Stacks: save your automorphisms!

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These notes correspond to the talk given in the Motives Seminar on November 16 in 2021. The purpose was to give a brief introduction to algebraic spaces and algebraic stacks, discuss some of their properties and state a theorem of Totaro, followed by discussing deformations to the normal cone.

Setup: Throughout, we work over a field k and consider the category Sch_k of schemes which are quasi-separated and quasi-compact over k. A "scheme" without further explanation will always mean a scheme in Sch_k.

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1 Algebraic spaces and a bit of motivation

Suppose we have a scheme X and a group scheme G which acts on it. We would like to define the quotient scheme X/G corresponding to this situation. However, this is not always possible.

Example 1.1 (See [12], Tag 02Z0 and [1], Example 2.9.2)**.** Suppose that the characteristic of k is not equal to 2. Consider the action of $\{\pm 1\}$ on $\mathbb{A}^1$ which - after choosing coordinates - is defined by $-1 \cdot x = -x$. The only fixed point is 0. We can therefore consider the scheme

$$R = (\{\pm 1\} \times \mathbb{A}^1) \setminus \{(-1, 0)\}$$

and we find two maps $R \to \mathbb{A}^1$: the natural projection p_2 and the map σ which is defined by the action. This defines an equivalence relation on $\mathbb{A}^1$.

Claim: The quotient $X = \mathbb{A}^1/R$ of $\mathbb{A}^1$ by the above equivalence relation is *not* a scheme.

The reason for this is that for any scheme, the diagonal is a locally closed immersion (so it can be written as a composition of a closed immersion and an open immersion, see also [5]). But from the diagram

$$\begin{array}{ccc}
(\mathbb{A}^1 \setminus \{0\}) \sqcup \{0\} & \longrightarrow & \mathbb{A}^1 \\
\downarrow & & \downarrow{\scriptstyle x \mapsto (x:\, -x)} \\
R & \xrightarrow{(\sigma:\, p_2)} & \mathbb{A}^1 \times_k \mathbb{A}^1 \\
\downarrow & & \downarrow \\
X & \xrightarrow{\Delta} & X \times_k X
\end{array}$$

we see that this is not the case for X. This is also called the "bug eyed cover" as the fixed point "sticks out of the quotient space like a big bugs eye".

So morally, the automorphisms are in the way of constructing a quotient scheme here. However, we can construct a quotient of the above action if we would allow gluing over étale covers, not Zariski ones, see [[1], Corollary 2.1.9]. This is one motivation for the definition of an algebraic space.
So how can we extend the category of schemes? First, recall that by the Yoneda lemma, schemes can be identified with their functors of points. And for functors, we can define gluing, i.e. a sheaf property as follows.

Definition 1.1 (See [1], Definition 1.2.3)**.** A *sheaf* of sets/groupoids/... on a site $\mathcal{S}$ is a contravariant functor $F : \mathcal{S} \to$ Sets/Groupoids/... such that for every object S and covering $\{S_i \to S\}$ of S, the diagram

$$F(S) \to \prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \times_S S_j)$$

is an equalizer diagram.

Now even though schemes do not in general allow étale gluing, it is true that their functors of points satisfy fpqc descent, see [[1], Example 1.2.5]. That implies in particular that for a scheme X the functor of points $\mathrm{Hom}(-, X)$ defines a sheaf in the étale topology.

Definition 1.2 ([8], Chapter 2, Definition 1.1)**.** An *algebraic space* is a functor $\mathcal{A} : \mathrm{Sch}_k^{op} \to \mathrm{Sets}$ such that:

- $\mathcal{A}$ is a sheaf in the étale topology.
- *(Local representability)* There exists a scheme U and a map of sheaves $U \to \mathcal{A}$ (where we identify U with its functor of points) such that for all schemes V and maps $V \to \mathcal{A}$ the fibre product $U \times_{\mathcal{A}} V$ is representable and the map $U \times_{\mathcal{A}} V \to V$ is induced by an étale surjective map of schemes.
- *(Quasi-seperatedness)* For $U \to \mathcal{A}$ as in the second part, the map of schemes inducing $U \times_{\mathcal{A}} U \to U \times U$ is quasi-compact.

We will spell out the sheaf condition in somewhat more detail once we come to the definition of stacks.

Remark 1.1. Algebraic spaces form a category where the morphisms are given by natural transformations of functors.

Example 1.2. Any scheme X defines an algebraic space. We can take $U = X$ for the second condition, then the third condition precisely says that the scheme X is quasi-seperated.

Since the "bad quotient" in Example 1.1 does exist as an algebraic space, one might start to wonder now whether we can always construct quotients as algebraic spaces. Unfortunately, this is not true.

Example 1.3 (See [1], Example 2.9.14)**.** Let $\mathbb{G}_m$ act on $\mathbb{A}^1$ by scalar multiplication. The quotient consists of two points, and is neither a scheme nor an algebraic space.

Moreover, there is another reason to be willing to extend the category of schemes a little further. In topology, there exists a classifying space $B\mathrm{GL}_n$ for any $n \in \mathbb{Z}_{\geq 0}$ such that for any space T, the homotopy classes of maps $f : T \to B\mathrm{GL}_n$ correspond to isomorphism classes of vector bundles of rank n on T. It would be great if such a thing could also exist in algebraic geometry. However, a vector bundle on a scheme is locally trivial, implying that if such a thing would exist, every vector bundle would be globally trivial, which is not the case. Somehow passing to isomorphism classes does not seem to be a good idea.
One way to get around both of those problems is to introduce algebraic stacks.

2 Stacks in general

For a completely formal definition of stacks, using the language of fibered categories, I can highly recommend reading [14] or [2] (for a shortened version). For the purpose of this seminar, however, we stick to the following definition from [6]. Equip the category of schemes with an appropriate Grothendieck topology, mostly either the étale topology or the fppf topology, or, if we are working over the complex numbers, even the analytic topology.

Definition 2.1 ([6], Definition 1.1)**.** A *stack* is a sheaf of groupoids

$$\mathcal{M} : \mathrm{Sch}_k^{op} \to \mathrm{Groupoids}.$$

This means that:

- For every scheme T, we obtain a groupoid $\mathcal{M}(T)$.
- For every morphism of schemes $f : X \to Y$, we obtain a corresponding functor $f^* : \mathcal{M}(Y) \to \mathcal{M}(X)$.
- For every pair of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, we obtain a natural transformation $\phi_{f,g} : f^* \circ g^* \to (g \circ f)^*$, associative for composition.

These should satisfy the following gluing conditions:

- Given a covering $\{U_i \to T\}_{i\in I}$ of T, objects $\mathcal{E}_i \in \mathcal{M}(U_i)$ and isomorphisms $\phi_{ij} : \mathcal{E}_i|_{U_i\times U_j} \to \mathcal{E}_j|_{U_i\times U_j}$ satisfying the usual cocycle condition, there exists a $\mathcal{E} \in \mathcal{M}(T)$ together with isomorphisms $\psi_i : \mathcal{E}|_{U_i} \to \mathcal{E}_i$ such that $\phi_{ij} = \psi_j \circ \psi_i^{-1}$.
- Given a covering $\{U_i \to T\}_{i\in I}$ of T, objects $\mathcal{E}, \mathcal{F} \in \mathcal{M}(T)$ and morphisms $\phi_i : \mathcal{E}|_{U_i} \to \mathcal{F}|_{U_i}$ such that $\phi_i|_{U_i\times U_j} = \phi_j|_{U_i\times U_j}$, there is a unique morphism $\phi : \mathcal{E} \to \mathcal{F}$ such that $\phi|_{U_i} = \phi_i$.

Here, we denote $\mathcal{E}|_U$ for the pull-back of $\mathcal{E}$ to U, not specifying all maps for convenience.

Remark 2.1. To make this definition really precise, one would need to specify some more data. For example, one needs all pull-backs to be defined functorially, and this often means one has to make choices. This is why fibered categories could come into the more formal way of doing this which was mentioned before. On the other hand, using the language of ∞-categories, this precisely means we need to specify fillers of certain triangles, so there is some higher structure going on here.

Definition 2.2 ([6], Remark 1.7)**.** A *morphism of stacks* $F : \mathcal{M} \to \mathcal{N}$ is a collection of functors $F_T : \mathcal{M}(T) \to \mathcal{N}(T)$ for all T together with for every morphism $f : X \to Y$ in Sch_k a natural transformation $F_f : F_X \circ f^* \to f^* \circ F_Y$ satisfying some associativity condition.

Remark 2.2. Again, one sees that there is some higher structure going on here: we do not require the functors F_T to respect the pull-backs on the nose, but rather choose a filler for the corresponding square. In this way one can also make sense of the associativity condition.

Remark 2.3. Given two morphisms of stacks $f : \mathcal{M} \to \mathcal{N}$ and $f' : \mathcal{M}' \to \mathcal{N}$, one can define a fibre product $\mathcal{M} \times_{\mathcal{N}} \mathcal{M}'$ which will again be a stack. See [[1], Section 1.3.5]. Elements of $\mathcal{M} \times_{\mathcal{N}} \mathcal{M}'$ are triples (m, m', γ) where $m : S \to \mathcal{M}$ and $m' : S \to \mathcal{M}'$ are maps from a scheme S and $\gamma : f(m) \to f'(m')$ is an isomorphism in $\mathcal{N}(S)$. This has the "usual properties".

Example 2.1 ([6], Example 1.5)**.** Let X be a scheme. Then $\mathrm{Hom}(-, X)$ defines a stack (also called a *representable stack*). Here, for a scheme T, we view $\mathrm{Hom}(T, X)$ as a category in which all morphisms are identities and pullbacks are given by composition.

Example 2.2. Any algebraic space does define a stack.

Example 2.3 ([6], Example 1.3)**.** Let C be a smooth projective curve and let $\mathrm{Bun}_{n,C}$ be the stack given by

$$\mathrm{Bun}_{n,C}(T) = \{\text{vector bundles of rank } n \text{ on } C \times T\}.$$

Morphisms are isomorphisms of vector bundles and the functors f^* are given by pull-back. Note that this is a stack precisely because vector bundles can be defined from gluing locally!

Example 2.4 ([6], Example 1.6)**.** Let X be a scheme and let G be an algebraic group acting on X. Then there is a *quotient stack* $[X/G]$ defined by

$$[X/G](T) = \{T \xleftarrow{p} P \xrightarrow{g} X : p \text{ is a } G\text{-bundle and } g \text{ is a } G\text{-equivariant map}\}$$

Here, recall that a G*-bundle* over T is a scheme P with a map $P \to T$ and an action of G on P, such that there is a covering $\{U_i \to T\}_{i \in I}$ of T such that for all U_i, there is a G-equivariant isomorphism

$$\begin{array}{ccc} P|_{U_i} & \xrightarrow{\cong} & G \times_k U_i \\ & \searrow \quad \swarrow & \\ & U_i & \end{array}$$

see also [[11], Definition 7]. Morphisms in $[X/G](T)$ are isomorphisms of G-bundles which commute with the map to X. Note that in particular, this covers Example 1.3.

Remark 2.4. If in the above example, there exists a quotient X/G of X by G which is a scheme, such that $X \to X/G$ is a G-bundle, then one can complete any diagram

$$\begin{array}{ccc} P & \xrightarrow{g} & X \\ \downarrow{\scriptstyle p} & & \downarrow \\ T & \xrightarrow{\bar{g}} & X/G \end{array}$$

in such a way that P becomes isomorphic to $X \times_{X/G} T$ over T. So $[X/G](T)$ is equivalent to $(X/G)(T)$ (considered as a category in which all morphisms are identities again).

3 Algebraic stacks

Understanding stacks somehow becomes easier and more geometric if there exists a suitable "atlas by schemes". Stacks which have this property are called *algebraic stacks* or *Artin stacks*. We will now make this definition precise.

Definition 3.1 ([6], Definition 1.10)**.** A stack $\mathcal{M}$ is called *algebraic* or an *Artin stack* if:

- For all schemes X and Y and morphisms $X \to \mathcal{M}$ and $Y \to \mathcal{M}$ the fibre product $X \times_{\mathcal{M}} Y$ is representable.
- There exists a scheme $u : U \to \mathcal{M}$ such that for all schemes $X \to \mathcal{M}$ the projection $X \times_{\mathcal{M}} U \to X$ is a smooth surjection.

- The forgetful map $U \times_{\mathcal{M}} U \to U \times_k U$ is quasi-compact and quasi-separated.

A map $u : U \to \mathcal{M}$ is also called an *atlas* of $\mathcal{M}$.

Remark 3.1. If in the second point of the above definition, one replaces "smooth" by "étale", one gets the definition of a *Deligne-Mumford stack*.

Remark 3.2. There are some variations of the above definition around in the literature. In the first condition one may require representability by algebraic spaces rather than schemes, for example. For the purposes of our seminar, both versions of the definition will work out. Many authors also ask for an atlas by algebraic spaces rather than schemes. Also, in [6], one asks seperatedness rather than quasi-seperatednesss in the last condition, but for our purposes, the definition as above will suffice.

Example 3.1. Schemes are algebraic stacks: an atlas is given by the identity morphism. Actually, any algebraic space is an algebraic stack, and even a Deligne-Mumford stack.

Example 3.2 ([6], Example 1.13)**.** Quotient stacks are algebraic, given that the group scheme G acting on a scheme X is smooth and affine at least. If we consider the canonical map $X \to [X/G]$ which is given by the trivial G-bundle $G \times X$ then this defines an atlas. See also [[1], Theorem 2.1.8] for this.

Example 3.3 ([6], Example 1.14)**.** Let C be a smooth projective curve. Then $\mathrm{Bun}_{n,C}$ is an algebraic stack. See [[1], Theorem 2.1.15] for a proof.

4 Properties of algebraic stacks

We now discuss some basic properties of algebraic stacks.

Definition 4.1 ([6], Definition 2.1)**.** An algebraic stack $\mathcal{M}$ is called *smooth/ normal/reduced/locally of finite presentation/locally Noetherian/regular* if there exists an atlas $u : U \to \mathcal{M}$ with U being smooth/normal/reduced/locally of finite presentation/locally Noetherian/regular (respectively).

Remark 4.1. For schemes, we don't get anything new, because all the above properties can be checked locally on a smooth covering, for example the identity, which we could take as an atlas.

One can do a similar thing when considering properties of morphisms. We again want to derive these properties from properties of schemes. In order to do so, the notion of *representable morphism* is needed.

Definition 4.2 ([6], Definition 1.15)**.** A morphism $F : \mathcal{M} \to \mathcal{N}$ of stacks is called *representable* if for all $X \to \mathcal{N}$ with $X \in \mathrm{Sch}_k$ the fibre product $X \times_{\mathcal{N}} \mathcal{M}$ is representable.

Definition 4.3 ([6], Definition 2.2)**.** Let P be a property of morphisms of schemes $f : X \to Y$ such that f has P if and only if for some smooth surjective $Y' \to Y$ the induced morphism $f' : X \times_Y Y' \to Y'$ has P. A representable morphism $F : \mathcal{M} \to \mathcal{N}$ of algebraic stacks *has property* P if for some (or equivalently for any) atlas $u : U \to \mathcal{N}$ the morphism $M \times_{\mathcal{N}} U \to U$ has P.

Remark 4.2. Examples of such properties P are being a closed or open immersion, as well as being an affine, finite or proper morphism. In particular, it now makes sense to talk about *open substacks*.

There are other properties, however, which one can also define for morphisms which are not necessarily representable.

Definition 4.4 ([6], Definition 2.4)**.** Let P be a property of morphisms of schemes $f : X \to Y$ such that f has P if and only if there exists a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & Y \end{array}$$

with p and p' smooth such that f' has P. Then a morphism of algebraic stacks $F : \mathcal{M} \to \mathcal{N}$ has P if for some atlases $v : V \to \mathcal{M}$ and $u : U \to \mathcal{N}$ there exists a diagram

$$\begin{array}{ccc} V & \xrightarrow{F'} & U \\ \downarrow & & \downarrow \\ \mathcal{M} & \xrightarrow{F} & \mathcal{N} \end{array}$$

such that F' has P.

Remark 4.3. Now examples are smooth morphisms, flat morphisms and morphisms which are locally of finite presentation.

Similar to the category of schemes, one can also look at a notion of (quasi-) coherent sheaves on stacks. These are defined as follows.

Definition 4.5 ([6], Definition 2.8)**.** A *(quasi-) coherent sheaf* $\mathcal{F}$ on an algebraic stack $\mathcal{M}$ is the datum consisting of:

- For all smooth maps $x : X \to \mathcal{M}$ with X a scheme a (quasi-) coherent sheaf $\mathcal{F}_{X,x}$ on X.
- For all diagrams

$$\begin{array}{ccccc} V & & \xrightarrow{\quad f \quad} & & U \\ & {\scriptstyle v}\searrow & & \swarrow{\scriptstyle u} & \\ & & \mathcal{M} & & \end{array}$$

  together with an isomorphism $\phi : u \circ f \to v$, we want an isomorphism $\theta_{f,\phi} : f^*\mathcal{F}_{U,u} \to \mathcal{F}_{V,v}$ such that these are compatible under composition.

Remark 4.4. Note that this somehow mimics the definition of quasi-coherent sheaves on a scheme, which you can define by gluing as well. Also, note that one can again observe a hint of higher structure going on here.

Remark 4.5. It also makes sense to speak about a vector bundle on a stack now, meaning all the quasi-coherent sheaves in the above definition are locally free.

5 Totaro's theorem

So far the motivating example for defining stacks has been quotient stacks. There is a local version of this definition as well.

Definition 5.1 ([7], Definition 6.1)**.** An algebraic stack $\mathcal{M}$ is called a *local quotient stack* if there is a covering $\{\mathcal{U}_i\}$ of $\mathcal{M}$ by open substacks such that $\mathcal{U}_i \cong [U_i/G_i]$ for all i, where G_i is an algebraic group acting on a scheme U_i.

Remark 5.1. That $\{\mathcal{U}_i\}$ is a covering by open substacks means the following: for any scheme T over $\mathcal{M}$, we have that $\mathcal{U}_i \times_{\mathcal{M}} T$ is representable (because the $\mathcal{U}_i$ are open substacks) and $\{\mathcal{U}_i \times_{\mathcal{M}} T \to T\}_i$ is an open cover of T.

Remark 5.2. The above definition was first given in [[3], Appendix 2]. I recommend reading this in full for a completely precise definition (but be aware they work from groupoids and not directly from stacks, after which they use that any stack has some sort of groupoid representation). For the purposes of this talk, however, it will be more convenient to work with the definition as given above.

Example 5.1. All examples we have seen so far are local quotient stacks.

Given that many interesting stacks are local quotient stacks, one can wonder under what circumstances a stack is a global quotient stack. The following theorem of Totaro, the proof of which is quite complicated, gives an answer to that question.

Theorem 5.1 ([13], Theorem 1.1)**.** *Let $\mathcal{M}$ be a normal, locally Noetherian algebraic stack whose stabilizer groups at closed points are affine group schemes. The following are equivalent:*

- *$\mathcal{M}$ has the resolution property: every coherent sheaf on $\mathcal{M}$ is a quotient of a vector bundle on $\mathcal{M}$.*
- *$\mathcal{M}$ is isomorphic to the quotient stack of some quasi-affine scheme by an action of $GL(n)$ for some n.*

If $\mathcal{M}$ is, moreover, of finite type over our ground field k then these are also equivalent to

- *$\mathcal{M}$ is isomorphic to the quotient stack of some affine scheme over k by an action of an affine group scheme of finite type over k.*

Here, we note that for any morphism $x : T \to \mathcal{M}$ one can form the fibre product

$$\begin{array}{ccc} \mathrm{Aut}_{\mathcal{M}(T)}(x) & \longrightarrow & T \\ \downarrow & & \downarrow \\ \mathcal{M} & \xrightarrow{\Delta} & \mathcal{M} \times \mathcal{M} \end{array}$$

If $T = \mathrm{Spec}(K)$ for some field K, i.e. x is a closed point, then we call $\mathrm{Aut}_{\mathcal{M}(T)}(x)$ the *stabilizer* of x.

Remark 5.3. Totaro actually works with Noetherian algebraic stacks in his paper [13], meaning stacks which are locally Noetherian, quasi-compact and quasi-seperated. However, given that we only work with schemes which are quasi-compact and quasi-seperated, the last two conditions will be automatic in our case. See [[12], Tag 04YA] for the definition of quasi-compact algebraic stacks.

6 Deformation to the normal cone

Deformations to the normal cone are a very useful tool in intersection theory, and we will also need them later on in this seminar. We recall their construction in the world of schemes from [[4], Section 5.1]. Suppose that we have a scheme Y and a closed subscheme X.

Construction 6.1 (See [4], Appendix B.5.1)**.** Recall that if $\mathcal{I}$ is the ideal sheaf of X inside Y, then $C = C_X Y = \mathbf{Spec}(\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1})$ is the corresponding *normal cone* (where **Spec** indicates we are using a relative spectrum now). The morphism $\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1} \to \mathcal{O}_X$ which is the canonical isomorphism in degree zero and zero everywhere else determines a morphism $X \to C$ called the *zero section embedding.*

Construction 6.2 (See [4], Appendix B.5.1 and B.5.2)**.** Let $S = \bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ and let $C \oplus 1$ be the global spectrum of the graded algebra $S[z]$, where the n'th graded piece is $S^n \oplus S^{n-1}z \oplus \cdots \oplus S^0 z^n$. The cone $\mathbb{P}(C \oplus 1) = \mathbf{Proj}(S[z])$ (where **Proj** denotes a relative Proj-construction) is called the *projective completion* of C. We have that $\mathbb{P}(C)$ embeds into $\mathbb{P}(C\oplus 1)$ as the so-called *hyperplane at infinity.* The complement is canonically isomorphic to C.

Construction 6.3 ([4], Appendix B.6.3)**.** The *blowup* of Y along X is defined to be

$$\tilde{Y} = \mathbb{P}\left(\bigoplus_{n\geq 0} \mathcal{I}^n\right).$$

There is a natural projection $\pi : \tilde{Y} \to Y$. The exceptional divisor $\pi^{-1}(X)$ is now the projective cone of $\bigoplus_{n\geq 0} \mathcal{I}^n \otimes \mathcal{O}_X = \bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$, which means it is precisely $\mathbb{P}(C)$.

Proposition 6.1 (Deformation to the normal cone)**.** *One can construct a scheme $M = M_X Y$ together with a closed embedding $i : X \times \mathbb{P}^1 \to M$ and a flat morphism $\sigma : M \to \mathbb{P}^1$ such that the diagram*

$$\begin{array}{ccc} X \times \mathbb{P}^1 & \xrightarrow{\quad i \quad} & M \\ & \searrow \quad \swarrow_{\sigma} & \\ & \mathbb{P}^1 & \end{array}$$

commutes and moreover:

- *Over $\mathbb{P}^1 \setminus \{\infty\} = \mathbb{A}^1$ we have that $\sigma^{-1}(\mathbb{A}^1) \cong Y \times \mathbb{A}^1$ and the embedding i is the trivial embedding $X \times \mathbb{A}^1 \to Y \times \mathbb{A}^1$.*
- *Over $\{\infty\}$ the divisor $M_\infty = \sigma^{-1}(\{\infty\})$ can be written as the sum*

$$M_\infty = \mathbb{P}(C \oplus 1) + \tilde{Y}$$

of Cartier divisors. We have that $X = X \times \{\infty\}$ embeds into M_∞ by the zero section embedding of X into C followed by the canonical open embedding of C into $\mathbb{P}(C \oplus 1)$. Furthermore, we have that $\mathbb{P}(C \oplus 1)$ and $\tilde{Y}$ intersect in $\mathbb{P}(C)$ (hyperplane at infinity in $\mathbb{P}(C \oplus 1)$ and exceptional divisor in $\tilde{Y}$).

Remark 6.1. In particular, the image of X inside M_∞ is disjoint from $\tilde{Y}$. If we let $M_0 = M \setminus \tilde{Y}$ then we have a family of embeddings

$$\begin{array}{ccc} X \times \mathbb{P}^1 & \xrightarrow{\qquad} & M_0 \\ & \searrow \quad \swarrow & \\ & \mathbb{P}^1 & \end{array}$$

which deforms the given embedding of X into Y to the zero section embedding of X into the normal cone C. This explains the name.

Construction 6.4 (Sketch of the construction)**.** For the construction, one defines M to be the blowup of $Y \times \mathbb{P}^1$ along the subscheme $X \times \{\infty\}$. As we have a sequence of embeddings

$$X \times \{\infty\} \to X \times \mathbb{P}^1 \to Y \times \mathbb{P}^1$$

and as $X \times \{\infty\}$ is a Cartier divisor on $X \times \mathbb{P}^1$, there is a closed embedding $X \times \mathbb{P}^1 = \mathrm{Bl}_X(X \times \mathbb{P}^1) \to M$. And from the sequence

$$X \times \{\infty\} \to Y \times \{\infty\} \to Y \times \mathbb{P}^1$$

we see that $\tilde{Y}$ also embeds into M. Now we let σ be the composition of the blown-down map $M \to Y \times \mathbb{P}^1$ followed by the projection to $\mathbb{P}^1$. One can then show that this is flat, and that this setup will have the properties defined above.

We now claim that a similar construction can be done for algebraic stacks. Given a closed immersion (which is in particular representable) one defines the normal cones and blowups in pretty much the same way (but note that they may be stacks rather than schemes) and then follows the same reasoning. A good reference to consult for details about this is [9]. For more details about for example Proj-constructions on stacks, see also [10].

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