Research seminar SS2022 - Talk 7

Explicit Jacquet-Langlands correspondence ([Eme02])

§.1 The Hecke action on supersingular elliptic curves

Let $N \in \mathbb{Z}$, $N > 3$ be a prime.

Talk 4 $\Rightarrow$ over $\overline{\mathbb{F}}_N$ there are only finitely many isom. classes of supersingular elliptic curves, and such curves are all defined over $\mathbb{F}_{N^2}$.

We let $\mathcal{X} :=$ free abelian group on the set of isomorphism classes of supersingular elliptic curves over $\overline{\mathbb{F}}_N$.

There is a geometric interpretation of $\mathcal{X}$:

Consider the $\Gamma_0(N)$-level moduli problem

$$T \overset{F_0(N)}{\longmapsto} \left\{ (E, C) \;\middle|\; \begin{array}{l} E \text{ family of elliptic curves over } T \\ C \hookrightarrow E \text{ finite flat subgroup scheme} \\ \text{locally free of rank } N \text{ and cyclic} \\ (\text{i.e. fppf locally it admits a generator}) \end{array} \right\} \Big/ \cong$$

a scheme

DR73 + KM85 prove that:

there exists a scheme $Y_0(N) \to \operatorname{Spec}(\mathbb{Z})$ which satisfies:

(i) $Y_0(N)$ is flat over $\mathbb{Z}$, of rel. dim 1

(ii) $Y_0(N) \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Z}[1/N])$ is smooth over $\operatorname{Spec}(\mathbb{Z}[1/N])$

(iii) $Y_0(N)$ is a so-called coarse moduli space for the moduli problem $F_0(N)$; in particular "forgetting the $\Gamma_0(N)$-level structure" induces a (finite) morphism $Y_0(N) \xrightarrow{f} \mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec}(\mathbb{Z}[j])$ (degree $= \# \mathbb{P}^1(\mathbb{F}_N) = N+1$)

(iv) In particular if k is an alg. closed field (of $\underline{\text{any}}$ char.), there is a "natural" bijection $F_0(N)(k) = Y_0(N)(k)$

<u>Rmk</u> As $Y_0(N) \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Z}[1/N])$ is smooth over $\mathbb{Z}[1/N]$ we know that the possible singularities of $Y_0(N)$ are detected by the study of $Y_0(N) \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{F}_N) = Y_0(N)_{\mathbb{F}_N}$

Let S be an $\mathbb{F}_N$-scheme and let $E \xrightarrow{\pi} S$ be a family of elliptic curves. We let F_{abs} denote the absolute Frobenius on $\mathbb{F}_N$-schemes (i.e. $a \mapsto a^N$ on sections). Then we have a diagram:

$$\begin{tikzcd}
E \arrow[r, dashed, "\exists! \, F_E"] \arrow[rr, bend left, "F_{abs,E}"] \arrow[dr, "\pi"'] & E^{(N)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\pi"] \\
 & S \arrow[r, "F_{abs,S}"'] & S
\end{tikzcd}$$

$F_E : E \to E^{(N)}$ is the so-called relative Frobenius map

F_E is an isogeny of abelian schemes of deg N ($= \operatorname{rank}(\ker(F_E))$) whose dual is called Verschiebung $V_E : E^{(N)} \to E$ (also of deg N)

<u>Def</u>: $E \xrightarrow{\pi} S$ as above is called ordinary if every one of its geom. fibers is ordinary (equiv. V_E is étale)

the natural transf. $\quad [E] \overset{1}{\longmapsto} [(E, \ker(F_E))]$ of moduli problems over $\mathbb{F}_N$

$\quad [E] \overset{2}{\longmapsto} [(E^{(N)}, \ker(V_E))]$

induce morphisms among the corresp. coarse moduli spaces

$$\begin{tikzcd}
\mathbb{A}^1_{\mathbb{F}_N} \arrow[dr, "\psi_1"] \arrow[rr, bend left, "id"] & & \mathbb{A}^1_{\mathbb{F}_N} \\
 & Y_0(N)_{\mathbb{F}_N} \arrow[ur, "f"] \arrow[dr, "f \circ w"] & \\
\mathbb{A}^1_{\mathbb{F}_N} \arrow[ur, "\psi_2"] \arrow[rr, bend right, "id"'] & & \mathbb{A}^1_{\mathbb{F}_N}
\end{tikzcd}$$

w involution on $Y_0(N)$ induced by $[(E, C)] \mapsto [(E/C, E[p]/C)]$

If we look at $\overline{\mathbb{F}_N}$-points we get that $\psi_1(x) = \psi_2(y)$ for $x, y \in \mathbb{A}^1_{\mathbb{F}_N}(\overline{\mathbb{F}_N})$

$\Leftrightarrow$ the corresp. elliptic curves E_x, E_y are supersingular with $E_x \cong E_y^{(N)}$

↳ <u>sketch of proof</u>: ($\Rightarrow$) if $E/\overline{\mathbb{F}_N}$ is ordinary $\Rightarrow$ $\ker(F_E) \cong \mu_N$, $\ker(V_E) \cong \mathbb{Z}/N\mathbb{Z}$

as finite group schemes over $\overline{\mathbb{F}_N}$,

hence $\psi_1(x) = \psi_2(y) \Rightarrow E_x, E_y$ supersingular and $E_x \cong E_y^{(N)}$

($\Leftarrow$) since for E supersingular $E \cong E^{(N^2)}$ we have that

$E_x \cong E_y^{(N)} \Leftrightarrow E_y \cong E_x^{(N)}$ and remember that in this situation we have a diagram

$$\begin{tikzcd} E_x \arrow[r, "F"] \arrow[d, phantom, "\wr\wr"] & E_x^{(N)} \cong E_y \\ E_x^{(N^2)} \arrow[ur, "V"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

so that clearly $\psi_1(x) = \psi_2(y)$ ∎

<u>Moral</u>: $Y_0(N)_{\mathbb{F}_N}(\overline{\mathbb{F}_N})$ is given by two affine lines intersecting at the (finitely many!) points given by supersingular elliptic curves.

<u>Fact</u>: the crossings at supersingular points are transversal (nodes)

(one can prove that over $W(\overline{\mathbb{F}_p})$ the completed local ring at a point lying over a supersingular one is isomorphic to

$$W(\overline{\mathbb{F}_p})[[u,v]]/(uv - p^{\varepsilon})$$ $\varepsilon \in \mathbb{Z}$ depending on the point)

The morphism $Y_0(N) \to \mathbb{A}^1_{\mathbb{Z}}$ can be extended to a morphism $X_0(N) \to \mathbb{P}^1_{\mathbb{Z}}$ where $X_0(N)$ is the compactified modular curve over $\mathbb{Z}$ (proper, flat$/\mathbb{Z}$, $X_0(N) \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{Z}[1/N])$ smooth over $\mathbb{Z}[1/N]$, ...) and that looking at $X_0(N)_{\mathbb{F}_N}(\overline{\mathbb{F}_N})$ we find that it is given by two rational curves with crossings at the supersingular points.

Moreover since the arithmetic genus is constant in flat families (we work over $Spec(\mathbb{Z})$) we can label the isom. classes of supers. elliptic curves as $S = \{x_0, x_1, \dots, x_g\}$ where

g = arithm. genus of every geom. fiber of $X_0(N)$ = geom/top. genus of $X_0(N)(\mathbb{C})$

Hence $\mathcal{X}$ = group of divisors supported on the set of sing. points (nodes) of the curve $X_0(N)_{\mathbb{F}_N}$

$\cup$

$\mathcal{X}^0$ = degree zero divisors in $\mathcal{X}$

$$0 \to \mathcal{X}^0 \to \mathcal{X} \xrightarrow{\deg} \mathbb{Z} \to 0$$

For every p prime we get correspondences

$$\begin{tikzcd} & X_0(Np) \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\ X_0(N) & & X_0(N) \end{tikzcd}$$

where α, β are essentially induced by the assignments:

$[(E, C)] \overset{\alpha}{\longmapsto} [(E, C_N)]$ $\quad C_N \subseteq C$ unique cyclic subgr. of rank N

$[(E, C)] \overset{\beta}{\longmapsto} [(E/C_p, C_N/C_p)]$ $\quad C_p \subseteq C$ " " " " " p

inducing via Picard & Albanese functoriality morphisms

$$J_0(N) = Jac(X_0(N)_{\mathbb{Q}}) = Pic^0(X_0(N)_{\mathbb{Q}}) \xrightarrow[T_p = \alpha_* \circ \beta^*]{} J_0(N)$$

$\hookrightarrow$ an abelian variety $/\mathbb{Q}$ of dimension g

Alb. covariant $\quad$ $\hookrightarrow$ Picard contr.

In this way we know that T_p is defined over $\mathbb{Q}$.

If we look at $\mathbb{C}$-points one gets back the classical interpretation

$$T_p : Pic^0(X_0(N)(\mathbb{C})) \longrightarrow Pic^0(X_0(N)(\mathbb{C})) \qquad (*)$$

$$[(E, C)] \longmapsto \sum_{\substack{D \subseteq E \\ \text{cyclic of order } p}} [(E/D, C+D/D)]$$

if $p = N$ one asks that $D \neq C$ in the summation

We would like an integral version of T_p since we're interested in the Hecke action mod N.

Idea: replace $J_0(N)$ with its Néron model $\mathcal{J}_0(N)/\mathbb{Z}$

Fact: Let R be a Dedekind domain, $K = \mathrm{Frac}(R)$, let $A \to \mathrm{Spec}(K)$ be an abelian variety, then there exists a unique (up to unique iso) pair $(\mathcal{A}, \varphi)$ where $\mathcal{A} \to \mathrm{Spec}(R)$ is a commutative group scheme over R, $\varphi : \mathcal{A} \times_R \mathrm{Spec}(K) \to A$ is an isomorphism of K-schemes and $\mathcal{A}$ satisfies the "Néron mapping property", i.e. for every smooth R-scheme $\mathcal{B}$ there is a bijection

$$\mathrm{Hom}_R(\mathcal{A}, \mathcal{B}) \to \mathrm{Hom}_K(\mathcal{A}_K, \mathcal{B}_K) \cong \mathrm{Hom}_K(A, \mathcal{B}_K)$$

$\Rightarrow$ we get $T_p : \mathcal{J}_0(N) \to \mathcal{J}_0(N)$ induced by T_p and we try to reduce everything mod N. It is known that $\mathcal{J}_0(N)$ has semiabelian reduction mod N, in particular there is an exact seq. of comm. group schemes / $\mathbb{F}_N$

$$1 \to T \to \mathcal{J}_0(N)^0_{\mathbb{F}_N} \xrightarrow{w} \mathcal{J}_0(1)_{\mathbb{F}_N} \times \mathcal{J}_0(1)_{\mathbb{F}_N} \to 1$$

a torus such that

$$X(T) = \mathrm{Hom}_{\overline{\mathbb{F}}_N}(T, \mathbb{G}_m) \cong \mathcal{X}^0$$

obtained using π_1, π_2 appropriately

(so only in char N we have this map)

Facts: (i) the T_p-action on $\mathcal{J}_0(N)_{\mathbb{F}_N}$ induces an action on $\mathcal{J}_0(N)^0_{\mathbb{F}_N}$ and on T and the induced action on $\mathcal{X}^0$ viewed as $X(T)$ matches with the action of the T_p's on supersingular elliptic curves given by the formula $(*)$ essentially

(ii) Working over $\mathbb{C}$ we get isomorphisms of Hecke modules

$$S_2(\Gamma_0(N)) \cong H^0(X_0(N)_{\mathbb{C}}, \Omega^1_{X_0(N)_{\mathbb{C}}/\mathbb{C}}) \cong \mathrm{Cot}_0(\mathrm{Jac}(X_0(N)_{\mathbb{C}}))$$

$$f \longmapsto f\,dz$$

essentially using the classical constr. of Jac / $\mathbb{C}$

§2. Connections between $\mathcal{X}$ and $\mathcal{M}$ (to be defined)

$\mathbb{T}' := \mathbb{Z}[T_n, n \geq 1]$ (polynomial ring in the variables T_n $n\geq 1$)

$\mathbb{T} := \mathbb{T}(N) :=$ quotient of $\mathbb{T}'$ acting faithfully on $M_2(\Gamma_0(N))$

It is a finite $\mathbb{Z}$-algebra $\left(\mathbb{T} \subseteq \mathrm{End}_{\mathbb{C}}(M_2(\Gamma_0(N))) \right)$

If $f \in M_2(\Gamma_0(N))$ we write $f = \sum_{n=0}^{+\infty} a_n(f) q^n$ for the q-exp at ∞

$\mathcal{M} := \{ f \in M_2(\Gamma_0(N)) \mid a_n(f) \in \mathbb{Z} \ \forall n \geq 1,\ 2\cdot a_0(f) \in \mathbb{Z} \}$

Rmk: $\mathbb{T} \curvearrowright \mathcal{M}$ faithfully — use e.g. the Ramanujan conjecture (Deligne, 1971) to see this

$\mathcal{M}^0 = \{ f \in \mathcal{M} \mid f \text{ cuspidal} \} = \{ f \in \mathcal{M} \mid a_0(f) = 0 \}$

$$0 \to J' \hookrightarrow \mathbb{T}' \twoheadrightarrow \mathbb{T}^0 \to 0$$

$\mathbb{T}^0 =$ quotient of $\mathbb{T}'$ acting faithfully on $\mathcal{M}^0$

$$E(\tau) = \frac{N-1}{24} + \sum_{n=1}^{+\infty} \sigma^{(N)}(n) q^n = E_2(\tau) - N\, E_2(N\tau)$$

$E_2(\tau) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n) q^n$ "Eisenstein series of wt 2"

$\sigma(n) := \sum_{d \mid n} d \qquad \sigma^{(N)}(n) := \sum_{\substack{d\mid n \\ N \nmid d}} d \qquad$ write $\frac{N-1}{12} = \frac{\delta}{\Delta}$ $(\delta, \Delta) = 1$, then

$\Delta \cdot E \in \mathcal{M} \setminus \mathcal{M}^0$ (the only Eisenstein series of level $\Gamma_0(N)$) and defines

$\mathbb{T}' \twoheadrightarrow \mathbb{T} \twoheadrightarrow \mathbb{Z}$ (ring hom.) $\qquad \mathbb{T}^{\mathrm{Eis}} = \mathbb{T}/I' \cong \mathbb{Z} \qquad I' = \ker(\mathbb{T}' \twoheadrightarrow \mathbb{Z})$

$T_n \longmapsto \sigma^{(N)}(n)$

set $\mathcal{M}^{\mathrm{Eis}} = \langle \Delta \cdot E \rangle_{\mathbb{Z}} = \langle \Delta \cdot E \rangle_{\mathbb{T}}$

Thm 1 (thm 3.1 [Em02])

(i) The Hecke action on $\mathcal{X}/\mathcal{X}^0 \cong \mathbb{Z}$ factors through $\mathbb{T}^{Eis}$

(ii) " " " " $\mathcal{X}^0$ makes $\mathcal{X}^0$ a faithful $\mathbb{T}^0$ module

(iii) " " " " $\mathcal{X}$ " $\mathcal{X}$ " " $\mathbb{T}$ "

Proof: Since $\mathbb{T} \overset{(+)}{\hookrightarrow} \mathbb{T}^{Eis} \oplus \mathbb{T}^0$ is injective we know that

$\ker(\mathbb{T}' \twoheadrightarrow \mathbb{T}) = I' \cap J'$

Also $\operatorname{coker}(+)$ is a finite group (of order δ) and we have a surjection

$\operatorname{coker}(+) \twoheadrightarrow \operatorname{coker}(\mathbb{T}' \longrightarrow \mathbb{T}^{Eis} \oplus \mathbb{T}^0) = \dfrac{\mathbb{T}'}{I' + J'} \Rightarrow \delta \cdot \mathbb{T}' \subseteq I' + J'$ (i.e. $\delta \in I' + J'$)

$\Rightarrow \delta \cdot (I' \cap J') \subseteq I' \cdot J'$

(i) : expression (*) $\Longrightarrow$ $\deg(T_p \cdot x_i) = \begin{cases} 1+p = \# \mathbb{P}^1(\mathbb{F}_p) & \text{if } p \neq N \\ 1 & \text{if } p = N \end{cases} = \sigma^{(N)}(p)$

$\Longrightarrow$ get a faithful action of $\mathbb{T}^{Eis} = \mathbb{T}'/I' \cong \mathbb{Z}$ on $\mathcal{Z} = \dfrac{\mathcal{X}}{\mathcal{X}^0}$

(ii) The discussion in § 1 shows that $\operatorname{Ann}_{\mathbb{T}'}(\mathcal{X}^0) = \operatorname{Ann}_{\mathbb{T}'}(S_2(\Gamma_0(N))) \underset{\text{by def}}{=} J'$

(iii) $t \in \operatorname{Ann}_{\mathbb{T}'}(\mathcal{X})$ then clearly $t \in I' \cap J'$ by (i) + (ii)

Also if $t = t_1 \cdot t_2$ $t_1 \in I'$ $t_2 \in J'$ then $t \in \operatorname{Ann}_{\mathbb{T}'}(\mathcal{X})$.

indeed if $\Delta \in \mathcal{X}$ we know $t_1 \cdot \Delta = 0$ in $\mathcal{X}/\mathcal{X}^0 \Rightarrow$

$\exists \Delta' \in \mathcal{X}^0$ st $\Delta' = t_1 \Delta \Rightarrow t_1 t_2 \cdot \Delta = 0$ ∎

But $\delta \cdot (I' \cap J') \subseteq I' \cdot J'$ and $\mathcal{X}$ is $\mathbb{Z}$-torsion free

$\Longrightarrow I' \cap J' = \operatorname{Ann}_{\mathbb{T}'}(\mathcal{X})$ so (iii) holds. ∎

Cor. $\mathcal{X} \otimes \mathbb{Q}$ is a free $\mathbb{T} \otimes_{\mathbb{Z}} \mathbb{Q}$ - module of rank one (so $\cong M_2(\Gamma_0(N), \mathbb{Q})$ as Hecke mod.)

Proof: By thm 1 $\mathcal{X} \otimes \mathbb{Q}$ is a faithful $\mathbb{T} \otimes_{\mathbb{Z}} \mathbb{Q}$- module

(use e.g. the Petersson inner product over $\mathbb{R}$ - -) and $\mathbb{T} \otimes_{\mathbb{Z}} \mathbb{Q}$ is semisimple

self adjoint subalg. $\Downarrow$ no nilpotent elem $\Downarrow$ semisimple

$\dim_{\mathbb{Q}}(\mathcal{X} \otimes \mathbb{Q}) = \dim_{\mathbb{Q}}(\mathbb{T} \otimes_{\mathbb{Z}} \mathbb{Q}) = g+1$ ∎

For $x_i \in S$ we set $e_i = \frac{1}{2} \# \mathrm{Aut}(E_i)$

<u>Fact (A)</u>: $\prod_{i=0}^{g} e_i = \Delta$

since $N > 3$, $e_i = 1$ unless $j(E_i) = 0,\ 1728$

$$\Downarrow$$

$$e_i = 3,\ 2$$

but $j = 0$ (1728) is supersing $\iff N \equiv -1 \bmod 3$ $(N \equiv -1 \bmod 4)$.

Then one has also explicit formulas for g given by:

$$g + 1 = \frac{N-1}{12} + \frac{1}{2}\varepsilon_N(0) + \frac{1}{2}\varepsilon_N(1728) \qquad \varepsilon_N(0) = \begin{cases} 0 & N \equiv 1 \bmod 3 \\ 1 & N \equiv 2 \bmod 3 \end{cases}$$

$$\varepsilon_N(1728) = \begin{cases} 0 & N \equiv 1 \bmod 4 \\ 1 & N \equiv 3 \bmod 4 \end{cases}$$

<u>Def</u>: We define a pairing $\mathcal{X} \times \mathcal{X} \to \mathbb{Z}$ by

$$\langle x_i, x_j \rangle = e_i \delta_{i,j} = e_j \delta_{i,j}$$

<u>Recall</u> (Talk 4): letting $L_{i,j} = \mathrm{Hom}_{\overline{\mathbb{F}}_N}(x_i, x_j)$ (with quadratic form given by deg) we know it is a free $\mathbb{Z}$-module of rank 4 such that deg is a positive definite quadratic form.

Moreover

$$L_{i,j} \xrightarrow{\cong} L_{j,i} \quad \text{(as quadratic spaces)}$$
$$\varphi \longmapsto \hat{\varphi}$$

Let $r_n(L_{i,j}) = \# \deg^{-1}(n)$, then by $(*)$ it follows easily that

$$\langle T_p x_i, x_j \rangle = \frac{1}{2} r_p(L_{i,j}) \quad \left(\text{and more gen. } \langle T_n x_i, x_j \rangle = \frac{1}{2} r_n(L_{i,j})\right)$$

<u>Corollary</u> (3.9. [Em02])

The pairing $\langle\, , \rangle$ is $\mathbb{T}$ bilinear

<u>Proof</u>: Obvious since $r_n(L_{i,j}) = r_n(L_{j,i})$

The pairing $\langle\, ,\rangle$ induces a morphism of $\mathbb{T}$ modules

$$\mathcal{X} \otimes_{\mathbb{T}} \mathcal{X} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{T}, \mathbb{Z})$$
$$x_i \otimes x_j \longmapsto (T_n \longmapsto \langle T_n x_i, x_j \rangle)$$

By the duality between mod. forms and Hecke algebras we get

$$\mathcal{X} \otimes_{\mathbb{T}} \mathcal{X} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{T}, \mathbb{Z}) \cong \mathcal{N} := \{ f \in M_2(\Gamma_0(N)) \mid a_n(f) \in \mathbb{Z} \ \forall n \geq 1 \}$$

with the composite $\vartheta : \mathcal{X} \otimes_{\mathbb{T}} \mathcal{X} \to \mathcal{N}$,

where ϑ is uniquely determined by $a_n(\vartheta(x \otimes x')) = \langle T_n x, x' \rangle$ $\quad x, x' \in \mathcal{X}, \ n \geq 1$

so that in particular $a_n(\vartheta(x_i \otimes x_j)) = \frac{1}{2} r_n(L_{i,j})$

Def: Let $\mathfrak{X} := \Delta \cdot \sum_{i=0}^{g} \frac{x_i}{e_i} \in \mathcal{X}$ (by fact (Δ))

Then $\langle \mathfrak{X}, x_i \rangle = \Delta$ $\forall i = 0, \dots g$; let $\mathcal{X}^{Eis} \subseteq \mathcal{X}$ be the maximal $\mathbb{T}$-submodule of $\mathcal{X}$ such that the $\mathbb{T}$-action factors through $\mathbb{T}^{Eis}$

$\Rightarrow$ $\mathcal{X}^{Eis}$ must be a rank 1 $\mathbb{Z}$-submodule of $\mathcal{X}$ by thm 1

<u>Lemma</u> $\mathcal{X}^{Eis} = \langle \mathfrak{X} \rangle_{\mathbb{Z}}$

Proof: $\langle T_n \mathfrak{X}, x_j \rangle = \langle \mathfrak{X}, T_n x_j \rangle = \Delta \sum_{i=0}^{g} \frac{\langle x_i, T_n x_j \rangle}{e_i} =$

$= \Delta \cdot \sum_{i=0}^{g} \#\{n\text{-isog. } x_i \to x_j\} / e_i =$

$= \Delta \cdot \#\{n\text{-isog. with source } x_j\} = \Delta \cdot \sigma'(n)$

$\Longrightarrow T_n \mathfrak{X} = \sigma'(n) \cdot \mathfrak{X} \Rightarrow \mathfrak{X} \in \mathcal{X}^{Eis}$

But $\gcd \{ \Delta / e_i \mid i = 0, \dots, g \} = 1 \Rightarrow \langle \mathfrak{X} \rangle_{\mathbb{Z}} = \mathcal{X}^{Eis}$ ∎

<u>Cor.</u>: $\vartheta(\mathfrak{X} \otimes x_i) = \Delta \cdot E$ $\quad \forall i = 0, \dots g$

Proposition 2 (3.15 [Emo02]) The following hold and are mutually equivalent:

(i) $a_0(\vartheta(x_i \otimes x_j)) = 1/2 \quad \forall i,j \in \{0,\dots,g\}$

(ii) $x, y \in \mathcal{X}$ then $a_0(\vartheta(x \otimes y)) = \frac{\deg x \cdot \deg y}{2}$

(iii) $\vartheta(x_i \otimes x_j) = \frac{1}{2} \Theta(L_{ij})$

(iv) $a_0(E) = \sum_{i=0}^{g} \frac{1}{2e_i}$ (v) $\deg \mathfrak{X} = \delta$ (vi) $\sum_{i=0}^{g} \frac{1}{e_i} = \frac{N-1}{12}$

Proof: Here $\Theta(L_{ij})$ is the theta series attached to $L_{ij} = L$ viewed as a left ideal in a maximal order of B = the quaternion alg over $\mathbb{Q}$ ramified at N and ∞ (cf talk 4)

$\cong \mathrm{Hom}(x_i, x_i) \otimes \mathbb{Q} \quad \forall i = 0, \dots g)$

$\Theta(L)(\tau) := \sum_{z \in L} \exp(2\pi i \tau N(z)/N(L))$ where
- $N(z)$ reduced norm on B
- $N(L) \in \mathbb{Q}_{>0}$ such that $\langle N(a) \mid a \in I \rangle_{\mathbb{Z}} = N(L) \cdot \mathbb{Z}$

Hecke proved (with analytic methods) that $\Theta(L) \in M_2(\Gamma_0(p))$ and $\Theta(L_{ij}) = 1 + 2 \cdot \sum_{n=1}^{+\infty} a_n(\vartheta(x_i \otimes x_j)) q^n$

$\Rightarrow$ (iii) holds true

(i) $\Leftrightarrow$ (ii) obvious

(i) $\Leftrightarrow$ (iii) follows from $a_0(\Theta(L_{ij})) = 1$ and $a_n((\Theta)(L_{ij})) = r_n(L_{ij})$

(iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ since $\deg(\mathfrak{X}) = \Delta \cdot \sum_{i=0}^{g} \frac{1}{e_i} \qquad a_0(E_N) = \frac{N-1}{24} = \frac{\delta}{2\Delta}$

(i) $\Leftrightarrow$ (iv)

$$a_0(\vartheta(x_i \otimes x_j)) = \frac{1}{\deg(\mathfrak{X})}\left[a_0(\vartheta(\mathfrak{X} \otimes x_j)) + a_0(\vartheta((\deg(\mathfrak{X}) x_i - \mathfrak{X}) \otimes x_j))\right]$$

$D_i := \deg(\mathfrak{X}) \cdot x_i - \mathfrak{X} \in \mathcal{X}^0 \Rightarrow \vartheta(D_i \otimes x_j) \in \mathcal{M}^0 \Rightarrow a_0(D_i \otimes x_j) = 0$

We also know $\vartheta(\mathfrak{X} \otimes x_i) = \Delta \cdot E \Rightarrow$

$$a_0(E) = \frac{\deg(\mathfrak{X}) \; a_0(\vartheta(x_i \otimes x_j))}{\Delta} = a_0(\vartheta(x_i \otimes x_j)) \sum_{i=0}^{g} \frac{1}{e_i} \quad \blacksquare$$

Cor. $\mathrm{Im}(\vartheta) \subseteq \mathcal{M}$ (by (iii) + expl. descr. of $\Theta(L_{ij})$)

§ 3. Reinterpretation of the ϑ-correspondence

Recall from Talk 4: let B be the quaternion alg. $/\mathbb{Q}$ ramified at $\{N, \infty\}$, there is a bijection

$$\{\text{s.s. elliptic curves} / \overline{\mathbb{F}}_N\}/_{\simeq} \xrightarrow{\sim} \mathrm{Pic}(B) := \left\{\begin{array}{c}\text{oriented max}\\ \text{orders in } B\end{array}\right\}/_{\simeq} =: \mathcal{E}$$

$$x_i = [E] \longmapsto (\mathrm{End}(E), \varphi_E) = \vec{L}_i$$

$$\varphi_E : \mathrm{End}(E) \to \overline{\mathbb{F}}_N$$
$$\varphi \longmapsto a_\varphi$$
$$\varphi^* \omega_E = a_\varphi \, \omega_E$$

hence we can reinterpret ϑ as

$$\tilde{\vartheta} : \mathrm{Div}(\mathcal{E}) \otimes_{\mathbb{T}} \mathrm{Div}(\mathcal{E}) \longrightarrow M_2(\Gamma_0(N)) \quad \left(\begin{array}{l}\text{note that } \mathrm{Im}(\vartheta) \leq M\\ \text{by prop. 2}\end{array}\right)$$

we let $\Sigma_0 \in \mathrm{Div}(\mathrm{Pic}(B)) \otimes \mathbb{Q}$ the divisor corresponding to our $\frac{x}{\Delta} \in X \otimes \mathbb{Q}$ then we have an explicit descr. of ϑ

$$\vartheta(D_1 \otimes D_2) = \frac{1}{2} \langle D_1, \Sigma_0 \rangle \langle D_2, \Sigma_0 \rangle + \sum_{n \geq 1} \langle D_1, T_n D_2 \rangle q^n$$

where
$$\mathrm{Div}(\mathrm{Pic}(B)) \times \mathrm{Div}(\mathrm{Pic}(B)) \longrightarrow \mathbb{Z}$$
$$(\vec{L}_i, \vec{L}_j) \longmapsto w_i \delta_{ij} \qquad w_i = \frac{1}{2}(\# L_i^\times) \overset{!}{=} e_i$$

(extended by $\mathbb{Q}$-lin...)

Appendix: Classical Jacquet-Langlands for $\Gamma_0(N)$

Let $B/\mathbb{Q}$ be as above, $\mathcal{O} \leq B$ be a maximal order,
$\mathcal{I}$ = fractional left ideals, $I \sim J \iff \exists \alpha \in B^\times$ s.t. $I = J \cdot \alpha$

set $\mathcal{Cl}(B) = \mathcal{I}/\sim$, $\mathcal{Cl}(B) = \{[I_1], \ldots [I_h]\}$ $\quad [I_i] = y_i$

$\mathcal{O}_i := I_i^{-1} \cdot \mathcal{O} \cdot I_i$ is a max order in B, $w_i = \# R_i^\times$

$\mathcal{Cl}(\mathcal{O}) \cong B^\times \backslash B_{\mathbb{A}}^\times / \hat{\mathcal{O}}^\times \cdot B_\infty^\times \qquad B_{\mathbb{A}} = B \otimes_{\mathbb{Q}} \mathbb{A}, \; B_\infty^\times = B \otimes_{\mathbb{Q}} \mathbb{R} \qquad \hat{\mathcal{O}} = \prod_p \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$

$$S_2(B) = \left\{ \varphi : \mathcal{C}\ell(B) \to \mathbb{C} \;\middle|\; \sum_{i=1}^{h} (2e_i)^{-1} \varphi(y_i) = 0 \right\}$$

Then there is a Hecke equivariant iso

$$S_2(B) \cong S_2(\Gamma_0(N)) \qquad \text{uniquely det. by}$$

$$\varphi_{ij} \longmapsto \Theta(I_i I_j^{-1}) \qquad \text{where } \frac{\varphi_{ij}(y_i)}{2e_i} = 1, \; \frac{\varphi_{ij}(y_j)}{2e_i} = -1$$

$$\varphi(y_k) = 0 \quad k \neq i, j$$

(Hecke, 1955)