

•) $p\text{-CoCart} \subseteq \text{Ar}(\mathcal{E}) = \text{Fun}([1], \mathcal{E})$ full subcat. spanned by p -cocartesian maps.

Fact: $p\text{-CoCart} \xrightarrow{(\text{source}, p)} \sum_{p, \mathcal{E}, \text{source}} \mathcal{E} \times \text{Ar}(\mathcal{E})$ is fully faithful.

Informally: " p -cocartesian maps are uniquely determined by their source and their image in \mathcal{E} ."

•) p is cocartesian if $p\text{-CoCart} \rightarrow \sum_{\mathcal{E}} \text{Ar}(\mathcal{E})$ is essentially surjective.

•) $\text{CoCart}(\mathcal{E}) \subseteq \text{Cat}_{\infty}/\mathcal{E}$ (non-full) subcat. given as follows =

objects: $(p: \mathcal{E} \rightarrow \mathcal{E})$ s.t. p is cocartesian

maps: $(F: (\mathcal{E}, p) \rightarrow (\mathcal{E}', p'))$ s.t. F sends p -cocartesian maps in \mathcal{E} to p' -cocartesian maps in \mathcal{E}' .

Then ((Un-)Straightening): let $\mathcal{E} \in \text{Cat}_{\infty}$.

$$\leadsto \text{CoCart}(\mathcal{E}) \cong \text{Fun}(\mathcal{E}, \text{Cat}_{\infty})$$

Remark:

•) Suppose $\mathcal{E} \in \text{Cat}_1$. Then the above equivalence restricts to

$$\text{CoCart}_1(\mathcal{E}) \cong \text{Fun}(\mathcal{E}, \text{Cat}_1)$$

where $\text{CoCart}_1(\mathcal{E}) \subseteq \text{CoCart}(\mathcal{E})$ is the full subcategory spanned by those $(p: \mathcal{E} \rightarrow \mathcal{E})$ with $\mathcal{E} \in \text{Cat}_1$:

- To $(p: \mathcal{E} \rightarrow \mathcal{C}) \in \text{CoCart}_1(\mathcal{C})$ one associates

$$F: \mathcal{C} \longrightarrow \text{Cat}_1$$

$$\bar{x} \longmapsto p^{-1}(\bar{x}) := [0]_{\bar{x}, \mathcal{C}, p}^x \mathcal{E}, \text{ pullback in } \text{Cat}_1.$$

$$(\bar{f}: \bar{x} \rightarrow \bar{y}) \longmapsto \left(\begin{array}{c} p^{-1}(\bar{x}) \longrightarrow p^{-1}(\bar{y}) \\ x \longmapsto (y, \text{ where } f: x \rightarrow y \text{ is } \\ \text{p-cocart. lift of } \bar{f}) \end{array} \right)$$

- To $F: \mathcal{C} \rightarrow \text{Cat}_1$ one associates $p: \mathcal{E} \rightarrow \mathcal{C}$ with \mathcal{E} given by:

objects: $(\bar{x}, x), \bar{x} \in \mathcal{C}, x \in F(\bar{x})$.

maps: $\text{Hom}_{\mathcal{E}}((\bar{x}, x), (\bar{y}, y)) = \left\{ (\bar{f}, f) \mid \begin{array}{l} \bar{f}: \bar{x} \rightarrow \bar{y} \text{ in } \mathcal{C} \\ f: F(\bar{f})(x) \rightarrow y \text{ in } F(\bar{y}) \end{array} \right\}$

•) If $p: \mathcal{E} \rightarrow \mathcal{C}$ is an arbitrary cocart. map of ∞ -cats., then the associated functor $F: \mathcal{C} \rightarrow \text{Cat}_{\infty}$ still satisfies $F(\bar{x}) = p^{-1}(\bar{x})$ for $\bar{x} \in \mathcal{C}$.

Example: Define the category Mod :

objects: $(R, M), R$ ring, M R -module.

maps: $\text{Hom}_{\text{Mod}}((R, M), (S, N)) := \left\{ (\varphi, f) \mid \varphi: R \rightarrow S, f: M \rightarrow N \text{ } R\text{-linear} \right\}$.

Then $p: \text{Mod} \rightarrow \text{Rings}, (R, M) \mapsto R$ is cocartesian:

Given $(R, M) \in \text{Mod}, \varphi: R \rightarrow S$, a p -cocartesian lift is given by

$$(R, M) \longrightarrow (S, S \otimes_R M), \quad (\varphi, m \mapsto 1 \otimes m)$$

Under Thm., p corresponds to the functor

$$R \mapsto \text{Mod}(R)$$

$$\text{Rings} \longrightarrow \text{Cat}_1, \quad (\varphi: R \rightarrow S) \mapsto (S \otimes_R - : \text{Mod}(R) \rightarrow \text{Mod}(S))$$

2. Symmetric monoidal ∞ -categories:

Def:

•) A commutative monoid is a pair (M, \cdot) with M a set and $\cdot: M \times M \rightarrow M$ s.t.:

$$- \forall x, y \in M: x \cdot y = y \cdot x$$

$$- \forall x, y, z \in M: (x \cdot y) \cdot z = x \cdot (y \cdot z)$$

$$- \exists e \in M: \forall x \in M: e \cdot x = x$$

•) A symmetric monoidal 1-category is a tuple $(\mathcal{M}, \otimes, \beta, \gamma)$ consisting of

$$- \mathcal{M} \in \text{Cat}_1$$

$$- \otimes: \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$$

$$- \text{with' isos. } \beta = (\beta_{x,y}: x \otimes y \cong y \otimes x)_{x,y}, \quad \gamma = (\gamma_{x,y,z}: (x \otimes y) \otimes z \cong x \otimes (y \otimes z))_{x,y,z}$$

s.t.

- compatibilities of β, γ

$$- \exists (e, \alpha) = e \in \mathcal{M}, \alpha = (\alpha_x: e \otimes x \cong x)_x \text{ s.t. } \underline{\text{compatibilities}}$$

•) SymMonCat_1 the category of s.m. 1-cats.

Example: $R \in \text{Ring} \rightsquigarrow \text{Mod}(R)$ is a s.m. 1-cat. wrt. $- \otimes_R -$

Problem: If we want to define SymMonCat_∞ , we need an efficient way of organizing "higher compatibilities".

Def:

•) Define the 1-category Fin_* :

objects: $\langle n \rangle = \{*, 1, \dots, n\}$ for $n \in \mathbb{Z}_{\geq 0}$.

maps: maps of pointed sets.

•) For $1 \leq i \leq n$, define $S_i^n: \langle n \rangle \rightarrow \langle 1 \rangle$, $i \mapsto 1$, $i \neq j \mapsto *$.

Notation: Write $\langle n \rangle^\circ = \{1, \dots, n\} \subseteq \langle n \rangle$. We can think of maps $\langle m \rangle \rightarrow \langle n \rangle$ in Fin_* as partially defined maps of sets $\langle m \rangle^\circ \rightarrow \langle n \rangle^\circ$.

Construction: Let $(\mathcal{M}, \otimes, \beta, \gamma) \in \text{SymMonCat}_1$.

•) Define $X_{\mathcal{M}}: \text{Fin}_* \rightarrow \text{Cat}_1$, $\left(\begin{array}{l} \langle n \rangle \mapsto \mathcal{M}^n \\ \mathcal{M}^m \longrightarrow \mathcal{M}^n \\ (\alpha: \langle m \rangle \rightarrow \langle n \rangle) \mapsto \left((x_i)_{i=1}^m \mapsto \left(\bigotimes_{j \in \alpha^{-1}(i)} x_j \right)_{i=1}^n \right) \end{array} \right)$

•) Let $(\mathcal{M}^\otimes \rightarrow \text{Fin}_*) \in \text{CoCat}_1(\text{Fin}_*)$ be the unstraightening of $X_{\mathcal{M}}$.

Concretely this means that \mathcal{M}^\otimes is given as follows:

objects: $(\langle n \rangle, x_1, \dots, x_n)$, $\langle n \rangle \in \text{Fin}_*$, $x_1, \dots, x_n \in \mathcal{M}$

maps: $\text{Hom}_{\mathcal{M}^\otimes}((\langle m \rangle, x_1, \dots, x_m), (\langle n \rangle, y_1, \dots, y_n))$
 $= \left\{ (\alpha, f_1, \dots, f_n) \mid \begin{array}{l} \alpha: \langle m \rangle \rightarrow \langle n \rangle \text{ in } \text{Fin}_* \\ f_i: \bigotimes_{j \in \alpha^{-1}(i)} x_j \rightarrow y_i \end{array} \right\}$

This construction gives rise to a fully faithful functor

$$\text{SymMonCat}_1 \subseteq \text{Fun}(\text{Fin}_*, \text{Cat}_1)$$

with essential image given by those functors $X: \text{Fin}_* \rightarrow \text{Cat}_1$ s.t.

$$\forall n \in \mathbb{Z}_{\geq 0}: (\mathcal{S}_{i=1}^n)_{i=1}^n = X_n \xrightarrow{\sim} \prod_{i=1}^n X_1$$

Def:

•) A s.m. ∞ -cat. \mathcal{M} is a map of ∞ -cats $X_{\mathcal{M}}: \text{Fin}_* \rightarrow \text{Cat}_{\infty}$ s.t.

$$\forall n \in \mathbb{Z}_{\geq 0}: (\mathcal{S}_{i=1}^n)_{i=1}^n = X_{\mathcal{M},n} \xrightarrow{\sim} (X_{\mathcal{M},1})^n$$

•) $\text{SymMonCat}_{\infty} \subseteq \text{Fun}(\text{Fin}_*, \text{Cat}_{\infty})$ full subcat. spanned by s.m. ∞ -cats.

•) Given $X_{\mathcal{M}} \in \text{SymMonCat}_{\infty}$ we have associated $(\mathcal{M}^{\otimes} \rightarrow \text{Fin}_*) \in \text{CoCat}(\text{Fin}_*)$,

$\mathcal{M} := X_{\mathcal{M},1} \in \text{Cat}_{\infty}$. We call either of $X_{\mathcal{M}}, \mathcal{M}^{\otimes}, \mathcal{M}$ a s.m. ∞ -cat.

Example:

•) $\text{SymMonCat}_1 \subseteq \text{SymMonCat}_{\infty}$ is a full subcategory.

•) Given an ∞ -cat. with finite products, we can equip it with the cartesian symmetric monoidal structure:

- Define a 1-category Γ^x as follows:

objects: $(\langle n \rangle, T)$, $\langle n \rangle \in \text{Fin}_*$, $T \subseteq \langle n \rangle^\circ$

maps: $\text{Hom}_{\Gamma^x}((\langle m \rangle, S), (\langle n \rangle, T)) := \{\alpha: \langle m \rangle \rightarrow \langle n \rangle \mid \alpha^{-1}(T) \subseteq S\}$

Then we have a natural functor $\Gamma^x \rightarrow \text{Fin}_*$ and the fibers are given by

$$\Gamma_{\langle n \rangle}^x \cong \mathcal{P}(\langle n \rangle^\circ)^{\text{op}} \quad (\text{where } \mathcal{P}(-) \text{ denotes the power set}).$$

Now suppose we are given $\mathcal{C} \in \text{Cat}_\infty$ with finite products.

- Define $\tilde{\mathcal{C}}^x \in \text{Set}_\Delta$ by setting

$$\tilde{\mathcal{C}}_k^x := \bigsqcup_{[k] \rightarrow \text{Fin}_*} \text{Hom}_{\text{Set}_\Delta}([k]_{\text{Fin}_*}^x \Gamma^x, \mathcal{C}) \quad \text{for } k \in \mathbb{Z}_{\geq 0}.$$

Then we have a natural map $\tilde{\mathcal{C}}^x \rightarrow \text{Fin}_*$ with fibers given by

$$\tilde{\mathcal{C}}_{\langle n \rangle}^x \cong \text{Hom}_{\text{Set}_\Delta}(\mathcal{P}(\langle n \rangle^\circ)^{\text{op}}, \mathcal{C})$$

- Let $\mathcal{C}^x \subseteq \tilde{\mathcal{C}}^x$ be the full simplicial subset spanned by the 0-simplices

$$f: \mathcal{P}(\langle n \rangle^\circ)^{\text{op}} \rightarrow \mathcal{C} \text{ s.t.}$$

$$\forall S \subseteq \langle n \rangle^\circ: f(S) \xrightarrow{\sim} \prod_{j \in S} f(\{j\})$$

Proposition:

•) \mathcal{C}^x is an ∞ -category

•) $(\mathcal{C}^x \rightarrow \text{Fin}_*)$ is a s.m. ∞ -cat. with underlying ∞ -cat. \mathcal{C} .

\mathcal{C}^x is called the cartesian s.m. ∞ -cat. associated to \mathcal{C} .

3. Operads and Algebras:

Def: A 1-operad \mathcal{O} consists of the following data:

•) A class of objects (write $a \in \mathcal{O}$)

•) \forall finite sets I , objects $(a_i)_{i \in I}, b$: a set of "multimaps"

$$\text{Mul}_{\mathcal{O}}((a_i)_{i \in I}, b)$$

•) \forall map of finite sets $\alpha: I \rightarrow J$, objects $(a_i)_{i \in I}, (b_j)_{j \in J}, c$:

a composition map

$$\text{Mul}_{\mathcal{O}}((b_j)_{j \in J}, c) \times \prod_{j \in J} \text{Mul}_{\mathcal{O}}((a_i)_{i \in \alpha^{-1}(j)}, b_j) \rightarrow \text{Mul}_{\mathcal{O}}((a_i)_{i \in I}, c).$$

We want these data to satisfy the following conditions:

•) Associativity

•) Identity maps

We obtain a category \mathcal{O}_{p_1} of 1-operads. Also, given $\mathcal{O}, \mathcal{O}' \in \mathcal{O}_{p_1}$ we have a functor category $\text{Fun}(\mathcal{O}, \mathcal{O}') \in \text{Cat}_1$.

Remark: We have a forgetful functor $\mathcal{O}_{p_1} \rightarrow \text{Cat}_1$ and an inclusion $\text{Cat}_1 \subseteq \mathcal{O}_{p_1}$ that make up an adjunction

$$\text{Cat}_1 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \mathcal{O}_{p_1}$$

Example:

•) Let $(\mathcal{M}, \otimes, \beta, \gamma) \in \text{SymMonCat}_1$. Then we can define a 1-operad $\tilde{\mathcal{M}}$ having the same objects as \mathcal{M} and with maps given by

$$\text{Mul}_{\tilde{\mathcal{M}}}((x_i)_{i \in I}, \gamma) := \text{Hom}_{\mathcal{M}}\left(\bigotimes_{i \in I} x_i, \gamma\right).$$

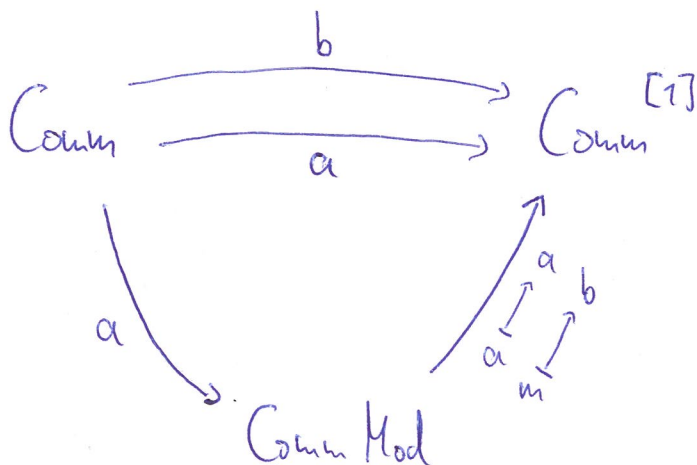
This gives a (non-full!) inclusion $\text{SymMonCat}_1 \subseteq \mathcal{O}p_1$.

•) Comm : objects: a
 maps: $\text{Mul}_{\text{Comm}}((a)_{i \in I}, a) = \{*\}$

•) $\text{Comm}^{[1]}$: objects: a, b
 maps: $\text{Mul}_{\text{Comm}^{[1]}}((x_i)_{i \in I}, \gamma) = \begin{cases} \{*\} & \forall i: x_i = a, \gamma = a \\ \{*\} & \gamma = b \\ \emptyset & \text{else} \end{cases}$

•) CommMod : objects: a, m
 maps: $\text{Mul}_{\text{CommMod}}((x_i)_{i \in I}, \gamma) = \begin{cases} \{*\} & \forall i: x_i = a, \gamma = a \\ \{*\} & \exists ! i: x_i = m, \gamma = m \\ \emptyset & \text{else} \end{cases}$

We also have various maps between the last three of these examples:



Def: let $\mathcal{O} \in \text{Op}_1$, $\mathcal{M} \in \text{SymMonCat}_1$. An \mathcal{O} -algebra in \mathcal{M} is a map of 1-operads $\mathcal{O} \rightarrow \tilde{\mathcal{M}}$. We set $\text{Alg}_{\mathcal{O}}(\mathcal{M}) := \text{Fun}(\mathcal{O}, \tilde{\mathcal{M}})$.

Remark: let $\mathcal{O}_1 \rightarrow \mathcal{O}_2$ be a map in Op_1 and let $\mathcal{M} \in \text{SymMonCat}_1$.

Then we have a restriction functor $\text{Alg}_{\mathcal{O}_2}(\mathcal{M}) \rightarrow \text{Alg}_{\mathcal{O}_1}(\mathcal{M})$.

Example: let $\mathcal{M} := (\text{Ab}, \otimes)$. Then we have:

1) $\text{Alg}_{\text{Comm}}(\mathcal{M}) \cong \text{Rings}$

2) $\text{Alg}_{\text{Comm}^{(1,2)}}(\mathcal{M}) \cong \{ \varphi: A \rightarrow B \mid A, B \in \text{Rings} \} = \text{Ar}(\text{Rings})$

3) $\text{Alg}_{\text{CommMod}}(\mathcal{M}) \cong \{ (R, M) \mid R \in \text{Rings}, M \text{ R-module} \} = \text{Mod}$

Now we also want to define $\text{Alg}_{\mathcal{O}}(\mathcal{M})$ when $\mathcal{M} \in \text{SymMonCat}_{\infty}$.

Construction: let $\mathcal{O} \in \text{Op}_1$.

1) Define a 1-category \mathcal{O}^{\otimes} .

objects: $(\langle n \rangle, a_{11}, \dots, a_n)$, $\langle n \rangle \in \text{Fin}_*$, $a_{11}, \dots, a_n \in \mathcal{O}$

maps: $\text{Hom}_{\mathcal{O}^{\otimes}}((\langle m \rangle, a_{11}, \dots, a_m), (\langle n \rangle, b_{11}, \dots, b_n))$

$$= \left\{ (\alpha, f_{11}, \dots, f_n) \mid \begin{array}{l} \alpha: \langle m \rangle \rightarrow \langle n \rangle \text{ in } \text{Fin}_* \\ f_i \in \text{Mod}_{\mathcal{O}}((a_j)_{j \in \alpha^{-1}(i)}, b_i) \end{array} \right\}$$

2) We have a natural map $\mathcal{O}^{\otimes} \rightarrow \text{Fin}_*$.

(note: For $\mathcal{M} \in \text{SymMonCat}_1$, $\tilde{\mathcal{M}}^{\otimes} \cong \mathcal{M}^{\otimes}$, compatible with map to Fin_*).

[Def: $\alpha: \langle m \rangle \rightarrow \langle n \rangle$ in Fin_* is inert if $|\alpha^{-1}(i)|=1 \forall i \in \langle n \rangle^\circ$]

•) $\mathcal{O}^\otimes \rightarrow \text{Fin}_*$ is in general not cocartesian, but inert maps always have cocartesian lifts:

Given $(\langle m \rangle, a_{11}, \dots, a_{1m}) \in \mathcal{O}^\otimes$ and $\alpha: \langle m \rangle \rightarrow \langle n \rangle$ inert, such a lift is given by $(\alpha, \text{id}, \dots, \text{id}): (\langle m \rangle, a_{11}, \dots, a_{1m}) \rightarrow (\langle n \rangle, a_{\alpha^{-1}(1)1}, \dots, a_{\alpha^{-1}(n)})$

•) Given $\mathcal{O}_1, \mathcal{O}_2^* \in \text{Op}_1$ we have a fully faithful inclusion

$$\text{Fun}(\mathcal{O}_1, \mathcal{O}_2) \subseteq \text{Fun}_{\text{Fin}_*}(\mathcal{O}_1^\otimes, \mathcal{O}_2^\otimes)$$

with essential image given by those functors preserving cocartesian maps lying over inert maps in Fin_* .

(compare this to the statement that $\text{SymMonCat}_1 \subseteq \text{Op}_1$ is not full).

Remark: One can give a definition of ∞ -operad as a map of ∞ -cats.

$\mathcal{O}^\otimes \rightarrow \text{Fin}_*$ that is modelled on the above construction.

Def: Let $\mathcal{O} \in \text{Op}_1, \mathcal{M} \in \text{SymMonCat}_\infty$. Then we define $\text{Alg}_{\mathcal{O}}(\mathcal{M}) \subseteq \text{Fun}_{\text{Fin}_*}(\mathcal{O}^\otimes, \mathcal{M}^\otimes)$

as the full subcategory spanned by those maps $(\mathcal{O}^\otimes, p_{\mathcal{O}}) \rightarrow (\mathcal{M}^\otimes, p_{\mathcal{M}})$ that send $p_{\mathcal{O}}$ -cocartesian maps in \mathcal{O}^\otimes that lie over inert maps in Fin_* to $p_{\mathcal{M}}$ -cocartesian maps in \mathcal{M} .

We call objects in $\text{Alg}_{\mathcal{O}}(\mathcal{M})$ \mathcal{O} -algebras in \mathcal{M} .

Remark:

•) This definition extends the definition of $\text{Alg}_0(\mathcal{M})$ for $\mathcal{M} \in \text{Sym Mon Cat}_1$ and in particular does not lead to a clash of notation.

•) As before we have contravariant functoriality of $\text{Alg}_0(\mathcal{M})$ with respect to maps of \mathcal{T} -operads.

We are now in a place where we can effortlessly define the following notions (for $\mathcal{M} \in \text{Sym Mon Cat}_\infty$):

•) $\text{CAlg}(\mathcal{M}) := \text{Alg}_{\text{Comm}}(\mathcal{M}) = \underline{\text{commutative algebra objects in } \mathcal{M}}$

•) Given $A \in \text{CAlg}(\mathcal{M})$:

- $\text{CAlg}(A) := [0]_{A, \text{CAlg}(\mathcal{M}), a} \times_{\text{Alg}_{\text{Comm}}^{[1]}(\mathcal{M})}$:

commutative A -algebras

- $\text{Mod}(A) := [0]_{A, \text{CAlg}(\mathcal{M})} \times_{\text{Alg}_{\text{CommMod}}(\mathcal{M})}$:

A -modules

The map of \mathcal{T} -operads $\text{CommMod} \rightarrow \text{Comm}^{[1]}$ induces a natural forgetful functor

$\text{CAlg}(A) \rightarrow \text{Mod}(A)$