Lecture 2

0. Recollections:

A. Kan, Cat_∞^Δ, $\mathcal{S}$ and Cat_∞.

B. The adjoint pair $\mathfrak{C} \dashv N_H$

C. Categorical equivalence of simplicial sets and ∞-categories

1. Cartesian fibrations and right fibrations

A. Grothendieck construction - fibered and cofibered categories

B. The ∞ category of ∞-categories, the functors $\mathfrak{C} \rightleftarrows N_H$

C. Cartesian fibrations, right fibration

D. Main equivalence theorem

2. Presheaves

A. Kan extensions

B. The ∞-category of presheaves

C. The Yoneda embedding

3. Adjoint functors

A. Adjunction for functors of ∞-categories

B localization

C. Adjointable squares, Fun^{LAd}, Fun^{RAd}

4. Filtered ∞-categories, Ind obj, accessible presentable ∞-categories

A. Basic definition

B Presentable ∞-cats as localization of presheaves

C Examples

D Adjoint functor theorem and the categories $\mathcal{P}r^L$, $\mathcal{P}r^R$

5. Stable ∞-categories

A Pointed ∞-categories, fibers and cofibers

B Stable ∞-categories, examples

C Suspension and loop functors

D existence of small limits in $\mathcal{P}r^{stb}$

0) Cat_∞, Cat_∞^Δ and equivalence

Recall $\mathrm{Path} : \Delta \to Cat_\infty^\Delta$

$$[n] \mapsto \mathrm{Path}[n]$$

$$\begin{tikzcd}
\Delta \arrow[r, "\mathrm{Path}"] \arrow[d, "j"'] & Cat_\infty^\Delta \\
(Set)^{\Delta^{op}} = Set_\Delta \arrow[ur, "\mathfrak{C}"'] &
\end{tikzcd}$$

$\mathfrak{C}$: left Kan extension

i.e. $\mathfrak{C}(K) = \operatorname{colim}_{([n],\, \Delta^n \xrightarrow{\alpha} K)} \mathrm{Path}[n]$

$\mathfrak{C}$ is left adjoint to N_H

$$\begin{tikzcd}
Kan \arrow[d] & \subset & Cat_\infty^\Delta \arrow[d] & \varepsilon & (Cat^\Delta)^{lK} \arrow[d] & \subset & Cat^\Delta \arrow[d, "N_H"] \\
\mathcal{S} & \subset & Cat_\infty & \varepsilon & \{\infty\text{-cats}\} & \subset & Set_\Delta
\end{tikzcd}$$

call $f : K \to K'$ in Set_Δ a categorical eqm
if $\mathfrak{C}(f)$ is an equivalence in Cat_∞^Δ

for $f : \mathcal{C} \to \mathcal{C}'$, $\mathcal{C}, \mathcal{C}'$ ∞-cats
f is an equivalence $\Leftrightarrow$ $[f]$ is an iso in $hCat_\infty$

and

$$K \to N_H \mathfrak{C}(K)$$ is a categorical

equivalence of K with an ∞-category.

§1 (b) Cartesian fibrations and (left) right fibrations

A Grothendieck construction

Def Let $p : A \to B$ be a functor (of categories)

i) A morphism $x \xrightarrow{f} y$ in A is p-Cartesian

if for x' in A the map

$$A(x', x) \xrightarrow{(f_*, p)} A(x', y) \times_{B(p(x'), p(y))} B(p(x'), p(x))$$

is a bijection

ii) p is a Grothendieck fibration if for each

$\bar{x} \xrightarrow{\bar{f}} \bar{y}$ in B and $y \in A$ with $p(y) = \bar{y}$, $\exists$

p-Cartesian morphism $x \xrightarrow{f} y$ with $p(f) = \bar{f}$

A p-coCartesian morphism & Grothendieck opfibration

is defined ~~dually~~ (for $p^{op} : A^{op} \to B^{op}$)

Note 1) Let $p : A \to B$ be a Grothendieck fibration and take $\bar{x} \xrightarrow{\bar{f}} \bar{y}$. Let $A_{\bar{x}} = p^{-1}(id_{\bar{x}})$, $A_{\bar{y}} = p^{-1}(id_{\bar{y}})$.

For each $y \in A_{\bar{y}}$, choose a p-Cartesian $x \xrightarrow{f_y} y$. Define

$\bar{f}^* : A_{\bar{y}} \to A_{\bar{x}}$ by $f^!(y) = x$, $f^!(\beta) = \alpha$:

$$\begin{tikzcd} A_{\bar{x}} & A_{\bar{y}} \\ x' \arrow[r, "f_{y'}"] \arrow[d, "\alpha"'] & y' \arrow[d, "\beta"] \\ x \arrow[r, "f_y"'] & y \end{tikzcd}$$

$\exists!\ \alpha : x' \to x$ with

$f_y \circ \alpha = \beta \circ f_{y'}$

$p(\alpha) = id$

$\bar{x}$ $\bar{y}$

- A different choice of $\{y \mapsto f_y\}$ gives an equivalent functor $\bar{f}^!$
- $\bar{f} \mapsto \bar{f}^*$ defines a pseudo-functor

$$p^! : B^{op} \to Cat$$

2) Suppose $A_{\bar{x}}$ is a groupoid for all $\bar{x}$ in A. Then each morphism $x \xrightarrow{f} y$ is p-Cartesian

3) For $p : A \to B$ a Grothendieck opfibration, get

$$p_! : B \to \mathrm{Cat} \qquad \bar{x} \rightsquigarrow A_{\bar{x}}$$

$$\bar{x} \xrightarrow{\bar{f}} \bar{y} \rightsquigarrow \bar{f}_! : A_{\bar{x}} \to A_{\bar{y}}$$

Conversely: Let $F : B^{op} \to \mathrm{Cat}$ be a (pseudo) functor. Define $p_F : A_F \to B$ where

A_F has objects $(\bar{x} \in B,\ x \in F(\bar{x}))$

morphisms $(\bar{x}, x) \xrightarrow{(\bar{f}, \alpha)} (\bar{y}, y)$

$\bar{f} : \bar{x} \to \bar{y}$ in B

$\alpha : \bar{x} \to F(\bar{f})(y)$ in $F(\bar{x})$

Then p_F is a Grothendieck fibration and $p_F^!$ is equivalent to F.

B. Cartesian fibrations

Def $p: X \to S$ map of simplicial sets.

1) p is a i) inner fibration if p has RLP for all inner horns $\Lambda^n_i \hookrightarrow \Delta^n$ $0<i<n$

ii) right fibration " $0<i\leq n$

iii) left fibration " $0\leq i<n$

iv) Kan fibration " $0\leq i\leq n$

2) an edge $x \xrightarrow{f} y$ in X_1 is p-Cartesian

if $X_{/f} \to X_{/y} \times_{S_{/p(y)}} S_{/p(f)}$

is a trivial Kan fibration (so admits a section, unique up to contractible choice)

3) p is a Cartesian fibration if

i) p is an inner fibration

ii) for each $\bar{x} \xrightarrow{\bar{f}} \bar{y} \in S_1$ $\exists$ a p-Cartesian edge $x \xrightarrow{f} y$ with $\bar{f} = p(f)$

4) coCartesian fibration is defined dually.

Note • inner / right / left / Cartesian fibrations are stable under pullback by $T \xrightarrow{a} S$ and composition

• The fibers of an inner fibration are ∞-categories

• The fibers of a left / right / Kan fibration are Kan complexes

• p is a right fibration $\Leftrightarrow$ p is a Cartesian fibration and every edge is p-Cartesian, (dually for left fibration)

co Cartesian fibration

Given a Cartesian fibration $p : X \to S$ there is a corresponding map of simplicial sets

$$p^! : S^{op} \to Cat_\infty$$

sending $s \in S \longmapsto X_s = p^{-1}(s)$

Roughly speaking, one chooses for each $s \xrightarrow{f} s'$ in S and $x' \in X_{s'}$ a Cartesian edge $x \xrightarrow{\bar{f}} x'$ over f and sends x' to x

More specifically, there is a "straightening functor"

$$St : \{\text{Cartesian fibrations}/S\} \to \{\text{functors } \mathfrak{C}[S^{op}] \to (Cat_\infty^\Delta)^{lK}\}$$

with $St(p: X \to S)(s \in S) = X_s$

Then $p' : S^{op} \to Cat_\infty$ is the adjoint to

$$St(p) : \mathfrak{C}[S^{op}] \to (Cat_\Delta^\infty)^{lK}$$

If $p: X \to S$ is a right fibration, this gives

$$p' : S^{op} \to \mathcal{S} = N_H(Kan)$$

Dually for coCartesian fibrations / left fibrations

To go in the other direction, construct the universal Cartesian fibration $\widetilde{Cat}_\infty \to Cat_\infty^{op}$ and take pullback by $g^{op} : S \to Cat_\infty^{op}$ for $g : S^{op} \to Cat_\infty$

A special case: Cartesian fibration $X \xrightarrow{p} \Delta^1$

These correspond to $p^! : \Delta^{1\,op} \to \mathrm{Cat}_\infty$ i.e. functors $f : X_1 \to X_0$. Using model structure on "marked simplicial sets $/\Delta^1$"

one shows: for $X \xrightarrow{p} \Delta^1$ Cartesian fibration, $\exists\ \tilde{p} : X_1 \times \Delta^1 \to X$ over Δ^1

such that $\cdot\ \tilde{p}|_{X_1 \times 1}$ = inclusion $X_1 \hookrightarrow X$

$\cdot\ \tilde{p}(y \times \Delta^1)$ = p-Cartesian edge with target y

then $p^! = \tilde{p}|_{X_1 \times 0}$. Conversely, given $f : X_1 \to X_0$

take $\bar{X} = X_1 \times \Delta^1 \cup_{X_1 \times 0 \xrightarrow{f} X_0} X_0 \to \Delta_1$ and let

$\bar{X} \to X$ over Δ^1, p, be a fibrant model ($\Rightarrow p$ is a Cartesian fibration)

This recovers f up to equivalence. Dually for coCartesian fibration

§2 Problems

A. Kan extensions

Def Let $\mathcal{C}$ be an ∞-category and $\mathcal{C}^0 \subset \mathcal{C}$ a full subcategory and a commutative diagram
[4.3.2.2, Lur09]

$$\begin{tikzcd} \mathcal{C}^0 \arrow[r, "F_0"] \arrow[d, hook] & \mathcal{D} \\ \mathcal{C} \arrow[ur, "F"'] & \end{tikzcd}$$

. Call F a left Kan extension of F_0 at $C \in \mathcal{C}$ if the diagram

$$\begin{tikzcd} \mathcal{C}^0_{/C} \arrow[r, "F_C"] \arrow[d, hook] & \mathcal{D} \\ \left(\mathcal{C}^0_{/C}\right)^{\triangleright} \arrow[ur] & \end{tikzcd}$$

exhibits $F(C)$ as a colimit of F_C

F is a left Kan extension of F_0 if it is a LKE at C $\forall C \in \mathcal{C}$

i.e. "$F(C) = \operatorname{colim}_{\mathcal{C}^0/C} F_0$" $\forall\, C \in \mathcal{C}$

B. The ∞-category of presheaves

Def/Prop Let S be a simplicial set. The ∞-category of presheaves on S is $\mathcal{P}(S) = \mathrm{Fun}(S, \mathcal{S}^{op})$

$= \mathrm{Fun}(S^{op}, \mathcal{S})$

Theorem Let K and S be simplicial sets and let $\mathcal{C}$ be an ∞-category admitting K-indexed colimits. Then

(1) $\mathrm{Fun}(S, \mathcal{C})$ admits K-indexed colimits

(2) $K^{\triangleright} \to \mathrm{Fun}(S, \mathcal{C})$ is a colimit $\iff \forall s \in S_0$ the induced map $K^{\triangleright} \to \mathcal{C}$ is a colimit diagram

Cor Let S be a simplicial set. Then the ∞-category $\mathcal{P}(S)$ admits all small limits and colimits (computed "pointwise")

If $\mathcal{S}$ admit all small colimits and limits

Note "small" means: fix an uncountable inaccessible cardinal κ. Let $U(\kappa)$ = subcat of $\mathcal{S}ets$ of all S with $|S| < \kappa$

$\Rightarrow U(\kappa)$ is closed under $\bigcup_{i \in I} S_i$, 2^S, ... A set is small if $S \in U(\kappa)$

C The Yoneda lemma

Construction S a simplicial set. We have the functor of simplicial categories

$$\mathfrak{C}[S]^{op} \times \mathfrak{C}[S] \to \mathcal{K}an$$

$$\overset{\shortparallel}{\mathfrak{C}[S^{op}]}$$

$$(X, Y) \longmapsto \mathrm{Sing}\,|\mathrm{Hom}_{\mathfrak{C}[S]}(X,Y)|$$

we have a canonical functor (induced by $p_1 : S^{op} \times S \to S^{op}$, $p_2 : \to S$)

$$\mathfrak{C}[S^{op} \times S] \to \mathfrak{C}[S]^{op} \times \mathfrak{C}[S]$$

so

$$\mathfrak{C}[S^{op} \times S] \to \mathcal{K}an$$

(adjoint)

$$S^{op} \times S \to N_\Delta(\mathcal{K}an) = \mathcal{S}$$

$$S \xrightarrow[j]{} \mathrm{Fun}(S^{op}, \mathcal{S}) = \mathcal{P}(S)$$

the ∞-Yoneda embedding j

Prop ([5.1.3.1, Lur09] Let S be a simplicial set
5.1.3.2
$j: S \to \mathcal{P}(S)$ the Yoneda map. Then
j is fully faithful. If $S = \mathcal{C}$ is an ∞-category
j preserves all limits existing in $\mathcal{C}$

Def Let S be a simplicial set, $\mathcal{C}$ an ∞-category.
$\mathrm{Fun}^L(\mathcal{P}(S), \mathcal{C})$ is the full subcategory of $\mathrm{Fun}(\mathcal{P}(S), \mathcal{C})$
of functors that preserve small colimits

Theorem (Thm 5.1.5.6) Let S be a small simplicial set
and let $\mathcal{C}$ be an ∞-category admitting small colimits.
Then the restriction j^* induces an equivalence

$$\mathrm{Fun}^L(\mathcal{P}(S), \mathcal{C}) \to \mathrm{Fun}(S, \mathcal{C})$$

The inverse is given by taking left Kan
extensions

In fact a functor $f: P(S) \to \mathcal{C}$ is a left Kan extension of $f \circ j$ $\Leftrightarrow$ f preserves small colimits,
and each functor $f_0: S \to \mathcal{C}$ admits a left Kan extension $f: P(S) \to \mathcal{C}$, unique up to contractible choice

In words: $P(S)$ is "freely generated" over S by adjoining all small colimits

§3 Adjoint functors

A. (co) Cartesian fibrations and adjunction.

Recall $\{\text{Cartesian fibrations } p: X \to \Delta^1\}/\sim \;\hookrightarrow\; \{\text{functors } p^!: X_1 \to X_0\}/\sim$

$p \to p^!$

$\{\text{coCartesian fibrations } p: X \to \Delta^1\} \hookrightarrow \{\text{functors } p_!: X_0 \to X_1\}/\sim$

$p \mapsto p_!$

Note for $p : X \to \Delta^1$ a Cartesian fibration we have the equivalence

$$\mathrm{Map}_{X_0}(x, p^!(y)) \xrightarrow{\sim} \mathrm{Map}_X(x, y)$$

and dually for a coCartesian fibration

$$\mathrm{Map}_{X_1}(p_!(x), y) \xrightarrow{\sim} \mathrm{Map}(x, y)$$

Def let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories. F is left adjoint to G / G is right adjoint to F if $\exists$ $p : \mathcal{M} \to \Delta^1$, both Cartesian and coCartesian fibration, equivalences $\mathcal{C} \simeq \mathcal{M}_0$, $\mathcal{D} \simeq \mathcal{M}_1$ transforming F to $p_!$ and G to $p^!$

B Localization

Def Let $L : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Call L a *localization* if L is a left adjoint with a fully faithful right adjoint $R : \mathcal{D} \to \mathcal{C}$.

If $\mathcal{C}_0 \overset{i}{\subseteq} \mathcal{C}$ is a full subcategory of $\mathcal{C}$ say $\mathcal{C}_0$ is a localization of $\mathcal{C}$ if i admits a left adjoint $L : \mathcal{C} \to \mathcal{C}_0$

C. *Adjointable squares.* Let

$$\begin{tikzcd} \mathcal{C} \arrow[r, "U"] \arrow[d, "G"'] & \mathcal{C}' \arrow[d, "G'"] \\ \mathcal{D} \arrow[r, "V"] & \mathcal{D}' \end{tikzcd}$$

be $\Delta^1 \times \Delta^1 \to \mathrm{Cat}_\infty$. This is *left adjointable* if G, G' admit left adjoints F, F' and the composition

$$F' \circ V \xrightarrow[\sim]{} F' \circ V \circ G \circ F \simeq F' \circ G' \circ U' \circ F$$

$$\wr \text{ by } \eta$$

$$U' \circ F$$

is an equivalence

Dually: right adjointable

<u>Def</u> S a simplicial set define

$$\mathrm{Fun}^{LAd}(S, \mathrm{Cat}_\infty),\ \mathrm{Fun}^{RAd}(S, \mathrm{Cat}_\infty)$$

$$\cap$$

$$\mathrm{Fun}(S, \mathrm{Cat}_\infty)$$

by

$$\mathrm{Fun}^{LAd/RAd}(S, \mathrm{Cat}_\infty)_0 =$$

$$\left\{ f : S \to \mathrm{Cat}_\infty \;\middle|\; \begin{array}{l} \forall\ s \to s' \text{ in } S_1, \\ f(s) \to f(s') \text{ admits a} \\ \text{left/right adjoint} \end{array} \right\}$$

$$\mathrm{Fun}^{LAd/RAd}(S, \mathrm{Cat}_\infty)_1 = \left\{ \alpha : f \to f' \;\middle|\; \begin{array}{l} \forall\, s \to s' \text{ in } S, \text{ the square} \\ \begin{array}{ccc} f(s) & \to & f(s') \\ \downarrow & & \downarrow \\ f'(s) & \to & f'(s') \end{array} \text{ is left/right adjointable} \end{array} \right\}$$

Cor [4.7.4.18 Lurie] for S a simplicial set

1) $\mathrm{Fun}^{LAd/RAd}(S, \mathrm{Cat}_\infty)$ are presentable ∞-categories (see below)

2) The inclusions $\mathrm{Fun}^{LAd/RAd}(S, \mathrm{Cat}_\infty) \overset{i^L/i^R}{\hookrightarrow} \mathrm{Fun}(S, \mathrm{Cat}_\infty)$ admit left adjoints $\Rightarrow$ i^L/i^R preserve small limits

3) $\mathrm{Fun}^{LAd}(S, \mathrm{Cat}_\infty) \cong \mathrm{Fun}^{RAd}(S^{op}, \mathrm{Cat}_\infty)$

§4 Presentable ∞-categories

A. Some definitions

- A cardinal κ is regular if for a set I with $|I| < \kappa$ and a map of sets $S \xrightarrow{f} I$ with $|f^{-1}(i)| < \kappa$ then $|S| < \kappa$

Let κ be a regular cardinal

- A simplicial set K is κ-small if $|\coprod_n K_n| < \kappa$
- A simplicial set S is κ-filtered if for each κ-small K, each $K \xrightarrow{F} S$ extends to $K^{\triangleright} \xrightarrow{F^{\triangleright}} S$

- A κ-filtered colimit in an ∞-cat $\mathcal{C}$ is a colimit of some $K \to \mathcal{C}$ with K κ-filtered
- Suppose $\mathcal{C}$ admits all κ-filtered colimits. An object $c \in \mathcal{C}$ is κ-compact if $\mathcal{C}(x,-)$ commutes with κ-filtered colimits
- $\mathcal{C}$ is locally small if $\mathrm{Map}_{\mathcal{C}}(x,y)$ is essentially small for all $x, y \in \mathcal{C}_0$

<u>Def</u> • An ∞-category $\mathcal{C}$ is κ-accessible if $\mathcal{C}$ is locally small, admits κ-filtered colimits and there is a set of κ-compact objects of $\mathcal{C}$ that generate $\mathcal{C}$ under κ-filtered colimits

- A functor of κ-accessible ∞-cats $F: \mathcal{C} \to \mathcal{D}$ is κ-accessible if F preserves κ-filtered colimits (κ-continuous)

- $\mathcal{C}$ is κ-presentable if $\mathcal{C}$ is κ-accessible and admits small colimits

We drop the κ-( ) to mean κ-( ) for some κ (hence for all $\kappa \gg 0$)

Theorem
- If $\mathcal{C}$ is presentable, then $\mathcal{C}$ admits small limits
- $\mathcal{S}$ is presentable. If S is a small simplicial set then $P(S)$ is presentable
- If $\mathcal{C}$ is presentable and $p: K \to \mathcal{C}$ is a small diagram then $\mathcal{C}_{p/}$ and $\mathcal{C}_{/p}$ are presentable
- If $\mathcal{C}$ is presentable and K is a small simplicial set then $\mathrm{Fun}(K, \mathcal{C})$ is presentable
- If $\mathcal{C}$ and $\mathcal{D}$ are presentable $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ is presentable

- $\mathcal{C}$ is presentable $\iff$ $\mathcal{C}$ is an accessible localization of $P(\mathcal{D})$ for $\mathcal{D}$ a small ∞-category: $\exists\ P(\mathcal{D}) \xrightarrow{L} \mathcal{C}' \overset{i}{\subseteq} P(\mathcal{D})$ with iL accessible, L left adjoint to i, and $\mathcal{C} \simeq \mathcal{C}'$.

Another description of accessibility

Def Let S be a simplicial set, κ a regular cardinal. $\mathrm{Ind}_\kappa(S)$: the ∞-cat. of κ-filtered objects, is the subcategory of $P(S)$ generated by κ-filtered colimits in $j(S) \subset P(S)$

Prop/Def An ∞-category $\mathcal{C}$ is κ-accessible

$\iff$ $\mathcal{C} \simeq \mathrm{Ind}_\kappa(\mathcal{C}_0)$ for some small ∞-category $\mathcal{C}_0$.

B. Adjoint Functor Theorem for ∞ cats

$\mathcal{C}, \mathcal{D}$: presentable ∞-categories. Then

- a functor $F: \mathcal{C} \to \mathcal{D}$ admits a right adjoint $\iff$ F preserves small colimits
- a functor $G: \mathcal{D} \to \mathcal{C}$ admits a left adjoint $\iff$ G is accessible and preserves small limits

Def $\Pr^L \subseteq \widehat{\mathrm{Cat}}_\infty$ with objects the presentable ∞-categories
$\supseteq \Pr^R$

$\Pr^L$ has morphisms the $f: \mathcal{C} \to \mathcal{D}$ that preserve small colimits, i.e. f is a left adjoint

$\Pr^R$ has morphisms $g: \mathcal{D} \to \mathcal{C}$ that are accessible and preserve small limits, i.e. g is a right adjoint

__Theorem__ [5.5.3.13, 5.5.3.18 Lur09]

- $\Pr^L$ and $\Pr^R$ are ∞-categories
- $\Pr^L$ and $\Pr^R$ admit small limits and small colimits
- $\Pr^L \hookrightarrow \mathrm{Cat}_\infty$, $\Pr^R \hookrightarrow \mathrm{Cat}_\infty$ preserve small limits and colimits

§5 Stable ∞-categories

Def A zero object in an ∞-cat $\mathcal{C}$ is an object 0 that is initial and final. Call $\mathcal{C}$ **pointed** if $\mathcal{C}$ admits a 0-object.

Def Let $\mathcal{C}$ be a pointed ∞ category. A **triangle** in $\mathcal{C}$ is a $\Delta^1 \times \Delta^1$ diagram of the form

$$(*) \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

with 0 a zero object.

a **fiber sequence** if this is a pull-back sq; $X = \lim \left(\begin{tikzcd} & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd} \right)$

a **cofiber sequence** if a pushout sq $Z = \operatorname{colim} \left(\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ 0 & \end{tikzcd} \right)$

Call $(*)$ a fiber sequence call $(*)$ (or X) a fiber of g

" cofiber sequence " (or Z) a cofiber of f

Def A pointed ∞-category $(\mathcal{C}, 0)$ is **stable** if

- every morphism admits a fiber and a cofiber
- A triangle is a fiber sequence $\iff$ it is a cofiber sequence

Loops and suspension Let $(\mathcal{C}, 0)$ be a pointed ∞-cat admitting fibers and cofibers. Let $M^{\Sigma} \subset \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ be the full subcat of objects

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$

which are pushout squares.

Then $M^{\Sigma} \to \mathcal{C}$ is a trivial fibration

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd} \mapsto X$$

so $\exists$ section $s : \mathcal{C} \to M^{\Sigma}$ unique up to contractible choice.

The composite $\mathcal{C} \xrightarrow{s} M^{\Sigma} \to \mathcal{C}$ is a suspension functor

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd} \mapsto Y$$

$$\Sigma : \mathcal{C} \to \mathcal{C}$$

Dually, let $\mathcal{M}^{\Omega} \subset \left\{ \begin{array}{ccc} X & \to & 0 \\ \downarrow & & \downarrow \\ 0 & \to & Y \end{array} \right\}$ pullback. Then there's a section s' to $\left\{ \begin{array}{ccc} X & \to & 0 \\ \downarrow & & \downarrow \\ 0 & \to & Y \end{array} \right\} \to X$

and the composition

$$\mathcal{C} \xrightarrow{s'} \mathcal{M}^{\Omega} \to \mathcal{C} \quad \text{is the loops functor}$$

$$\begin{array}{ccc} X & \to & 0 \\ \downarrow & & \downarrow \\ 0 & \to & Y \end{array} \mapsto X \qquad \Omega : \mathcal{C} \to \mathcal{C}$$

Set $X[n] = \begin{cases} \Sigma^n X & \text{for } n \geq 0 \\ \Omega^{-n} X & \text{for } n \leq 0 \end{cases}$

Note • $\Sigma \dashv \Omega$, $\quad \mathcal{C}(\Sigma X, Y) \cong \Omega \operatorname{Maps}(X, Y)$

• A pointed ∞ category $\mathcal{C}$ is stable $\iff$ $\mathcal{C}$ admits cofibers and $\Sigma : \mathcal{C} \to \mathcal{C}$ is an equivalence

Def Let $(\mathcal{C}, 0)$ be a pointed ∞-category with cofibers. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in $h\mathcal{C}$ is a distinguished triangle if $\exists$ $\Delta^1 \times \Delta^2$ diagram in $\mathcal{C}$

$$\begin{tikzcd} X \arrow[r, "\tilde{f}"] \arrow[d] & Y \arrow[r] \arrow[d, "\tilde{g}"] & 0' \arrow[d] \\ 0 \arrow[r] & Z \arrow[r, "\tilde{h}"'] & W \end{tikzcd}$$

such that

- both squares are pushouts and $0, 0'$ are zero objects
- $\tilde{f}, \tilde{g}$ represent f, g,
- let $\alpha : W \to X[1]$ be the equivalence given by the outer rectangle. Then $\alpha \tilde{h}$ represents h

Theorem: Let $\mathcal{C}$ be a stable ∞-category

Then $h\mathcal{C}$ with the class of distinguished triangles as above is a triangulated category

Def Let $\mathcal{C}, \mathcal{D}$ be stable ∞-categories. A functor $F: \mathcal{C} \to \mathcal{D}$ is **exact** if

- F sends zero objects in $\mathcal{C}$ to zero objects in $\mathcal{D}$
- F sends cofiber sequences to cofiber sequences

Equivalently F commutes with finite limits and colimits

Def $Cat_\infty^{Ex} \subset Cat_\infty$ is the ∞-category of stable ∞-categories and exact functors

Prop

Theorem Cat_∞^{Ex} admits small limits and $Cat_\infty^{Ex} \hookrightarrow Cat_\infty$ preserves small limits. Same for small filtered colimits

Def $Pr^L_{stb} \subset Pr^L$ is the full subcategory of presentable stable ∞-categories

Note each morphism in Pr^L_{stb} is exact: 0 is the colimit over the empty index and each morphism in Pr^L_{stb} preserves small colimits hence cofiber sequences and 0

so Pr^L_{stb} is a full subcategory of Cat^{Ex}_∞

Cor • Pr^L_{stb} admits small limits and small filtered colimits.

• the inclusions

$$\begin{array}{ccc} Pr^L_{stb} & \subset & Pr^L \\ \cap & & \cap \\ Cat^{Ex}_\infty & \subset & Cat_\infty \end{array}$$

preserves these (co)limits

Pf $Pr^L_{stb} = Pr^L \cap Cat^{Ex}_\infty$ and all three properties hold for $Pr^L \hookrightarrow Cat_\infty \hookleftarrow Cat^{Ex}_\infty$