Descent Along Sections

Aims: Recall some basic notions about split-forks, with motivations, describe in detail the skeletal structures of Δ_+ and $\Delta_{-\infty}$, introduce split-simplicial objects in the ∞-categorical setting, generalizing split forks, and present the result about descent along morphisms admitting sections in this setting, that follows by a deep theorem proved by Lurie in HTT.

§ Split-forks

Here we just recall some basic notions:

- A fork in a category $\mathcal{C}$ is a diagram of the shape
$$a \xrightarrow{d_0^0} b \overset{d_0^1}{\underset{d_1^1}{\rightrightarrows}} c\,, \quad \text{where } d_0^1 d_0^0 = d_1^1 d_0^0$$
A fork is an equalizer if $a = \lim(b \rightrightarrows c)$

Examples: - If $f: A \to B$ morphism of commutative ring, then the diagram
$$A \to B \overset{b\otimes 1}{\underset{1\otimes b}{\rightrightarrows}} B \otimes_A B \quad \text{is a fork}$$

- A presheaf $F: \mathcal{C}^{op} \to Sets$ on a site $\mathcal{C}$ admitting products is, by definition, a sheaf if for all $\{U_i \to X\}_i \in Cov(\mathcal{C})$, the fork $F(X) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_X U_j)$

is an equalizer

- A fork $a \xrightarrow{d_0^0} b \overset{d_0^1}{\underset{d_1^1}{\rightrightarrows}} c$ is split if $\exists\ s_{-1}^{-1}: b \to a$ and $s_{-1}^0 : c \to b$ s.t.
$$\begin{cases} s_{-1}^{-1} d_0^0 = id_a \\ s_{-1}^0 d_0^1 = id_b \end{cases}$$

$s^0_{-1} : c \to b$ s.t. $\begin{cases} s^{-1}_{-1} d_0 = id_a \\ s^0_{-1} d^1_0 = id_b \\ s^0_{-1} d^1_1 = d^0_0 s^{-1}_{-1} \end{cases}$

Importance of split-forks relies in the following fact:

<u>Fact</u>: Split forks are equalizers

↦ We can easily verify the universal property of limits:

$$\begin{tikzcd} x \arrow[r, dashed, "h'"'] \arrow[rr, bend left, "h"] & a \arrow[r] & b \arrow[d, shift left] \arrow[d, shift right] \\ & & c \end{tikzcd}$$

If $h: x \to b$ s.t. $d^1_0 h = d^1_1 h$, we define $h' := s^{-1}_{-1} h : x \to a$ and we have

$d^0_0 h' = d^0_0 s^{-1}_{-1} h = s^0_{-1} d^1_1 h = s^0_{-1} d^1_0 h = h$.

And if we have $h'' : x \to a$ s.t. $d^0_0 h'' = h$ we see that

$s^{-1}_{-1} d^0_0 h'' = s^{-1}_{-1} d^0_0 h'$ that means $h' = h''$ since $s^{-1}_{-1} d^0_0 = id_a$. ↤

<u>Example</u>: If $\mathcal{C}$ is a site where coverings are just maps $U \to X$ admitting a section s, then any presheaf is a sheaf, since the diagram

$$F(X) \to F(U) \rightrightarrows F(U \times_X U)$$

with $F(s): F(U) \to F(X)$ and $F((id, U \to X \to U)): F(U\times_X U) \to F(U)$, is a split-fork, thus an equalizer.

<u>Digression</u> [Barr-Beck thm]

Split-forks are used in Lurie's HA to prove the following ∞-categorical version of Barr-Beck monadicity theorem:

<u>Thm</u>: $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ adjoint functors between ∞-categories. Then, the following are equivalent:

a) G is conservative (f in $\mathcal{D}$ is an equivalence $\Leftrightarrow$ $G(f)$ is an equivalence in $\mathcal{C}$), and, if is a G-split simplicial object of $\mathcal{D}$, V admits a colimit in $\mathcal{D}$ preserved by G.

b) $\exists$ monoidal ∞-category $\mathcal{E}^{\otimes}$, with a left action on $\mathcal{C}$, an algebra object $A \in Alg(\mathcal{E})$ and an equivalence $G' : \mathcal{D} \xrightarrow{\sim} LMod_A(\mathcal{C})$ s.t. G is equivalent to the composition $\mathcal{D} \xrightarrow{G'} LMod_A(\mathcal{C}) \xrightarrow{U} \mathcal{C}$.

In particular, to prove (a) $\Rightarrow$ (b) one needs the following:

Lemma: $\mathcal{C}$ monoidal ∞-category, $\mathcal{M}$ ∞-category which is left-tensored over $\mathcal{C}$, A algebra object, and $U: \mathrm{Mod}_A(\mathcal{M}) \to \mathcal{M}$ forgetful functor. Then:

1) Every U-split simplicial object of $\mathrm{LMod}_A(\mathcal{M})$ admits a colimit in $\mathcal{M}$.

2) The functor U preserves colimits of U-split simplicial objects.

§ Augmented Simplicial & Split Simplicial Category

- Augmented simplicial category: Is the category Δ_+ with:

- $\mathrm{obj}(\Delta_+) = \{[n]_+ := [n] \cup \{-\infty\} \mid n \geq 0\} \cup [-1]_+ := \{-\infty\}$

- $\mathrm{Hom}_{\Delta_+}([n]_+, [m]_+) := \{\alpha : [n]_+ \to [m]_+$ order preserving, and s.t. $\alpha^{-1}(-\infty) = \{-\infty\}\}$

It's trivial to see that Δ_+ is a category.

We can see Δ_+ as obtained by Δ by formally adjoining $[-1]_+$ as initial element. In particular Δ is a full subcategory of Δ_+.

Notation: we can write $[n]$ to denote $[n]_+$ ($\Delta \underset{\text{full}}{\subseteq} \Delta_+$).

Let's define face maps & degeneracy maps for Δ and Δ_+.

- Face maps $\quad d_i^n : [n-1] \to [n], \quad d_i^n(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases} \qquad \begin{matrix} n > 0 \\ 0 \leq j \leq n \end{matrix}$

slogan: the only injective map without i in the image

- Degeneracy maps : $\quad s_i^n : [n+1] \to [n] \quad s_i^n(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases} \qquad \begin{matrix} n \geq 0 \\ 0 \leq i \leq n \end{matrix}$

slogan: the only surjective map with two j's in the image.

Face and degeneracy maps are fundamental because of this proposition:

Prop [Mac Lane]: $f:[n]\to[m]$ in Δ (or Δ_+). Then f has a unique decomposition as:

$$(\star)\quad f = (d^{m}_{i_1} d^{m-1}_{i_2} \cdots d^{m-k+1}_{i_k})(s^{n-h}_{j_1} \cdots s^{n-1}_{j_h})$$

where: $k-h = m-n$
$m \geq i_1 > \cdots > i_k \geq 0$
$0 \leq j_1 < j_2 < \cdots < j_h < n$

$\mapsto$ f is uniquely determined by its image and by the set of elements in $[n]$ for which $f(i) = f(i+1)$ (this can easily seen by induction on n).

Let $\{j_1, \ldots, j_h\}$ be the set of elements in $[n]$ for which $f(j) = f(j+1)$, in increasing order, and let $\{i_1, \ldots, i_k\}$ be the set of elements in $[m]$ that do not belong to $Im(f)$, in reverse order.

Then the RHS and LHS of $(\star)$ have the same image and the same "stationary points".

The decomposition is unique because the sets of subscripts and superscripts are tied. $\dashv$

In particular, any arbitrary composition of face and degeneracy maps can be written in the canonical form $(\star)$.

By checking it for compositions of 2 maps, one obtains the following identities, that we will denote by (SI):

Simplicial Identities :

$$\left.\begin{array}{ll}
d_i^{n+1} d_j^n = d_{j+1}^{n+1} d_i^n & i \leq j \\[1em]
s_j^n s_i^{n+1} = s_i^n s_{j+1}^{n+1} & i \leq j \\[1em]
s_j^n d_i^{n+1} = \begin{cases} id_{[n]} & \text{if } i = j, j+1 \\ d_i^n s_{j-1}^{n-1} & \text{if } i < j \\ d_{i-1}^n s_j^{n-1} & \text{if } i \geq j+2 \end{cases} &
\end{array}\right] \qquad (S1)$$

$\mapsto$ For instance, one can verify that both sides of the first identity are monotone injections $[n-1] \to [n+1]$ with the same image (i.e. $[n] - \{i, j+1\}$).
Similarly, one can verify all the others. $\dashv$

Corollary [Presentation of Δ]: The class of objects $\{[n]\}_{n \geq 0}$ and a family of maps $\{d_i^n, s_j^m\}_{\substack{i \geq 1, j \geq 0 \\ 0 \leq i \leq n \\ 0 \leq j \leq m-1}}$ subjected to the relations (S1) provide a presentation of Δ.

$\mapsto$ By using the relations (S1), one can write all possible finite compositions of maps d_i^n, s_j^m in the canonical form (✱). $\dashv$

We want to obtain the analog for the category Δ_+.
Morphisms in Δ_+ are the ones of Δ and the ones of the form $[-1] \to [n]$.
In particular there is an "extra" face map $d_0^0 : [-1] \to [0]$, $-\infty \mapsto -\infty$, and it obviously satisfy:

$$d_0^1 d_0^0 = d_1^1 d_0^0 \qquad (D1)$$

Note that (D1) says that the diagram $[-1] \xrightarrow{d_0^0} [0] \overset{d_0^1}{\underset{d_1^1}{\rightrightarrows}} [1]$ is a fork.

Cor [Presentation of Δ_+]: The objects $\{[n]\}_{n \geq -1}$ and arrows $\{d_i^n, s_j^m\}_{n \geq -1}$ satisfying (S1) and (D1) provide a presentation for Δ_+

$\mapsto$ Same proof as for Δ. $\dashv$

The <u>skeletal description</u> of Δ_+ (and Δ) looks like:

$$[-1] \longrightarrow \left([0] \begin{smallmatrix} \longrightarrow \\ \dashleftarrow \\ \longrightarrow \end{smallmatrix} [1] \begin{smallmatrix} \longrightarrow \\ \dashleftarrow \\ \longrightarrow \\ \dashleftarrow \\ \longrightarrow \end{smallmatrix} [2] \begin{smallmatrix} \longrightarrow \\ \dashleftarrow \\ \longrightarrow \\ \dashleftarrow \\ \longrightarrow \\ \dashleftarrow \\ \longrightarrow \end{smallmatrix} \cdots \right)$$

- <u>Split Simplicial Category</u>: Is the category $\Delta_{-\infty}$ with:

$Obj(\Delta_{-\infty}) = Obj(\Delta_+)$

$Hom_{\Delta_{-\infty}}([n],[m]) := \{\alpha : [n] \to [m] \text{ order preserving, s.t. } \alpha(-\infty) = -\infty\}$

$\Delta_{-\infty}$ has many more maps than Δ_+, and every map $[-1] \to [n]$ has a section $const_{-\infty} : [n] \to [-1]$

In order to obtain a presentation for $\Delta_{-\infty}$ we have to define another special class of maps in $\Delta_{-\infty}$.

<u>Splitting maps</u> $\quad s_{-1}^{n} : [n+1] \to [n] \quad s_{-1}^{n}(j) = \begin{cases} -\infty & \text{if } j = -\infty, 0 \\ j-1 & \text{otherwise} \end{cases}$

<u>Prop</u>: $f : [n] \to [m]$ in $\Delta_{-\infty}$ has a unique decomposition as:

$$(\star\star) \quad f : \left(d_{i_1}^{m} \cdots d_{i_k}^{m-k+1}\right)\left(s_{j_1}^{n-h} \cdots s_{j_h}^{n'-1}\right)\left(s_{-1}^{n'} \cdots s_{-1}^{n-1}\right)$$

where: $n' = n - \underline{n}$, $\quad \underline{n} := |\{j \in [n] : f(j) = -\infty, j \neq -\infty\}|$

$k - h + n' = m$

$m \geq i_1 > i_2 > \dots > i_k \geq 0$

$0 \leq j_1 < j_2 < \dots < j_h < n$

$\mapsto$ Since $f(j) = -\infty$ for $0 \leq j < \underline{n}$, f can be factorized as:

$$f : [n] \xrightarrow{f'} [n'] \xrightarrow{f''} [m]$$

where $f' = s_{-1}^{n'} \cdots s_{-1}^{n-1}$

and f'' is s.t. $f''^{-1}(-\infty) = \{-\infty\}$.

... canonical form $(\star)$ then we have

and f'' is s.t. $f''^{-1}(-\infty) = \{-\infty\}$.
Then f'' has a decomposition in the canonical form $(\star)$. Then we have a decomposition of f in the form $(\star\star)$.
Since the description of f' is intrinsic to f, and the decomposition of f'' is unique, this decomposition of f is unique. $\dashv$

In particular, one may verify, as before, the following identities:

$$\left.\begin{aligned}
&s_{-1}^{m} d_0^{m+1} = id_{[m]}, \quad m \geq -1\\
&s_{-1}^{m} d_j^{m+1} = d_{j-1}^{m} s_{-1}^{m-1}, \quad m \geq 0,\ 0 < j \leq m+1\\
&s_{-1}^{m-1} s_j^{m} = s_{j-1}^{m-1} s_{-1}^{m}, \quad m > 0,\ 0 \leq j \leq m
\end{aligned}\right] \qquad (S2)$$

As before, we obtain the following:

__Cor__ [Presentation of $\Delta_{-\infty}$]: The relations (S1), (S2) and (D1) provide a presentation of $\Delta_{-\infty}$.

The __skeletal description__ of $\Delta_{-\infty}$ looks like:

$$\begin{tikzcd}
{[-1]} \arrow[r] & {[0]} \arrow[l, dashed, bend right, "s_{-1}^{-1}"'] \arrow[r, shift left] \arrow[r, shift right] & {[1]} \arrow[l, dashed] \arrow[l, dashed, bend right, "s_{-1}^{0}"'] \arrow[r, shift left] \arrow[r] \arrow[r, shift right] & \cdots \arrow[l, dashed, shift left] \arrow[l, dashed, shift right]
\end{tikzcd}$$

Note that
$$\begin{tikzcd}
{[-1]} \arrow[r] & {[0]} \arrow[l, bend right] \arrow[r, shift left] \arrow[r, shift right] & {[1]} \arrow[l, bend right]
\end{tikzcd}$$
is a split fork $\Rightarrow$ It is an equalizer.

§ Split Simplicial Objects in an ∞-Category

__Def__: Let $\mathcal{C}$ be an ∞-category. Then

- A __simplicial object__ of $\mathcal{C}$ is a functor $N(\Delta)^{op} \to \mathcal{C}$
- An __augmented simplicial object__ " $N(\Delta_+)^{op} \to \mathcal{C}$
- A __split simplicial object__ " $N(\Delta_{-\infty})^{op} \to \mathcal{C}$

__Rmk__: Split simplicial objects are a generalization of split forks in the simplicial setting.

Rmk: Split simplicial objects are a generalization of [illegible]
in the simplicial setting.

Dold-Kan Correspondence: In this setting, the DK functor provides an equivalence between split simplicial objects and augmented exact cochain complexes. This extends the ordinary DK correspondence (as explained in [Lurie, HA, sec. 4.7.2]).

Example: Let $\mathcal{C}$ be an ordinary category admitting products, and $f: X \to Y$ a morphism in $\mathcal{C}$ admitting a section $s: Y \to X$. We can construct a split-simplicial object $X^{+}_{\bullet\, Y,f,s}$ as the nerve $N(F)$ of a functor $F: \Delta_{-\infty}^{op} \to \mathcal{C}$ of the underlying ordinary categories defined in the following way:

- $F([n]) = \begin{cases} Y, & \text{if } n = -1 \\ X_Y^n := \underbrace{X \times_Y X \times_Y \cdots \times_Y X}_{n+1 \text{ times}}, & \text{otherwise} \end{cases}$

- For $p: [m] \to [n]$ in $\Delta_{-\infty}$, $F(p): X_Y^n \to X_Y^m$ is defined by:

  - For $n = -1$, $F(p): Y \to X_Y^m$, $y \mapsto (s(y), \dots, s(y))$

  - For $m = -1$, $F(p): X_Y^n \to Y$, $(x_0, \dots, x_n) \mapsto f(x_0)$. Note that $f(x_0) = f(x_i)\ \forall i$

$$\begin{tikzcd} X \times_Y X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

  - For $n, m > -1$, $F(p)(x_0, \dots, x_n) := (x_0', \dots, x_m')$ where

$$x_i' := \begin{cases} s \circ f(x_0) & \text{if } p(i) = -\infty \\ x_{p(i)} & \text{otherwise} \end{cases}$$

By a direct check one may verify that F is compatible with compositions

$$[m] \to [n] \to [\ell] \quad \rightsquigarrow \quad X_Y^{\ell} \to X_Y^n \to X_Y^m$$

$$(x_0, \dots, x_\ell) \mapsto (x_0', \dots, x_n') \mapsto (\underbrace{x_0'', \dots, x_m''}_{})$$

x_i'' are coherent with the definition.

$X^{+}_{\bullet\, Y,f,s}$ looks like:

$X^+_{\bullet, Y, f, s}$ looks like: definition.

$$\cdots X \times_Y X \times_Y X \;\substack{\to\\ \leftarrow\\ \to\\ \leftarrow\\ \to}\; X \times_Y X \;\substack{\to\\ \leftarrow\\ \to}\; X \xrightarrow{f} Y$$

One could also define F only for splitting, face and degeneracy maps, and verify the conditions (S1), (D1), (S2). We gave a more concrete construction.

Rmk: In this example $N(\mathcal{C})$ could be replaced by any ∞-category $\mathcal{C}$. [Lurie, HA, Prop 4.7.2.9].

Importance of Split Simplicial Objects: the following lemma:

Lemma: Let $X: N(\Delta_{-\infty})^{op} \to \mathcal{C}$ be a split simplicial object. Then X is a colimit diagram.

Proving this result is very hard. A proof can be found in [Lurie, HTT, lemma 6.1.3.16].

Cor: $F: N(\mathcal{C}) \to \mathcal{D}$ functor between ∞-categories. $X^+_{\bullet, Y, f, s}$ as before. Then

$$F(Y) = \operatorname{Colim}\left(\cdots \;\substack{\to\\ \leftarrow\\ \to\\ \leftarrow\\ \to}\; F(X \times_Y X) \;\substack{\to\\ \leftarrow\\ \to}\; F(X) \right)$$

$\vdash$ Since $X^+_{\bullet, f, Y, s}: N(\Delta_{-\infty})^{op} \to \mathcal{C}$ is a split-simplicial object, so is the composition $F(X^+_{\bullet, f, Y, s}): N(\Delta_{-\infty})^{op} \to \mathcal{D}$. Thus, $F(X^+_{\bullet, Y, f, s})$ is a colimit diagram by previous lemma.

$\dashv$