Triangulated categories of motives

We have defined the triangulated category of effective geometric motives $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ as a localization of $K^b(\mathrm{Cor}_{\mathrm{fin}}(k))$, giving us motivic cohomology with good structural properties.

It is very difficult to make computations, however, for instance, to see that one recovers the (co)homology we have defined using cycle complexes.

For this, we need a sheaf-theoretic extension of $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$.

Definition

Let X be a k-scheme of finite type. A *Nisnevich cover* $\mathcal{U} \to X$ is an étale morphism of finite type such that, for each finitely generated field extension F of k, the map on F-valued points $\mathcal{U}(F) \to X(F)$ is surjective.

Using Nisnevich covers as covering families gives us the *small Nisnevich site on X*, X_{Nis}, with underlying category the finite type étale X-schemes $U \to X$.

Notation $\mathrm{Sh}^{\mathrm{Nis}}(X) :=$ Nisnevich sheaves of abelian groups on X
For a presheaf $\mathcal{F}$ on $\mathbf{Sm}/k$ or X_{Nis}, we let $\mathcal{F}_{\mathrm{Nis}}$ denote the associated sheaf.

For the record: A presheaf on $\mathcal{C}$ is just a functor $P : \mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}$. A presheaf F on X_{Nis} is a sheaf if for each $U \to X$ in X_{Nis} and each covering family $\{f_\alpha : U_\alpha \to U\}$, the sequence

$$0 \to F(U) \xrightarrow{\prod f_\alpha^*} \prod_\alpha F(U_\alpha) \xrightarrow{\prod p_{U_\alpha}^* - p_{U_\beta}^*} \prod_{\alpha,\beta} F(U_\alpha \times_X U_\beta)$$

is exact.

We now return to motives.

The sheaf-theoretic construction of mixed motives is based on the notion of a *Nisnevich sheaf with transfer*.

Definition

(1) The category $\mathrm{PST}(k)$ of presheaves with transfer is the category of presheaves of abelian groups on $\mathrm{Cor}_{\mathrm{fin}}(k)$ which are additive as functors $\mathrm{Cor}_{\mathrm{fin}}(k)^{\mathrm{op}} \to \mathbf{Ab}$.

(2) The category of Nisnevich sheaves with transfer on $\mathbf{Sm}/k$, $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$, is the full subcategory of $\mathrm{PST}(k)$ with objects those F such that, for each $X \in \mathbf{Sm}/k$, the restriction of F to X_{Nis} is a sheaf.

Example. For $X \in \mathbf{Sm}/k$, we have the representable presheaf with transfers $\mathbb{Z}^{tr}(X) := \mathrm{Cor}_{\mathrm{fin}}(-, X)$. This is in fact a Nisnevich sheaf.

For $X \in \mathbf{Sm}/k$, $\mathbb{Z}^{tr}(X)$ is the free sheaf with transfers generated by the representable sheaf of sets $\mathrm{Hom}(-, X)$. Thus:

there is a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))}(\mathbb{Z}^{tr}(X), F) = F(X)$$

In fact: For $F \in \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$ there is a canonical isomorphism

$$\mathrm{Ext}^n_{\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))}(\mathbb{Z}^{tr}(X), F) \cong H^n(X_{\mathrm{Nis}}, F)$$

and for $C^* \in D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$ there is a canonical isomorphism

$$\mathrm{Hom}_{D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))}(\mathbb{Z}^{tr}(X), C^*[n]) \cong \mathbb{H}^n(X_{\mathrm{Nis}}, C^*).$$

Definition

Let F be a presheaf of abelian groups on $\mathbf{Sm}/k$. We call F *homotopy invariant* if for all $X \in \mathbf{Sm}/k$, the map

$$p^* : F(X) \to F(X \times \mathbb{A}^1)$$

is an isomorphism.

The main foundational result on homotopy invariant PST's is:

Theorem (PST)

Let F be a homotopy invariant PST. Then all the Nisnevich cohomology sheaves $\mathcal{H}^q_{\mathsf{Nis}}(F)$ are homotopy invariant sheaves with transfers.
Additionally: For $X \in \mathbf{Sm}/k$, $H^*(X_{\mathrm{Zar}}, F_{\mathrm{Zar}}) \cong H^*(X_{\mathsf{Nis}}, F_{\mathsf{Nis}})$.

Definition

Inside the derived category $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$, we have the full subcategory $DM^{\mathrm{eff}}_-(k)$ consisting of complexes whose cohomology sheaves are homotopy invariant.

Proposition

$DM^{\mathrm{eff}}_-(k)$ is a triangulated subcategory of $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$.

This follows from the PST theorem: F a homotopy invariant sheaf with transfer $\Longrightarrow$ all cohomology sheaves are homotopy invariant sheaves with transfer, so homotopy invariance "makes sense in the derived category".

We can promote the Suslin complex construction to an operation on $D^{-}(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$.

Definition

Let F be a presheaf on $\mathrm{Cor}_{\mathrm{fin}}(k)$. Define the presheaf $\mathcal{C}_n^{\mathrm{Sus}}(F)$ by

$$\mathcal{C}_n^{\mathrm{Sus}}(F)(X) := F(X \times \Delta^n)$$

The *Suslin complex* $\mathcal{C}_*^{\mathrm{Sus}}(F)$ is the complex with differential

$$d_n := \sum_i (-1)^i \delta_i^* : \mathcal{C}_{n+1}^{\mathrm{Sus}}(F) \to \mathcal{C}_n^{\mathrm{Sus}}(F).$$

Remarks (1) If F is a sheaf with transfers on $\mathbf{Sm}/k$, then $\mathcal{C}_*^{\mathrm{Sus}}(F)$ is a complex of sheaves with transfers.

(2) The homology presheaves $h_i(F) := \mathcal{H}^{-i}(\mathcal{C}_*^{\mathrm{Sus}}(F))$ are homotopy invariant: Triangulating

$$\mathbb{A}^1 \times \Delta^n = \Delta^1 \times \Delta^n = \cup_{i=0}^n \Delta^{n+1}.$$

defines a chain homotopy of $\mathrm{id}_{\mathcal{C}_*^{\mathrm{Sus}}(F)(X\times\mathbb{A}^1)}$ with

$$\mathcal{C}_*^{\mathrm{Sus}}(F)(X \times \mathbb{A}^1) \xrightarrow{i_0^*} \mathcal{C}_*^{\mathrm{Sus}}(F)(X) \xrightarrow{p^*} \mathcal{C}_*^{\mathrm{Sus}}(F)(X \times \mathbb{A}^1),$$

so i_0^* is the homotopy inverse to p^*.

Thus, by Voevodsky's PST theorem, the associated Nisnevich sheaves $h_i^{\mathrm{Nis}}(F)$ are homotopy invariant. We thus have the functor

$$\mathcal{C}_*^{\mathrm{Sus}} : \mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)) \to DM_-^{\mathrm{eff}}(k).$$

Remark The Suslin complex $C_*^{\mathrm{Sus}}(X)$ is just $\mathcal{C}_*^{\mathrm{Sus}}(\mathbb{Z}^{tr}(X))(\mathrm{Spec}\, k)$.

We denote $\mathcal{C}_*^{\mathrm{Sus}}(\mathbb{Z}^{tr}(X))$ by $\mathcal{C}_*^{\mathrm{Sus}}(X)$.

Let $\mathcal{A}$ is the localizing subcategory of $D^-(\mathrm{Sh_{Nis}}(\mathrm{Cor_{fin}}(k)))$ generated by complexes

$$\mathbb{Z}^{tr}(X \times \mathbb{A}^1) \xrightarrow{p_1} \mathbb{Z}^{tr}(X); \quad X \in \mathbf{Sm}/k,$$

and let

$$Q_{\mathbb{A}^1} : D^-(\mathrm{Sh_{Nis}}(\mathrm{Cor_{fin}}(k))) \to D^-(\mathrm{Sh_{Nis}}(\mathrm{Cor_{fin}}(k)))/\mathcal{A}$$

be the quotient functor.

Since $\mathbb{Z}^{tr}(X) = \mathcal{C}_0^{\mathrm{Sus}}(X)$, we have the canonical map

$$\iota_X : \mathbb{Z}^{tr}(X) \to \mathcal{C}_*^{\mathrm{Sus}}(X)$$

This acts like an "injective resolution" of $\mathbb{Z}^{tr}(X)$, with respect to the localization $\mathbb{Q}_{\mathbb{A}^1}$.

Theorem

1. The functor

$$\mathcal{C}_*^{\mathsf{Sus}} : \mathsf{Sh}^{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)) \to DM_-^{\mathrm{eff}}(k).$$

extends to an exact functor

$$\mathbf{R}\mathcal{C}_*^{\mathsf{Sus}} : D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))) \to DM_-^{\mathrm{eff}}(k),$$

left adjoint to the inclusion $DM_-^{\mathrm{eff}}(k) \to D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))$.

2. $\mathbf{R}\mathcal{C}_*^{\mathsf{Sus}}$ *identifies* $DM_-^{\mathrm{eff}}(k)$ *with* $D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))/\mathcal{A}$

Theorem

There is a commutative diagram of exact tensor functors

$$\begin{tikzcd} K^b(\mathrm{Cor}_{\mathrm{fin}}(k)) \arrow[r, "\mathbb{Z}^{tr}"] \arrow[d] & D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))) \arrow[d, "\mathbf{R}C_*"] \\ DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \arrow[r, "i"'] & DM^{\mathrm{eff}}_{-}(k) \end{tikzcd}$$

such that

1. *i is a full embedding with dense image.*
2. $\mathbf{R}\mathcal{C}^{\mathrm{Sus}}_*(\mathbb{Z}^{tr}(X)) \cong \mathcal{C}^{\mathrm{Sus}}_*(X)$.

Explanation: Sending $X \in \mathbf{Sm}/k$ to $\mathbb{Z}^{tr}(X) \in \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$ extends to an additive functor

$$\mathbb{Z}^{tr} : \mathrm{Cor}_{\mathrm{fin}}(k) \to \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$$

and then to an exact functor

$$\mathbb{Z}^{tr} : K^b(\mathrm{Cor}_{\mathrm{fin}}(k)) \to K^b(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))) \to D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))).$$

One shows

1. Sending X to $\mathcal{C}_*^{\mathrm{Sus}}(X)$ sends the complexes

$$[X \times \mathbb{A}^1] \to [X]; \quad [U \cap V] \to [U] \oplus [V] \to [U \cup V]$$

to "zero". Thus i exists.

2. Using results of Ne'eman, one shows that i is a full embedding with dense image.

Corollary

For X and $Y \in \mathbf{Sm}/k$,

$$\mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}(m(Y), m(X)[n]) \cong \mathbb{H}^n(Y_{\mathrm{Nis}}, \mathcal{C}^{\mathrm{Sus}}_*(X)) \cong \mathbb{H}^n(Y_{\mathrm{Zar}}, \mathcal{C}^{\mathrm{Sus}}_*(X)).$$

Because:

$$\begin{aligned}\mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}&(m(Y), m(X)[n]) \\ &= \mathrm{Hom}_{DM^{\mathrm{eff}}_{-}(k)}(\mathcal{C}^{\mathrm{Sus}}_*(Y), \mathcal{C}^{\mathrm{Sus}}_*(X)[n]) \\ &= \mathrm{Hom}_{D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))}(\mathbb{Z}^{tr}(Y), \mathcal{C}^{\mathrm{Sus}}_*(X)[n]) \\ &= \mathbb{H}^n(Y_{\mathrm{Nis}}, \mathcal{C}^{\mathrm{Sus}}_*(X))\end{aligned}$$

plus the PST theorem: $\mathbb{H}^n(Y_{\mathrm{Nis}}, \mathcal{C}^{\mathrm{Sus}}_*(X)) = \mathbb{H}^n(Y_{\mathrm{Zar}}, \mathcal{C}^{\mathrm{Sus}}_*(X))$.

Taking $Y = \mathrm{Spec}\, k$, the corollary yields

$$H_n^{\mathrm{mot}}(X, \mathbb{Z}) = \mathrm{Hom}_{DM_{\mathrm{gm}}^{\mathrm{eff}}(k)}(\mathbb{Z}[n], m(X))$$
$$\cong H_n(\mathcal{C}_*^{\mathrm{Sus}}(X)(k)) = H_n(C_*^{\mathrm{Sus}}(X)) = H_n^{\mathrm{Sus}}(X, \mathbb{Z}).$$

Since $m(\mathbb{P}^q) = \oplus_{n=0}^q \mathbb{Z}(n)[2n]$ we have

$$\mathcal{C}_*^{\mathrm{Sus}}(\mathbb{Z}^{tr}(q)[2q])(Y) \cong C_*^{\mathrm{Sus}}(\mathbb{P}^q/\mathbb{P}^{q-1})(Y) = \Gamma_{FS}(q)(Y)[2q]$$

Applying the corollary with $X = \mathbb{Z}^{tr}(q)$ gives

$$\begin{aligned} H^p_{\mathrm{mot}}(Y, \mathbb{Z}(q)) &:= \mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}(m(Y), \mathbb{Z}(q)[p]) \\ &\cong \mathbb{H}^p(Y_{\mathrm{Zar}}, \mathcal{C}_*^{\mathrm{Sus}}(\mathbb{Z}(q))) = \mathbb{H}^p(Y_{\mathrm{Zar}}, \Gamma_{FS}(q)) \\ &\cong H^p(\Gamma_{FS}(q)(Y)) = H^p(Y, \mathbb{Z}(q)). \end{aligned}$$

Thus, we have identified motivic (co)homology with universal (co)homology.

Tate motives

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Outline

- Duality
- Tate motives
- Differential graded algebras and Hopf algebras
- Bloch's cycle algebra
- Examples: polylog.

Duality

Duality

Let M be an object in a tensor category $\mathcal{C}$. A dual of M is a triple

$$(M^\vee, i : \mathbb{1} \to M \otimes M^\vee, tr : M^\vee \otimes M \to \mathbb{1})$$

such that the composition

$$M \cong \mathbb{1} \otimes M \xrightarrow{i \otimes \mathrm{id}} M \otimes M^\vee \otimes M \xrightarrow{\mathrm{id} \otimes tr} M \otimes \mathbb{1} \cong M$$

is the identity.

One can easily show:
if M has a dual $(M^\vee, i, tr)$, then for each $A, B \in \mathcal{C}$ there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(A \otimes M, B) \cong \mathrm{Hom}_{\mathcal{C}}(A, B \otimes M^\vee)$$

$M^\vee$ has dual $(M, i^t : \mathbb{1} \to M^\vee \otimes M, tr^t : M \otimes M^\vee \to \mathbb{1})$.

Duality

A tensor category such that each object admits a dual is called rigid: a rigid tensor category has a canonical duality involution

$$(-)^\vee : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$$

since two duals are canonically isomorphic.

If $\mathcal{C}$ is a rigid triangulated tensor category the duality involution is automatically exact.

Suppose $\mathcal{C}$ is a triangulated tensor category, containing a collection of objects $\mathcal{S}$ such that

1. each $M \in \mathcal{S}$ has a dual
2. The smallest full triangulated subcategory of $\mathcal{C}$ containing $\mathcal{S}$ is $\mathcal{C}$.

Then $\mathcal{C}$ is rigid.

Duality for motives

$M^{\rm eff}_{\rm rat}(k)$ is not a rigid tensor category; we need to invert the Lefschetz motive.

Definition

$M_\sim(k) := M^{\rm eff}_\sim(k)[\otimes\mathbb{L}^{-1}]$, that is

$M_\sim(k)$ has objects $M(p)$, $M \in M^{\rm eff}_\sim(k)$, $p \in \mathbb{Z}$

$$\mathrm{Hom}_{M_\sim(k)}(M(p), N(q)) := \varinjlim_n \mathrm{Hom}_{M^{\rm eff}_\sim(k)}(M\otimes\mathbb{L}^{\otimes n+p}, N\otimes\mathbb{L}^{\otimes n+q}).$$

Sending M to $M(0)$ defines the functor of tensor categories $M^{\rm eff}_\sim(k) \to M_\sim(k)$.

Note. 1. $M^{\rm eff}_\sim(k) \to M_\sim(k)$ is fully faithful.

2. $M(n)\otimes\mathbb{L} \cong M(n+1)$.

Duality for motives

Recall that

$$\mathrm{CH}^q(Y)_{\mathbb{Q}} \cong \mathrm{Hom}_{M^{\mathrm{eff}}_{\mathrm{rat}}(k)}(\mathbb{L}^q, \mathfrak{h}(Y)).$$

Thus, the diagonal $\Delta_X \subset X \times X$ corresponds to $\delta_X : \mathbb{L}^{d_X} \to \mathfrak{h}(X \times X)$ in $M^{\mathrm{eff}}_{\mathrm{rat}}(k)$, i.e.

$$i_X : \mathbb{1} \to \mathfrak{h}(X) \otimes \mathfrak{h}(X)(-d_X)$$

in $M_{\mathrm{rat}}(k)$. Similarly

$$\mathrm{CH}_q(Y) \cong \mathrm{Hom}_{M^{\mathrm{eff}}_{\mathrm{rat}}(k)}(\mathfrak{h}(Y), \mathbb{L}^q)$$

so Δ_X also gives us

$$tr_X : \mathfrak{h}(X) \otimes \mathfrak{h}(X)(-d_X) \to \mathbb{1}$$

One computes: $(\mathfrak{h}(X)(-d_X), i_X, tr_X)$ is a dual of $\mathfrak{h}(X)$ in $M_\sim(k)$, hence

Proposition

$M_\sim(k)$ is a rigid tensor category.

Definition

$DM_{\mathrm{gm}}(k) := DM^{\mathrm{eff}}_{\mathrm{gm}}(k)[\otimes\mathbb{Z}(1)^{-1}]$; $DM_{\mathrm{gm}}(k)$ is a triangulated tensor category and $DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \to DM_{\mathrm{gm}}(k)$ is an exact tensor functor.

Theorem (Friedlander, Suslin, Voevodsky)

Suppose k has characteristic zero. Then

1. $DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \to DM_{\mathrm{gm}}(k)$ *is fully faithful.*
2. $DM_{\mathrm{gm}}(k)$ *is generated by objects $m(X)(n)$ (and taking summands), $X \in \mathbf{SmProj}/k$, $n \in \mathbb{Z}$.*
3. $DM_{\mathrm{gm}}(k)$ *is rigid; for $X \in \mathbf{SmProj}/k$, the dual of $m(X)(n)$ is $m(X)(-d_X - n)$.*

Tate motives

Definition

The triangulated category of Tate motives, $DMT(k) \subset \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$, is the full triangulated subcategory of $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ generated by objects $\mathbb{Q}(p)$, $p \in \mathbb{Z}$.

Note. $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}}(\mathbb{Q}, \mathbb{Q}(m)[n]) = H^n(k, \mathbb{Q}(m)) \cong K_{2m-n}(k)^{(n)}$, so Tate motives contain a lot of information.

Let $DMT(k)_{\leq n}$ be the full triangulated subcategory generated by the $\mathbb{Q}(p)$, $p \geq -n$. $DMT(k)_{\geq n}$ is the full triangulated subcategory generated by the $\mathbb{Q}(p)$, $p \leq n$.

Proposition

The inclusion $i_n : DMT(k)_{\leq n} \to DMT(k)$ *admits an exact tensor right adjoint* $r_n : DMT(k) \to DMT(k)_{\leq n}$.

Dually, the inclusion $i'_n : DMT(k)_{\geq n} \to DMT(k)$ *admits an exact tensor left adjoint* $l_n : DMT(k) \to DMT(k)_{\geq n}$.

Define the weight truncations

$$W_{\leq n}, W_{\geq n} : DMT(k) \to DMT(k)$$

as $W_{\leq n} := i_n \circ r_n$, $W_{\geq n} := i'_n \circ l_n$.

This gives the natural distinguished triangle

$$W_{\leq n}M \to M \to W_{\geq n+1}M \to W_{\leq n}M[1].$$

and tower

$$0 = W_{\leq N-1}M \to W_{\leq N}M \to \ldots \to W_{\leq N'-1}M \to W_{\leq N'}M = M.$$

Define $\mathrm{gr}_n^W M := W_{\le n} W_{\ge n} M$.

Note that $\mathrm{gr}_n^W M$ is in the triangulated subcategory $W_{=n}DMT(k)$ generated by $\mathbb{Q}(-n)$. In fact

$$W_{=n}DMT(k) \cong D^b(\mathbb{Q}\text{-Vec})$$

since

$$\mathrm{Hom}_{DTM(k)}(\mathbb{Q}(-n), \mathbb{Q}(-n)[m]) = H^m(k, \mathbb{Q}(0)) = \begin{cases} 0 & \text{if } m \neq 0 \\ \mathbb{Q} & \text{if } m = 0 \end{cases}$$

Thus, it makes sense to take $H^p(\mathrm{gr}_n^W M)$.

Definition

Let $MT(k)$ be the full subcategory of $DMT(k)$ with objects those M such that

$$H^p(\mathrm{gr}_n^W M) = 0$$

for $p \neq 0$ and for all $n \in \mathbb{Z}$.

Theorem

Suppose that k satisfies the $\mathbb{Q}$-Beilinson-Soulé vanishing conjectures:

$$H^p(k, \mathbb{Q}(q)) = 0$$

for $q > 0$, $p \leq 0$. Then $MT(k)$ is an abelian rigid tensor category, where a sequence $0 \to A \to B \to C \to 0$ is exact if and only if $A \to B \to C$ extends to a distinguished triangle in $DMT(k)$. The tensor structure is induced from $DMT(k)$.

In addition:

1. $MT(k)$ is closed under extensions in $DMT(k)$: if $A \to B \to C \to A[1]$ is a distinguished triangle in $DMT(k)$ with $A, C \in MT(k)$, then B is in $MT(k)$.
2. $MT(k)$ contains the Tate objects $\mathbb{Q}(n)$, $n \in \mathbb{Z}$, and is the smallest additive subcategory of $DMT(k)$ containing these and closed under extension.
3. The weight filtration on $DMT(k)$ induces a exact weight filtration on $MT(k)$, with

$$\mathrm{gr}_n^W M \cong \mathbb{Q}(-n)^{r_n}$$

Finally: send $M \in MT(k)$ to $\oplus_n \mathrm{gr}_n^W M \in \mathbb{Q}$-Vec defines an exact faithful tensor functor

$$\omega_W : MT(k) \to \mathbb{Q}\text{-Vec}$$

i.e. $MT(k)$ is a Tannakian category.

Theorem

Suppose that k satisfies the $\mathbb{Q}$-Beilinson-Soulé vanishing conjectures. Let $\mathcal{G}(k) := \mathrm{Aut}^{\otimes}(\omega_W)$. Then

1. *$MT(k)$ equivalent to the category of finite dimensional $\mathbb{Q}$-representations of $\mathcal{G}(k)$.*
2. *There is a pro-unipotent group scheme $\mathcal{U}(k)$ over $\mathbb{Q}$ with $\mathcal{G}(k) \cong \mathcal{U}(k) \ltimes \mathbb{G}_m$*

(1) is Tannakian duality.

$\mathcal{G}(k) \to \mathbb{G}_m$ is dual to $\mathrm{Gr}\mathbb{Q}$-Vec $\to MT(k)$ sending $\oplus_n V_n$ to $\oplus_n V_n \otimes \mathbb{Q}(-n)$.

$\mathbb{G}_m \to \mathcal{G}(k)$ is dual to $\oplus_n \mathrm{gr}_n^W : MT(k) \to \mathrm{Gr}\mathbb{Q}$-Vec.

$\mathcal{U}(k) := \ker[\mathcal{G}(k) \to \mathbb{G}_m]$ is uni-potent because $\mathcal{G}(k)$ preserves the weight filtration W_*M for all M and $\mathcal{U}(k)$ acts trivially on the associated graded $\oplus_n \mathrm{gr}_n^W M$.

Let $\mathcal{L}(k)$ be the pro-nilpotent Lie algebra of $\mathcal{U}(k)$. The $\mathbb{G}_m$-action gives $\mathcal{L}(k)$ a (negative) grading and $MT(k)$ is equivalent to the category of finite dimensional graded $\mathbb{Q}$-representations of $\mathcal{L}(k)$.

Let k be a number field. Borel's theorem tells us that k satisfies B-S vanishing.

In fact $H^p(k,\mathbb{Q}(n)) = 0$ for $p \neq 1$ ($n \neq 0$). This implies

Proposition

Let k be a number field. Then $\mathcal{L}(k)$ is the free graded pro-nilpotent Lie algebra on $\oplus_{n\geq 1} H^1(k,\mathbb{Q}(n))^\vee$, with $H^1(k,\mathbb{Q}(n))^\vee$ in degree $-n$.

Note. $H^1(k,\mathbb{Q}(n)) = \mathbb{Q}^{d_n}$ with $d_n = r_1 + r_2$ ($n > 1$ odd) or r_2 ($n > 1$ even).

$H^1(k,\mathbb{Q}(1)) = \oplus_{\mathfrak{p}\subset\mathcal{O}_k \text{ prime}}\mathbb{Q}$.

Let $\mathcal{L}(k)_{\mathbb{Z}} := \mathcal{L}(k)/{<}\mathcal{L}(k)^{(-1)}{>}$, $MT(k)_{\mathbb{Z}} := \mathrm{GrRep}\mathcal{L}(k)_{\mathbb{Z}}$.

Example. $\mathcal{L}(\mathbb{Q}) = \mathrm{Lie}_{\mathbb{Q}}<[2],[3],[5],\ldots,s_3,s_5,\ldots>$, with $[p]$ in degree -1 and with s_{2n+1} in degree $-(2n+1)$.

$\mathcal{L}(\mathbb{Q})_{\mathbb{Z}} = \mathrm{Lie}_{\mathbb{Q}}<s_3,s_5,\ldots>$, with s_{2n+1} in degree $-(2n+1)$.

$$\mathrm{Lie}_{\mathbb{Q}}<[2],[3],[5],\ldots,s_3,s_5,\ldots>) \twoheadrightarrow \mathrm{Lie}_{\mathbb{Q}}<s_3,s_5,\ldots>.$$

$$\begin{aligned} MT(\mathbb{Q}) &= \mathrm{GrRep}(\mathrm{Lie}_{\mathbb{Q}}<[2],[3],[5],\ldots,s_3,s_5,\ldots>) \\ &\supset MT(\mathbb{Q})_{\mathbb{Z}} = \mathrm{GrRep}(\mathrm{Lie}_{\mathbb{Q}}<s_3,s_5,\ldots>). \end{aligned}$$

A $\mathbb{Q}$-mixed Tate Hodge structure (V, W, F) consists of

1. a finite dimenisonal $\mathbb{Q}$ vector space with a finite exhaustive increasing filtration W_*
2. a finite exhaustive decreasing filtration F^* on $V_\mathbb{C}$

such that, for each n, the filtration induced by F^* on $\mathrm{gr}_n^W V_\mathbb{C}$ satisfies

$$F^m(\mathrm{gr}^n_W V_\mathbb{C}) = \begin{cases} 0 & \text{for } m > n \\ \mathrm{gr}^n_W V_\mathbb{C} & \text{for } m \leq n. \end{cases}$$

Note. The indexing for W_* disagrees with the usual conventions by a factor of 2.

If we choose a grading $V_{\mathbb{C}} = \oplus_{n=A}^{B} V_n$ on $V_{\mathbb{C}}$ so that $F^pV = \oplus_{n\geq -p} V_n$, then expressing W_* in terms of the chosen basis gives the period matrix $P(V, W, F)$, with basis elements for F^* the columns of P.

Choosing bases for W_* appropriately, we can assume that $P(V, W, F)$ is block lower triangular, with the diagonal block corresponding to $\mathrm{gr}^n_W V$ equal to $(2\pi i)^{-n}$ times an identity matrix. The remaining entries of P are the periods of (V, W, F).

Tate motives

Hodge realization

$$
\begin{array}{cc}
 & \qquad\qquad\qquad\qquad\qquad\qquad | \leftarrow \qquad W_m \qquad \rightarrow | \\
\begin{array}{c} - \\ \uparrow \\ F^n \\ \downarrow \\ - \end{array} &
\begin{pmatrix}
(2\pi i)^{-B} I_{n_B} & 0 & \dots & & 0 \\
* & (2\pi i)^{1-B} I_{n_{B-1}} & 0 & \dots & 0 \\
* & * & (2\pi i)^{2-B} I_{n_{B-2}} & \dots & 0 \\
* & * & * & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
* & * & * & \dots & (2\pi i)^{-A} I_{n_A}
\end{pmatrix}
\end{array}
$$

There is a functor from $DMT(k)$ to the derived category of $\mathbb{Q}$-mixed Tate Hodge structures.

Thus, each $M \in MT(k)$ has a period matrix: how to calculate it?

Differential graded algebras and Hopf algebras

Modules over a cdga

Let (A, d) be a commutative differential graded algebra over $\mathbb{Q}$:

- $A = \oplus_n A^n$ as a graded-commutative $\mathbb{Q}$-algebra
- d has degree $+1$, $d^2 = 0$ and $d(xy) = dx \cdot y + (-1)^{\deg x} x \cdot dy$.

A dg module over A, (M, d) is

- $M = \oplus_n M^n$ a graded A-module
- d has degree $+1$, $d^2 = 0$ and
  $d_M(xm) = d_A x \cdot +(-1)^{\deg x} x \cdot d_M m$.

This gives the category $\mathbf{d.g.Mod}_A$.

Let $f : M \to N$ be a map of dg A-modules. Then $\mathrm{cone}(f)$ is a dg A-module and the cone sequence

$$M \xrightarrow{f} N \to \mathrm{cone}(f) \to M[1]$$

is a sequence in $\mathbf{d.g.Mod}_A$.

Modules over a cdga

Let $D\mathbf{d.g.Mod}_A := \mathbf{d.g.Mod}_A[\text{q-iso}^{-1}]$: The derived category of dg A-modules.

$D\mathbf{d.g.Mod}_A$ is a triangulated category, with distinguished triangles those triangles isomorphic to a cone sequence.

Define a derived tensor product $\otimes_A^L$: each M admits a quasi-isomorphism

$$F(M) \to M$$

with $F(M)$ a free A-module. Set

$$M \otimes_A^L N := F(M) \otimes_A F(N).$$

This makes $D\mathbf{d.g.Mod}_A$ a triangulated tensor category.

Adams grading

An Adams graded cdga $A(*)$ is a cdga with an additional grading:

$$A(*) = \mathbb{Q} \cdot \mathrm{id} \oplus \oplus_{q \geq 1} A(q)$$

such that $d(A(q)) \subset A(q)$ and product is bi-graded. The category of Adams graded dg $A(*)$-modules is defined similarly $\rightsquigarrow$ the triangulated tensor category $D\mathbf{d.g.Mod}_{A(*)}$

$|m|$ = Adams degree, $\deg m$ = cohomological degree.

Definition

An Adams graded cdga $A(*)$ is c-connected if $H^n(A(q)) = 0$ for $n < 0$ and $q > 0$. $A(*)$ is connected if $A^n(q) = 0$ for $n \leq 0$, $q > 0$.

A morphism of Adams graded cdgas $\phi : A(*) \to B(*)$ induces an exact functor

$$\phi_* : D\mathbf{d.g.Mod}_{A(*)} \to D\mathbf{d.g.Mod}_{B(*)}$$

by $\phi_*(M) = M \otimes^L_{A(*)} B(*)$.

Finite modules

Definition

A dg module $(M(*), d_M)$ over $A(*)$ is cell finite if $M(*)$ is a free, finitely generated bi-graded $A(*)$ module. $(N(*), d_N)$ is a finite dg $A(*)$ module if $N(*) \cong M(*)$ in $D\mathbf{d.g.Mod}_{A(*)}$ for some cell finite $M(*)$.

$D\mathbf{d.g.Mod}^f_{A(*)}$ is the full subcategory of $D\mathbf{d.g.Mod}_{A(*)}$ with objects the finite dg $A(*)$ modules, a triangulated tensor subcategory.

Note. If $M(*)$ is a cell finite dg $A(*)$ module, $N(*)$ any dg $A(*)$ module, then

$$\mathrm{Hom}_{D\mathbf{d.g.Mod}_{A(*)}}(M, N) = \mathrm{Hom}_{\mathbf{d.g.Mod}_{A(*)}}(M, N)/\text{homotopy}.$$

Tate modules

Let $\mathbb{Q}_A(n) := A(*) \cdot e$, with $|e| = -n$, $\deg e = 0$ and $de = 0$.

$$\begin{aligned}\mathrm{Hom}_{D\mathbf{d.g.Mod}_{A(*)}}&(\mathbb{Q}_A(n), \mathbb{Q}_A(n')[m]) \\ &= \mathrm{Hom}_{\mathbf{d.g.Mod}_{A(*)}}(\mathbb{Q}_A(n), \mathbb{Q}_A(n')[m])/\text{homotopy} \\ &= H^m(A(n'-n)).\end{aligned}$$

Since $A(0) = \mathbb{Q} \cdot \mathrm{id}$

$$\mathrm{Hom}_{D\mathbf{d.g.Mod}_{A(*)}}(\mathbb{Q}_A(n), \mathbb{Q}_A(n)[m]) = \begin{cases} 0 & \text{for } m \neq 0 \\ \mathbb{Q} \cdot \mathrm{id} & \text{for } m = 0. \end{cases}$$

Since $A(q) = 0$ for $q < 0$,

$$\mathrm{Hom}_{D\mathbf{d.g.Mod}_{A(*)}}(\mathbb{Q}_A(n), \mathbb{Q}_A(n')[m]) = 0$$

for $n > n'$

Weight filtration

Let $M(*)$ be a cell finite dg $A(*)$ module, with basis $\{e_\alpha\}$. Write

$$de_\alpha = \sum_\beta a_{\alpha\beta} e_\beta; \quad a_{\alpha\beta} \in A(*)^*.$$

Then $|e_\alpha| = |de_\alpha| = |a_{\alpha\beta} e_\beta| = |a_{\alpha\beta}| + |e_\beta|$. As $|a_{\alpha\beta}| \geq 0$, we have

$$|e_\beta| \leq |e_\alpha|$$

if $a_{\alpha\beta} \neq 0$.
Let

$$W_{\leq n} M(*) := \oplus_{\alpha, |e_\alpha| \leq n} A(*) e_\alpha \subset M(*),$$

a dg $A(*)$-submodule of $M(*)$, and

$$W_{\geq n} M(*) := \oplus_{\alpha, |e_\alpha| \geq n} A(*) e_\alpha,$$

a dg $A(*)$-quotient module of $M(*)$.

Weight filtration

The operations $W_{\leq n}$, $W_{\geq n}$ pass to $\mathcal{H}^f_{A(*)}$, we have the functorial distinguished triangle

$$W_{\leq n}M \to M \to W_{\geq n+1}M \to W_{\leq n}M[1]$$

and the (finite) tower

$$0 = W_{\leq N-1}M \to W_{\leq N}M \to \ldots \to W_{\leq N'-1}M \to W_{\leq N'}M = M$$

in $\mathcal{H}^f_{A(*)}$.

$\mathrm{gr}^W_n M := W_{\leq n}W_{\geq n}M \cong \oplus_i \mathbb{Q}_A(-n)^{r_i}[m_i]$:

$$W_{=n}D\mathbf{d.\,g.\,Mod}_{A(*)} \cong D^b(\mathbb{Q}\text{-Vec}).$$

Abelian subcategory

Definition

Let $\epsilon : A(*) \to \mathbb{Q}$ be the augmentation, so we have

$$\epsilon_* : D\mathbf{d.g.Mod}^f_{A(*)} \to D\mathbf{d.g.Mod}^f_{\mathbb{Q}} \cong \oplus_{n\in\mathbb{Z}} D^b(\mathbb{Q}\text{-Vec})$$

Let $\mathcal{H}^f_{A(*)} \subset D\mathbf{d.g.Mod}^f_{A(*)}$ be the full subcategory with objects those M such that

$$H^p(\epsilon_*(M)) = 0$$

for $p \neq 0$.

Note. $\epsilon_* = \mathrm{gr}^W_*$.

Abelian subcategory

Theorem

If $A(*)$ is c-connected, then $\mathcal{H}^f_{A(*)}$ is a rigid tensor abelian category, closed under extensions in $D\mathbf{d.g.Mod}^f_{A(*)}$. The weight filtration on $D\mathbf{d.g.Mod}^f_{A(*)}$ induces an exact weight filtration on $\mathcal{H}^f_{A(*)}$, with graded pieces finite dimensional $\mathbb{Q}$-vector spaces.

$\mathcal{H}^f_{A(*)}$ contains all Tate modules $\mathbb{Q}_A(n)$ and is the smallest additive subcategory of $D\mathbf{d.g.Mod}^f_{A(*)}$ containing all $\mathbb{Q}_A(n)$ and closed under extensions in $D\mathbf{d.g.Mod}^f_{A(*)}$.

ϵ_* induces the fiber functor

$$\epsilon_* : \mathcal{H}^f_{A(*)} \to \mathbb{Q}\text{-Vec}.$$

making $\mathcal{H}^f_{A(*)}$ a Tannakian category.

Abelian subcategory

Note.

The Tannaka group $\mathcal{G}_A$ of $\mathcal{H}^f_{A(*)}$ is a semi-direct product

$$\mathcal{G}_A = \mathcal{U}_A \ltimes \mathbb{G}_m$$

with $\mathcal{U}_A$ pro-nilpotent. Let $\mathcal{L}_A$ be the pro-nilpotent graded Lie algebra of $\mathcal{U}_A$.
We can write down $\mathcal{L}_A$ in two ways:

- Using the 1-minimal model of $A(*)$
- Using the Hopf algebra $\chi_{A(*)} := H^0(\bar{B}A(*))$.

The bar construction

Form the (double) complex

$$\tilde{B}A(*) := A(*)^{\otimes n+2} \xrightarrow{\partial_{n-1}} A(*)^{\otimes n+1} \xrightarrow{\partial_{n-2}} \ldots A(*)^{\otimes 3} \xrightarrow{\partial_0} A(*)^{\otimes 2}$$

with

$$\partial_{n-1}(a_0 \otimes a_1 \otimes \ldots \otimes a_{n+1}) = \sum_{i=0}^{n} a_0 \otimes \ldots a_i a_{i+1} \otimes \ldots \otimes a_{n+1}$$

∂_{n-1} is an $A(*) \otimes A(*)$ module map: let

$$\bar{B}A(*) := \mathrm{Tot}[\tilde{B}A(*) \otimes_{A(*)\otimes A(*)} \mathbb{Q}].$$

$\bar{B}A(*)$ is an Adams graded differential Hopf algebra, giving us the graded (commutative) Hopf algebra $\chi_{A(*)} := H^0(\bar{B}A(*))$

The bar construction

Theorem

$\mathcal{U}_A \cong \mathrm{Spec}\,(\chi_A)$ and $\mathcal{L}_A \cong [\mathfrak{m}_{\chi_A}/\mathfrak{m}^2_{\chi_A}]^\vee$.

In particular $\mathcal{H}^f_{A(*)}$ is equivalent to the category of graded χ_A co-modules which are finite dimensional as $\mathbb{Q}$-vector spaces.

Bloch's cycle algebra

We modify Bloch's cycle complex to form a cdga whose dg modules are Tate motives.

The cubical complex

We want to construct a strictly commutative dg algebra, so it's better to use cubes instead of simplices.

$\square^1 := (\mathbb{A}^1, 0, 1)$, $\square^n := (\mathbb{A}^1, 0, 1)^n$. $\square^n$ has faces $t_{i_1} = \epsilon_1 \ldots t_{t_r} = \epsilon_r$.

Definition

$\tilde{z}^q(k, n)^c :=$ the free abelian group on the irreducible codimension q closed $W \subset \square^n$ which intersect each face in codimension q.

$z^q(k, n)^c := \tilde{z}^q(k, n)/\sum_{i=1}^n p_i^*(\tilde{z}^q(k, n-1))$, $p_i : \square^n \to \square^{n-1}$ the projections.

$z^q(k, *)^c$ is a complex with differential the alternating sum of restrictions to faces:

$$d_{n-1} := \sum_{j=1}^n (-1)^j i_{t_j=1}^* - \sum_{j=1}^n (-1)^j i_{t_j=0}^*.$$

The cubical complex

External product gives a well-defined product

$$z^q(k,*)^c \otimes z^{q'}(k,*)^c \to z^{q+q'}(k,*)^c$$

which we make graded-commutative by taking $\mathbb{Q}$-coefficients and taking the alternating projection.

Definition

S_n acts on $\square^n$ by permuting the coordinates. Let $\Pi_{\mathrm{Alt}} \in \mathbb{Q}[S_n]$ be the alternating idempotent $(1/n!)\sum_\sigma \mathrm{sgn}(\sigma)\sigma$. Let

$$z^q(k,n)^{\mathrm{Alt}} := \Pi_n^{\mathrm{Alt}}(z^q(k,n)^c_{\mathbb{Q}}).$$

This gives us the complex $z^q(k,*)^{\mathrm{Alt}}$.

Bloch's cycle algebra is

$$\mathcal{N}_k := \mathbb{Q} \oplus \oplus_{q\geq 1} z^q(k, 2q-*)^{\mathrm{Alt}}$$

The cycle algebra

Proposition

1. $\mathcal{N}_k$ *is an Adams graded cdga over* $\mathbb{Q}$

2. $\mathcal{N}_k$ *is c-connected iff* k *satisfies the* $\mathbb{Q}$*-Beilinson-Soule' vanishing conjectures*

In fact $H^p(\mathcal{N}_k(q)) = H^p(k, \mathbb{Q}(q))$.

Theorem (Spitzweck)

Let k be a field.

1. There is a natural equivalence of triangulated tensor categories

$$D\mathbf{d.g.Mod}^f_{\mathcal{N}_k} \sim DMT(k)$$

compatible with the weight filtrations.

2. If k satisfies the $\mathbb{Q}$-Beilinson-Soule' vanishing conjectures, then the equivalence in (1) induces an equivalence of (filtered) Tannakian categories

$$\mathcal{H}^f_{\mathcal{N}_k} \sim MT(k)$$

and $\mathcal{U}(k) \cong \mathrm{Spec}\,(\chi_{\mathcal{N}_k})$.

One writes down a tilting module $\mathcal{N}^{\mathsf{mot}}(*)$ in $\mathrm{Gr}\, C^{-}(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))_{\mathbb{Q}}$:

- $\mathcal{N}^{\mathsf{mot}}(q) \cong \mathbb{Q}(q)$ in $DM^{\mathrm{eff}}_{-}(k)$
- $\mathcal{N}^{\mathsf{mot}}(*)$ is a dg $\mathcal{N}_k$ module

Sending a finite cell module $M \in \mathbf{d.g.Mod}_{\mathcal{N}_k}$ to $\mathcal{N}^{\mathsf{mot}}(*) \otimes_{\mathcal{N}_k} M$ defines the functor

$$\phi : D\mathbf{d.g.Mod}^{f}_{\mathcal{N}_k} \to DMT(k).$$

By calculation, the Hom's agree on Tate objects $\rightsquigarrow$ ϕ is an equivalence.

The cycle algebra and Tate motives

We can use this result to

- construct interesting Tate motives
- compute the Hodge realization of these Tate motives

The polylog motive

The Kummer motive

As a warm-up, we construct the "Kummer motive" associated to a unit $t \in k^\times = H^1(k, \mathbb{Z}(1))$.

Lift t to an element $\tilde{t} \in \mathcal{N}_k^1(1)$ with $d\tilde{t} = 0$. Let $\log(t)$ be the dg $\mathcal{N}_k$ module with basis e_0, e_1, $|e_1| = -1$, $|e_0| = 0$, $\deg e_i = 0$ and

$$de_1 = 0,\ de_0 = \tilde{t} \cdot e_1$$

We have the exact sequence of dg $\mathcal{N}_k$ modules

$$0 \to \mathbb{Q}(1) \to \log^{\mathrm{mot}}(t) \to \mathbb{Q}(0) \to 0$$

If k satisfies the B-S vanishing conjectures, this is a short exact sequence in $\mathcal{H}^f_{\mathcal{N}_k}$

The motive of a unit

The same construction applied to a class $\alpha \in H^1(k, \mathbb{Q}(n))$ gives us exact sequence of dg $\mathcal{N}_k$ modules

$$0 \to \mathbb{Q}(n) \to \log_n^{\mathrm{mot}}(\alpha) \to \mathbb{Q}(0) \to 0$$

Note that (assume $A(*)$ c-connected)

$$\mathrm{Ext}^1_{\mathcal{H}^f_{A(*)}}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong H^1(A(n))$$

via a similar construction.

The Hodge realization of $\log_1$

We work over $\mathbb{A}^1 - \{0\}$ with canonical unit t.
We have the local system given by the trivial rank two vector bundle with basis e_0, e_1 and connection

$$\nabla e_1 = 0$$
$$\nabla e_0 = -\frac{dt}{t} e_1$$

The flat sections are $f(t) = A(e_0 + \log(t)e_1) + Be_1$.
This gives us the variation of MHS over $\mathbb{A}^1 - \{0\}$

$$t \mapsto \begin{pmatrix} 1 & 0 \\ \log(t) & 2\pi i \end{pmatrix}$$

This fits into the exact sequence

$$0 \to \mathbb{Q}(1) \to \log(t) \to \mathbb{Q}(0) \to 0$$

Evaluating at $t = a$ gives the Hodge realization of $\log^{\mathrm{mot}}(a)$.

To make the formulas simpler, we identify $(\square^1, 0, 1) \cong (\mathbb{P}^1 - 1, 0, \infty)$, and express our formulas using the standard coordinate x on $\mathbb{P}^1$ as (rational) coordinate for $\square^1$.

Definition

Let $\hat{\rho}_n$ be the cycle on $\mathbb{A}^1 \times \square^{2n-1}$ given in parametric form as the locus (in parameters $t, x_1, \ldots, x_{n-1}$)

$$(t, x_1, \ldots, x_{n-1}, 1 - x_1, 1 - \frac{x_2}{x_1}, \ldots, 1 - \frac{x_{n-1}}{x_{n-2}}, 1 - \frac{t}{x_{n-1}}),$$

and let

$$\rho_n := (-1)^{n(n-1)/2} \Pi_{\mathrm{Alt}}(\hat{\rho}_n) \in \mathcal{N}_{\mathbb{A}^1_{\mathbb{Q}}}(n)^1$$

Let $[1 - t] = \rho_1 := \mathrm{locus}(t, 1 - t)$, $[t] := \mathrm{locus}(t, t)$.

Let $\rho_n(a) \in \mathcal{N}_{k(a)}(n)^1$ be the restriction of ρ_n to $a \times \square^n$. One computes:

$$d\rho_n = \rho_{n-1} \cdot [t]$$

for $n \geq 2$ and $d[t] = d[1-t] = 0$.
Since $[t] \rightsquigarrow \emptyset$ as $t \to 1$, we have

$$d\rho_n(1) = 0$$

giving us a class $\rho_n(1) \in H^1(\mathbb{Q}, \mathbb{Q}(n))$.

The polylog motive

Let Poly_n be the dg $\mathcal{N}_{\mathbb{A}^1_{\mathbb{Q}}}$-module with basis $e_0, \ldots, e_n$, $|e_i| = -i$, $\deg e_i = 0$ and

$$
\begin{aligned}
de_n &= 0 \\
de_i &= [t]e_{i+1} \quad \text{ for } i = 1, \ldots n-1 \\
de_0 &= [1-t]e_1 + \rho_2 e_2 + \ldots + \rho_n e_n
\end{aligned}
$$

Note that $\mathrm{gr}^W_i \mathrm{Poly}_n(t) = \mathbb{Q}(i)$ for $i = 0, \ldots, n$, and the first extension data

$$0 \to \mathrm{gr}^W_{-i-1}\mathrm{Poly}_n(t) \to W_{\leq -i}W_{\geq -i-2}\mathrm{Poly}_n(t) \to \mathrm{gr}^W_{-i}\mathrm{Poly}_n(t) \to 0$$

is $\log(t) \otimes \mathbb{Q}(i)$ for $i = 1, \ldots, n-1$ and $\log(1-t)$ for $i = 0$.

So, for each a $\mathrm{Poly}_n(a)$ gives an object in $MT(k(a))$.

The Polylog local system

We transform Poly_n to a flat connection on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

If $\mathcal{L}$ is the corresponding flat connection, then $\mathcal{L}$ is uni-potent with first extension data

$$[0 \to \mathbb{Q}(1) \to \log(t) \to \mathbb{Q} \to 0] \otimes \mathbb{Q}(i)$$

except for $i = 0$, where we have $\log(1-t)$. This characterizes $\mathcal{L}$, giving us the rank $n+1$ flat connection with basis $e_0, \ldots, e_n$ and with

$$\begin{aligned}
\nabla e_n &= 0 \\
\nabla e_i &= -\frac{dt}{t} e_{i+1} \text{ for } i = 1, 2, \ldots, n-1 \\
\nabla e_0 &= -\frac{dt}{1-t} e_1
\end{aligned}$$

The Polylog local system

The flat sections are given by the columns of the matrix

$$\begin{pmatrix} 1 & 0 & \dots & & \\ Li_1(t) & 2\pi i & 0 & \dots & \\ Li_2(t) & 2\pi i \log(t) & (2\pi i)^2 & \dots & \\ Li_3(t) & 2\pi i \frac{1}{2}\log^2(t) & (2\pi i)^2 \log(t) & \dots & \\ \vdots & \vdots & \vdots & \vdots & \\ Li_n(t) & 2\pi i \frac{1}{(n-1)!}\log^{n-1}(t) & (2\pi i)^2 \frac{1}{(n-2)!}\log^{n-2}(t) & \dots & (2\pi i)^n \end{pmatrix}$$

$$Li_n(t) := \sum_{i=1}^{\infty} \frac{t^i}{i^n}.$$

This gives us a variation $\mathcal{V}$ of MHS. The underlying vector bundle is the trivial bundle with basis $e_0, \dots, e_n$, the F-filtration is

$$F^{-m}\mathcal{V} = \text{ span of } e_0, \dots, e_m$$

and the weight filtration is $W_{-m}\mathcal{V} =$ the span of the columns $m, m+1, \dots, n$.

The Polylog local system

Thus the limit MHS $\mathcal{V}(1)$ has period matrix

$$\begin{pmatrix} 1 & 0 & \dots & & \\ 0 & 2\pi i & 0 & \dots & \\ Li_2(1) & 0 & (2\pi i)^2 & \dots & \\ Li_3(1) & 0 & 0 & \dots & \\ \vdots & \vdots & \vdots & \vdots & \\ Li_n(1) & 0 & 0 & \dots & (2\pi i)^n \end{pmatrix}$$

so $W_{\leq -2}\mathcal{V}(1) = \oplus_{i=1}^n \mathbb{Q}(i)$ and $\mathcal{V}(1)$ fits in a short exact sequence

$$0 \to \oplus_{m=1}^n \mathbb{Q}(m) \to \mathcal{V}(1) \to \mathbb{Q} \to 0.$$

The Polylog local system

The individual extensions

$$0 \to \mathbb{Q}(m) \to \mathcal{V}(1)_m \to \mathbb{Q} \to 0 \in \mathrm{Ext}^1_{MHS}(\mathbb{Q}, \mathbb{Q}(m)) \cong \mathbb{C}/(2\pi i)^m\mathbb{Q}$$

give the mixed Hodge realization of $\rho_m(1) \in H^1(k, \mathbb{Q}(m))$. This is thus given by

$$\zeta_{\mathbb{Q}}(m) = Li_m(1) \in \mathbb{C}/(2\pi i)^m\mathbb{Q}.$$

As $\zeta_{\mathbb{Q}}(m) \neq 0 \mod (2\pi i)^m\mathbb{Q}$ for m odd, this implies:

Corollary

The cycle $\rho_m(1)$ is an explicit generator for $H^1(\mathbb{Q}, \mathbb{Q}(m)) = K_{2m-1}(\mathbb{Q})_{\mathbb{Q}}$ for m odd.