are close to being Verdier which ought to be sent to fiber seq. by $K(-)$

Example Let $R = k[x, y] / (y^2 - x^3 - x^2)$, the nodal.
Then the functor

$$\text{Perf}_{R[t]} \rightarrow \text{Perf}_{R[t, t^{-1}]}$$

is not surjective because K_0 isn't surj

However, $K(-)$ still ought to send

$$\text{Perf}_{R[t]}^{\text{R}[t][t^{-1}]\text{-tors}} \rightarrow \text{Perf}_{R[t]} \rightarrow \text{Perf}_{R[t, t^{-1}]}$$

to a fiber seq. of Anims (NOT SPECTRA!)

Def A morphism $f: E \rightarrow D$ is a Karabi-
equivalence if it is fully faithful & every
object in D is a retract of an object in E .

Lemma $E \rightarrow D$ is a Karabi-
 $E \xrightarrow{f} D \xrightarrow{g} E$ is
idempotent completion

Def A seq. $E \rightarrow D \rightarrow \mathcal{E}$ is a Karabi-
if i is fully faithful
 $D/\mathcal{E} \rightarrow \mathcal{E}$ is a Karabi-
eq.

Theorem (Cofinality) If $A \rightarrow B$ is a dense
inclusion (i.e. Karabi eq.), then
 $K_1(A) \simeq K_1(B)$ if $i \gg 0$
 $K_0(B)/K_0(A) \simeq \pi_0(B^{\text{eq}})/\pi_0(A^{\text{eq}})$

Theorem The functor $K_0(-)^{\text{eq}}$ is Karabi-localizing