Properties of higher K-theory

Goal of the talk: as in the title, the goal is to present the main properties of K-theory, and some of their applications

Sections of the talk:

1) Additivity $\to$ for Waldhausen and exact cat.

2) Resolution $\rightsquigarrow$ for exact cat.

3) Devissage $\to$ for abelian cat. ($\rightsquigarrow$ Quillen constr. for exact cat: talk of Louisa)

4) Localization

Elementary properties of $K(\mathcal{A})$ for $\mathcal{A}$ exact category

<u>Recall:</u> (from Luisa's talk)

- given an exact category $\rightsquigarrow$ construct a category $Q\mathcal{A}$
  $\rightsquigarrow$ apply $B: \mathrm{Cat} \xrightarrow[\text{nerve}]{N} \mathrm{sSet} \xrightarrow[\text{geom. realization}]{1:1} \mathrm{Top}$ classifying space

  to get the CW complex $BQ\mathcal{A}$

  then define $K\mathcal{A} := \Omega BQ\mathcal{A}$ as the loop space of this CW complex

  and $K_n(\mathcal{A}) := \pi_n(K\mathcal{A})$

<u>Rmk:</u>

- an <u>exact functor</u> $\mathcal{A} \longrightarrow \mathcal{B}$ induces a map
  sends exact seq. to exact seq.

  $K\mathcal{A} \longrightarrow K\mathcal{B}$

- $K(\mathcal{A})$ has an <u>H-space</u> structure because it is a loop space
  topological space with an operation and neutral element up to homotopy

Graded ring structure on $K_*(R)$ and $K_*(X)$

- $F(A,-): \mathcal{B} \to \mathcal{C}$ & $F(-,B): \mathcal{A} \to \mathcal{C}$ are exact functors
- $F(A,0) = F(0,B) = 0$ (0 stands for the distinguished zero objects of $\mathcal{A}, \mathcal{B}, \mathcal{C}$)

A <u>biexact</u> functor $F(-,-): \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$ induces bilinear maps:

$$K_i(\mathcal{A}) \times K_j(\mathcal{B}) \longrightarrow K_{i+j}(\mathcal{C})$$

In particular, if there is an associative pairing $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, then $K_*(\mathcal{A})$ becomes a graded ring.

<u>Examples:</u>

K-theory of a commutative ring

- $K(R) = K(\mathcal{P}(R))$ R comm. ring, $\mathcal{P}(R)$ cat. of f.g. proj R-modules

  ↳ Tensor product $\mathcal{P}(R) \times \mathcal{P}(R) \longrightarrow \mathcal{P}(R)$ makes $K_*(R)$ a graded commutative ring with unit

K-theory of a scheme

- $K(X) = K(VB(X))$ X scheme, $VB(X)$ cat. of vector bundles on X

  ↳ tensor product $VB(X) \times VB(X) \longrightarrow VB(X)$

§1. Additivity

↳ Same statement for Waldhausen and exact categories

Idea: given a short exact sequence of exact functors $F' \to F \to F''$ then the induced maps on the K-groups satisfy $F_* = F'_* + F''_*$

Def: (s.e.s. of exact functors)

(a) If $\mathcal{B}$ and $\mathcal{C}$ are exact categories, the sequence $F' \to F \to F''$ of exact functors $F, F', F'' : \mathcal{B} \to \mathcal{C}$ is a short exact sequence of exact functors if

$$0 \to F'(B) \to F(B) \to F''(B) \to 0 \text{ is an}$$

exact sequence in $\mathcal{C}$ for every $B \in \mathcal{B}$.

And we denote it: $F' \rightarrowtail F \twoheadrightarrow F''$

(b) If $\mathcal{B}$ and $\mathcal{C}$ are Waldhausen categories, we say that $F' \rightarrowtail F \twoheadrightarrow F''$ is a short exact sequence or cofibration sequence of exact functors if:

- $F'(B) \rightarrowtail F(B) \twoheadrightarrow F''(B)$ is a cofibration sequence in $\mathcal{C}$, $\forall B \in \mathcal{B}$
- $\forall A \rightarrowtail B$ cofibration in $\mathcal{B}$, the map $F(A) \cup_{F'(A)} F'(B) \rightarrowtail F(B)$ is a cofibration in $\mathcal{C}$

Thm: (Additivity theorem)

Let $F' \rightarrowtail F \twoheadrightarrow F''$ be a short exact sequence of exact functors from $\mathcal{B}$ to $\mathcal{C}$

Then $\underset{\text{homotopic}}{F_* \simeq F'_* + F''_*}$ as H-space maps $K(\mathcal{B}) \longrightarrow K(\mathcal{C})$.

Therefore on the homotopy groups we have

$$F_* = F'_* + F''_* : K_i(\mathcal{B}) \longrightarrow K_i(\mathcal{C})$$

We can extend this result also to <u>admissibly exact sequences</u>

↳ i.e. $0 \to F_0 \to F_1 \to \dots \to F_n \to 0$

s.t. • we have decompositions

$$\begin{tikzcd} F_i \arrow[rr] \arrow[dr, two heads] & & F_{i+1} \\ & G_i \arrow[ur, tail] & \end{tikzcd} \quad \forall i$$

• $G_i \rightarrowtail F_i \twoheadrightarrow G_{i+1}$ is exact

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Corollary (Additivity for long exact sequences)

If

$$0 \to F_0 \to F_1 \to \dots \to F_n \to 0$$

is an admissibly exact sequence of exact functors $\mathcal{B} \to \mathcal{C}$, then

$$\sum (-1)^p F^p_* = 0 \quad \text{as maps} \quad K_i(\mathcal{B}) \to K_i(\mathcal{C})$$

Applications: PROJECTIVE BUNDLE THM

Let $\mathcal{E}$ be a vector bundle of rank $r+1$ over a quasi projective scheme X. Consider the projective bundle $\mathbb{P}(\mathcal{E}) \xrightarrow{\pi} X$, and the exact functors

$$u_i : VB(X) \longrightarrow VB(\mathbb{P}(\mathcal{E}))$$
$$\mathcal{N} \longmapsto \pi^*(\mathcal{N})(-i)$$

Then $K_0(\mathbb{P}(\mathcal{E}))$ is a free $K_0(X)$-module with basis the twisting line bundles $\{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-i)\}_{i=0}^{r}$.

In particular, if $\mathcal{E}$ is the trivial bundle $\rightsquigarrow$ then $K_0(\mathbb{P}^r_X) \cong \dfrac{K_0(X)[z]}{(z^{r+1})}$

Thm: (Projective bundle thm) $\longrightarrow$ (gives a relation between the K-theory of a projective scheme and K-theory of a projective bundle on it.)

Let X and $\mathcal{E}$ as above.

Then the u_i induce an homotopy equivalence,

$$K(X)^{r+1} \simeq K(\mathbb{P}(\mathcal{E}))$$

Thus $\quad K_*(X) \otimes_{K_0(X)} K_0(\mathbb{P}(\mathcal{E})) \cong K_*(\mathbb{P}(\mathcal{E}))$ as rings

Special case: if $\mathcal{E}$ is the trivial bundle $\rightsquigarrow \mathbb{P}(\mathcal{E}) = \mathbb{P}^n_X$, then

$$K_*(\mathbb{P}^n_X) \cong \frac{K_*(X)[z]}{(z^{r+1})} \quad \text{as rings}$$

idea of the proof :

- We would like to apply the direct image functor π_* to a vector bundle and get a vector bundle, this is not always true,

$\hookrightarrow$ need the following definition:
Mumford regular vector bundles

Def: A quasi-coherent $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$ - module $\mathcal{F}$ is called Mumford-regular if $\forall q > 0$,
the higher derived sheaves $R^q \pi_* \mathcal{F}(-q)$ vanish.

Notation: MR is the cat. of M-R vector bundles

- Quillen's Resolution thm

$\rightsquigarrow$ use Additivity thm

Resolution:

idea: given an isom. btw the K-groups of an exact cat. and of the K-groups of a subcategory in the case where objects in the bigger category admit a finite resolution by obj. in the subcategory

<u>Def</u>: (II.7.0.1 pag 141)

Let $\mathcal{B}, \mathcal{C}$ be exact categories, $\mathcal{B} \subset \mathcal{C}$ full subcategory

- We say that $\mathcal{B}$ is a <u>full exact subcategory</u> of $\mathcal{C}$ if the exact sequences in $\mathcal{B}$ are precisely the sequences in $\mathcal{B}$ which are exact in $\mathcal{C}$ (equivalently, if the functor $\mathcal{B} \hookrightarrow \mathcal{C}$ is exact)

- We say that $\mathcal{B}$ is closed under kernels of admissible surjections in $\mathcal{C}$ if whenever $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{C}$ is an exact sequence with $B, C \in \mathcal{B}$ then A is also in $\mathcal{B}$

Why do we need to add this?

<u>Thm</u>: (3.1) (Resolution theorem for exact categories)

Let $\mathcal{H}$ be an exact cat. and $\mathcal{P} \subset \mathcal{H}$ a full exact subcategory s.t.

- $\mathcal{P}$ is closed under extensions in $\mathcal{H}$
- $\mathcal{P}$ is closed under kernels of admissible surjections in $\mathcal{H}$
- $\forall M \in \mathcal{H}$ has a <u>finite $\mathcal{P}$-resolution</u>:

$$0 \to P_n \to \dots \to P_1 \to P_0 \to M \to 0$$

Then

$$K(\mathcal{P}) \simeq K(\mathcal{H}) \quad \text{(homotopy equivalence)}$$

$$\Rightarrow K_i(\mathcal{P}) \cong K_i(\mathcal{H}) \quad \forall i \quad \text{(isom.)}$$

Applications to resolution thm:

Corollary:

If R is a Noeth. regular ring, then

$$K_i(R) \cong G_i(R) \quad \forall i$$

proof:

recall the definitions:

- $K(R) = K(\mathcal{P}(R))$ where $\mathcal{P}(R)$ is the cat. of finitely gen. projective R-modules

- $G(R) = K(R\text{-mod})$ where R-mod is the cat. of finitely gen. R-modules

Key fact:

- R regular $\Rightarrow$ every f.g. R-mod has a finite projective resolution

then apply Resolution Thm to $\underbrace{R\text{-mod} \subseteq \mathcal{P}(R)}$.

Rmk: (see Lemma II.7.7.1 to see why $\mathcal{P}(R)$ is closed under kernel of surjections in R-mod

$\rightsquigarrow$ for schemes:

<u>Cor</u>:

If X is a <u>separated regular Noeth.</u> scheme

then
$$K(X) \simeq G(X)$$

<u>proof</u>: apply resolution thm to

$$\underbrace{VB(X)}_{\text{vector bundles on } X} \subseteq Coh(X)$$

use <u>Thm</u> $\left\{ \begin{array}{l} X \text{ separated Noeth. regular scheme} \\ \Rightarrow \forall \mathcal{F} \in Coh(X) \text{ has a finite resolution by vector bundles} \end{array} \right.$

<u>Rmk</u>: (DA CAPIRE MEGLIO)
This does <u>not</u> hold for non separated regular noeth. schemes.

$\hookrightarrow$ X is the affine line with double origin over a field.

$$G_0(X) = \mathbb{Z} \oplus \mathbb{Z} \qquad K_0(X) = \mathbb{Z}$$

§3. Devissage:

→ for abelian categories (why not for exact categories? ↳ because we need quotients)

(idea: isom. btw K-groups of an ab. cat. and an ab. sub cat. provided that every obj has a finite filtration with quotients in the subcat.)

Thm: (V.4.1)

Let $i: \mathcal{A} \subset \mathcal{B}$ be an inclusion of abelian categories s.t.

- $\mathcal{A}$ is an exact abelian subcategory of $\mathcal{B}$ → difference between this and exact subcat.? this means exact subcat. which is abelian
- $\mathcal{A}$ is closed in $\mathcal{B}$ under subobjects and quotients

Suppose also that every object B of $\mathcal{B}$ has a finite filtration

$$0 = B_r \subset \dots \subset B_1 \subset B_0 = B \quad \text{of objects in } \mathcal{B}$$

s.t. every (sub) quotient B_i/B_{i+1} lies in $\mathcal{A}$

then $K(\mathcal{A}) \simeq K(\mathcal{B})$ homotopy equivalence

and $K_*(\mathcal{A}) \cong K_*(\mathcal{B})$ isom. as rings

why? Do ab. cats always have an associative pairing $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$?

Applications to Devissage:

↳ 2 applications

(1)

Corollary (4.2) (G-theory of a ring doesn't see nilpotent elements)

If I is a nilpotent ideal in a Noeth. ring R, then

$$G(R/I) \simeq G(R)$$

$$\Longrightarrow G_*(R/I) \cong G_*(R)$$

proof:

Devissage applies to (R/I)-mod (f.g.) $\subset$ R-mod (f.g.) because every f.g. R-module M has a finite filtration by submodules MI^k

(if $I^n = 0$,

$0 = MI^n \subset MI^{n-1} \subset \dots \subset MI \subset M$ is a finite filtr. of M with quotients $\frac{MI^k}{MI^{k+1}}$ finite R/I-modules

$\square$

⇝ scheme version:

Corollary: X Noetherian scheme, then

$$G(X_{red}) \simeq G(X)$$

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<u>Corollary</u> (4.4.) [R-modules with support]

R Noeth. ring, $I \subset R$ ideal, and
$\mathcal{M}_I(R) \subset R$-mod be the ab. subcat. of all M s.t. $MI^n = 0$ for some n,
then

$$G_*(R/I) \cong K_*(\mathcal{M}_I(R))$$

<u>proof</u>: want to apply Devissage to $\mathcal{M}(R/I) \subset \mathcal{M}_I(R)$

inclusion given by restriction of scalars

if $M \in \mathcal{M}_I(R)$, then the filtration

$$M \supset MI \supset MI^2 \supset \dots \supset MI^n = 0$$

is a finite filtration and the quotients $\dfrac{MI^k}{MI^{k+1}}$
are (R/I)-modules. $\square$

The analogous result for schemes is the following:

<u>Corollary</u>: (K-theory of sheaves with support on a closed subscheme)

Let $Z \overset{i}{\subset} X$ be a closed subscheme of a Noeth. scheme, then

$$G(Z) = K(\mathrm{Coh}(Z)) \cong K(\underbrace{\mathrm{Coh}_Z(X)})$$

coherent sheaves on X with support on Z

<u>proof</u>: by Devissage on $\mathrm{Coh}(Z) \subset \mathrm{Coh}_Z(X)$

4.) Localization:

Def: Let $\mathcal{A}$ be a small abelian category. We say that $\mathcal{B} \subset \mathcal{A}$ is a <u>Serre subcategory</u> if it is an abelian subcategory closed under subobjects, quotients, and extensions.
i.e. if $0 \to B \to C \to D \to 0$ exact in $\mathcal{A}$, then

$$C \in \mathcal{B} \iff B, D \in \mathcal{B}$$

In this case, you can define a <u>quotient abelian category</u> $\mathcal{A}/\mathcal{B}$ as the category with

<u>objs</u>: objects of $\mathcal{A}$

<u>morph.</u>: morphisms $A_1 \to A_2$ are equivalence classes of diagrams $A_1 \xleftarrow{f} A' \xrightarrow{g} A_2$ in $\mathcal{A}$

where f is a $\mathcal{B}$-iso, i.e. $\ker(f)$ and $\operatorname{coker}(f)$ are in $\mathcal{B}$

Such morphism is equivalent to
$A_1 \xleftarrow{f'} A'' \xrightarrow{g'} A_2 \iff \exists$ a commutative diagram

$$\begin{tikzcd} & A' \arrow[dl] \arrow[dr] & \\ A_1 & A \arrow[l] \arrow[r] \arrow[u] \arrow[d] & A_2 \\ & A'' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

where $A \to A'$ and $A \to A''$ are $\mathcal{B}$-isos

Thm: (Abelian Localization Theorem)

Let $\mathcal{B}$ be a Serre subcategory of a small abelian category $\mathcal{A}$. Then

$$K(\mathcal{B}) \longrightarrow K(\mathcal{A}) \xrightarrow{loc} K(\mathcal{A}/\mathcal{B})$$

is a homotopy fibration sequence.

Then, there is a long exact seq. of homotopy groups

$$\longrightarrow K_{n+1}(\mathcal{A}/\mathcal{B}) \xrightarrow{\partial} K_n(\mathcal{B}) \longrightarrow K_n(\mathcal{A}) \xrightarrow{loc} K_n(\mathcal{A}/\mathcal{B}) \xrightarrow{\partial} \cdots$$

ending in $K_0(\mathcal{B}) \to K_0(\mathcal{A}) \to K_0(\mathcal{A}/\mathcal{B}) \to 0$

Rmk:

R Noeth. $\Longrightarrow$ R-mod is an ab. cat.

X Noeth. $\Longrightarrow$ $\mathrm{Coh}(X)$ is an ab. cat.

Applications:

Corollary (Example 6.11): [Localization sequence for the G-theory of schemes]

X Noeth. scheme

$j: U \hookrightarrow X$ open subscheme, $Z = X \setminus U$

then
$$G(Z) \to G(X) \to G(U)$$
is a homotopy fibration

$\leadsto$ there is a l.e.s. on the homotopy groups:

$$\cdots \xrightarrow{\partial} G_n(Z) \to G_n(X) \xrightarrow{j^*} G_n(U) \to G_{n-1}(Z) \to \cdots$$

proof:
Apply the Localization thm to
$\mathcal{A} = \mathrm{Coh}(X)$ $\quad\quad \mathcal{B} = \mathrm{Coh}_Z(X)$

- Fact: $\dfrac{\mathrm{Coh}(X)}{\mathrm{Coh}_Z(X)} = \mathrm{Coh}(U)$

- By Devissage: $K(\mathrm{Coh}_Z(X)) \simeq G(Z)$

Thm (Thomason – Trobaugh)

X q.c., q.s. scheme $\quad U \xhookrightarrow[\text{q.c. open}]{} X \quad Z = X \setminus U$

$\Rightarrow \quad K(X \text{ on } Z) \longrightarrow K(X) \xrightarrow{j^*} K(U)$

is an homotopy fibration

- $K(X \text{ on } Z) = K(\underbrace{Ch_{perf,Z}(X)}_{\text{Waldhausen cat.}})$

of perfect complexes

on X which are exact on U

quasi iso up to bounded complexes