RESEARCH SEMINAR PROGRAM: THE MILNOR CONJECTURE

This is the outline program for the Research Seminar this semester. For details on the various lectures, with references, please see Program Details.

Leitfaden

Lecture 1 is an overview and introduces the main players: Milnor K-theory, étale cohomology and the Witt ring of quadratic forms. A sketch of the main thread of Voevodsky's argument is also presented.

Lectures 2, 3, and 4 build up the background in algebraic cycles, various categories of motives and motivic cohomology, as needed to go through Voevodsky's arguments and constructions. Lecture 4 also gives the construction of the Rost motive of a norm quadric, needed in Lecture 7. The remaining lectures follow [30] chapter by chapter. The background in the 3 introductory lectures is used throughout. Here is a Leitfaden of how the various main points are used in later lectures.

Lecture 5: [30, Chap 2] Besides introducing the unstable motivic homotopy category, the main result is Theorem 2.11, used in the proof of Corollary 3.8

Lecture 6: [30, Chap 3] This introduces the Steenrod operations (as a black box), the cosimplicial scheme $\check{C}(X)$ and Margolis homology. The main result is Corollary 3.8, used in the proof of Theorem 7.1, but the Čech construction is used in a number of other places, for example in the construction of the fundamental triangle of Theorem 4.4 of Lecture 7.

Lecture 7: [30, Chap 4] This uses Rost's work to construct the fundamental triangle of Theorem 4.4 and uses results of Lecture 6 on Margolis homology to prove the vanishing result Theorem 4.10.

Lecture 8: [30, Chap. 5] Here results relating Galois cohomology and Milnor K-theory are proven. The first inductive assumption $BK(w, \ell)$ is introduced. The main result is [30, Theorem 5.9], used in the proof of [30, Theorem 7.4], but for [30, Theorem 5.9], nearly all the preceeding results in this chapter are needed.

Lecture 9: [30, Chap 6]. This lecture introduces the second inductive assumption $H90(n, \ell)$. The results [30, Theorem 6.1, Theorem 6.6, Corollary 6.9, Lemma 6.11 and Lemma 6.12] are needed in Lecture 10.

Lecture 10: [30, Chap 7] The main result of the seminar is [30, Theorem 7.4]. In addition to work from previous lectures listed above, this needs [30, Theorem 7.1, Lemma 7.2, Lemma 7.3].

Lectures 11 and 12 are independent and present two papers [19, 22] proving the part of the Milnor Conjecture on quadratic forms, relying on the version relating Milnor K-theory and étale cohomology, as presented above.

1. Lecture 1 (April 6):Introduction

Introduce the main players: Milnor K-theory, galois cohomology, the Witt ring of quadratic forms, state the two main conjectures, give some examples, and give an outline of the strategy of proof of the main conjectures.

2. Lecture 2 (April 13): Chow motives and Voevodsky's triangulated category of geometric motives

i. Define the Chow ring $\mathrm{CH}^*(X)$ for $X \in \mathbf{Sm}_k$, and list its basic properties. Define the category of effective Chow motives $\mathrm{Mot}_{\mathrm{CH}}^{\mathrm{eff}}(k)$ and Chow motives $\mathrm{Mot}_{\mathrm{CH}}(k)$ (over a field k) [15, §1.2.1].

ii. Introduce Voevodsky's category of finite correspondences, $\mathrm{Cor}(k)$

iii. Construct Voevodsky's triangulated category of effective geometric motives over k, $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$ and the triangulated category of geometric motives over k, $\mathrm{DM}_{\mathrm{gm}}(k)$. Describe the main features of these categories iv. Define motivic cohomolgy, and its basic properties. This includes the isomorphism $H^{2n}(X, \mathbb{Z}(n)) \cong \mathrm{CH}^n(X)$ and the embedding of the category of (effective) Chow motives in the category of (effective) geometric motives.

3. Lecture 3 (April 20): Motivic complexes, Milnor K-theory

The definition of motivic cohomology given in Lecture 2 has the disadvantage that one cannot compute anything directly from the definition. The few explicit computations given there were cheating, as these really rely on a completely different description of motivic cohomology. A detailed discussion of this would take us too far afield, but in this lecture we give some hints as to this second construction, and more applications that follow from it. This gives a different view of motivic cohomology that is both more sophisticated and more concrete.

Main points:

i. The categories of presheaves with transfer ($\mathrm{PST}(k)$), Nisnevich sheaves with transfer ($\mathrm{NST}(k)$) and notions of homotopy invariance.

ii. The construction of the triangulated category of effective motives $\mathrm{DM}_-^{\mathrm{eff}}(k)$ as a localization of the derived category $D^-(\mathrm{NST}(k))$. The embedding of $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$ in $\mathrm{DM}_-^{\mathrm{eff}}(k)$.

iii. The Suslin complex construction and the resulting embedding of $\mathrm{DM}_-^{\mathrm{eff}}(k)$ in $D^-(\mathrm{NST}(k))$ via the Suslin complex.

iv. The motivic complexes $\mathbb{Z}(q)^*$ and their relation with motivic cohomology. A description of weight one motivic cohomology as the Zariski cohomology of $\mathbb{G}_m$.

v. The extension of Milnor K-theory of fields to a sheaf $\mathcal{K}_n^M$ on $\mathbf{Sm}_k$, and the theorem of Nestorenko-Suslin/Totaro [21, 29] giving an isomorphism $\mathcal{K}_n^M \cong \mathcal{H}^n(\mathbb{Z}(n))$.

4. Lecture 4 (April 27): Lichtenbaum motivic cohomology, the Rost motive and Rost's injectivity theorem

.

i. Lichtenbaum motivic cohomology. This is a version of motivic cohomology for the étale topology, and is used to give a comparison of mod n weight q motivic cohomology $H^*(-, \mathbb{Z}/n(q))$ with étale cohomology $H^q_{\text{ét}}(-, \mu_n^{\otimes q})$.

ii. Pfister forms. Recall some facts about the Grothendieck-Witt ring and the Witt ring. Introduce the Pfister forms and Pfister quadrics and some of their elementary properties.

iii. The Rost motive. Prove Rost's nilpotence theorem (see [5]). Use this to construct the Rost motive of a quadric and its relation with some Tate motives following [12, 13].

iv. Rost's injectivity theorem: Present the Gersten resolution for $\mathcal{K}_n^M$, and use this to give a concrete description of $H^n(X, \mathcal{K}_{n+1}^M)$ for $n = \dim X$, $X \in \mathbf{Sm}_k$. State and sketch a proof following [24].

If time permits, discuss some of the examples of the Milnor Conjecture from [7, Appendix A]

5. Lecture 5 (May 11): The unstable motivic homotopy category and motivic Alexander-Spanier duality

Follow [30, Chap. 2], see [20, §2] as detailed source. Give a brief sketch of the category of spaces over k, $\mathbf{Spc}(k)$, the $\mathbb{A}^1$ unstable homotopy category $\mathcal{H}(k)$ and the pointed version $\mathbf{Spc}_\bullet(k)$, $\mathcal{H}_\bullet(k)$. State the Morel-Voevodsky homotopy purity theorem (pg. 64, (5)) and present the discussion on pg. 64-5. (As reference see [20, Theorem 2.23]. State and prove the Lemmas 2.1, 2.2, 2.3. Before discussing Lemma 2.4, recall the definition of basic properties of the Thom class from [31, §4].

After this, the main results are [30, Proposition 2.7 (degree map)] and [30, Theorem 2.11]. Give as much of the proof of [30, Prop. 2.7] (i.e., the Lemmas 2.8, 2.9, 2.10) as time permits. The background on $\mathcal{H}(k)$, etc., and [30, Theorem 2.11] are needed in Lecture 6 on Margolis homology.

6. Lecture 6 (May 25): The cosimplicial scheme $\check{C}(X)$ and Margolis homology.

Present [30, Appendix B, Chap. 3].

Begin with a brief introduction of the motivic Milnor operations Q_i and their basic properties, especially [31, theorem 14.2(1), corollary 14.3, proposition 13.6], as black box. Then discuss [30, Appendix B] describing $\check{C}(X)$ and its main properties (Def. 9.1, Lemma 9.2, Lemma 9.3), and go to the main topic, Margolis homology.

The main results are [30, Theorem 3.2, Proposition 3.6, Cor. 3.8], but you should also mention [30, Lemma 3.1].

7. Lectures 7 (June 1): Norm quadrics and their motives

Present [30, Chap. 4]. This uses the results from Lecture 4 to construct Rost motive and maps in [30, Theorem 4.3] fundamental triangle of [30, Theorem 4.4], and the Margolis cohomology vanishing theorem [30, Theorem 4.9], the main result so this section.

The essential point in the proof of [30, Theorem 4.9] is to use Rost's injectivity theorem [30, Theorem 4.10] to allow one to make the necessary computation (using the triangle of [30, Theorem 4.4]) after passing to the algebraic closure of k, where it is straightforward.

8. Lecture 8 (June 15): Milnor K-theory and étale cohomology

Present [30, Chap 5]. Do this for $\ell = 2$, give the "elementary" proof of Prop 5.2 following Remark 5.5. For later discussion, you can state that these results also hold for general ℓ, but will not be needed for the main results. The main results here are [30, Prop. 5.2] and [30, Theorem 5.9], but you should also present Lemmas 5.6, 5.7 and 5.8.

This section introduces $BK(w, \ell)$, the first of two inductive assumptions that form the key to the proof of [30, Theorem 7.4].

9. Lecture 9 (June 22): Beilinson-Lichtenbaum conjectures

Present [30, Chap 6], where the second inductive hypothesis, $H90(n, \ell)$, is introduced. Mention [30, Theorem 6.5] only if time remains at the end, the other results are important enough to state and at least briefly discuss. You should prove [30, Theorem 6.1], state [30, Conj. 6.3, Def. 6.4], State and prove [30, Theorem 6.6]. This section uses numerous results about Nisnevich sheaves with transfer from [34] (also to be found in [16]). Most of these will have been mentioned in Lecture 3 (homotopy purity is mentioned in Lecture 5). Give a list of these results near the beginning of the lecture to recall results from previous lectures or cite ones that have not yet been mentioned.

In addition, you will need to introduce the $(-)_{-1}$ construction, and explain enough to at least state [34, Chap. 3, Proposition 4.34, p.124], which is needed for the proof of [30, Lemma 6.2].

10. Lecture 10 (June 29): Main results

Present [30, Chap. 7]. The main results here are [30, Theorem 7.4, Cor. 7.5]. If time permits, discuss the other theorems and corollaries further on in the section.

11. Lecture 11 (July 6): A proof of the Milnor conjecture by Orlov-Vishik-Voevodsky [22]

12. Lecture 12 (July 13): A proof of the Milnor conjecture by Morel [19]

References

[1] H. Bass, J. Tate, *The Milnor ring of a global field*. Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 349–446. Lecture Notes in Math., Vol. 342, Springer, Berlin, 1973.

[2] S. Bloch, *Algebraic cycles and higher K-theory*. Adv. in Math. 61 (1986), no. 3, 267–304.

[3] S. Bloch, K. Kato, Kazuya, *p-adic étale cohomology*, Inst. Hautes Études Sci. Publ. Math. No. 63 (1986).

[4] S. Bloch and S. Lichtenbaum, *A spectral sequence for motivic cohomology*, 1994. www.math.uiuc.edu/K-theory/062

[5] P. Brosnan, *A short proof of Rost nilpotence via refined correspondences*. Doc. Math. 8 (2003), 69–78.

[6] D.-C. Cisinski, F. Déglise, *Étale motives*. Compos. Math. 152 (2016), no. 3, 556–666.

[7] *Notes on the Milnor Conjecture* arxiv.org/abs/math/0408436.

[8] E. Friedlander, H.B. Lawson, *Moving algebraic cycles of bounded degree*. Invent. Math. 132 (1998), no. 1, 91–119.

[9] E. M. Friedlander and A. Suslin, *The spectral sequence relating algebraic K-theory to motivic cohomology*, Ann. Sci. École Norm. Sup. (4), 35 (6) (2002), 773–875.

[10] T. Geisser and M. Levine, *The Bloch-Kato conjecture and a theorem of Suslin–Voevodsky*, J. Reine Angew. Math., 530 (2001), 55–103.

[11] B. Kahn, *La conjecture de Milnor (d'après V. Voevodsky)*, Astérisque, (245): Exp. No. 834, 5 (1997), 379–418. Séminaire Bourbaki, Vol. 1996/97.

[12] N. Karpenko, *Characterization of minimal Pfister neighbors via Rost projectors*, J. Pure Appl. Algebra, 160 (2001), 195–227.

[13] N. Karpenko, *A shortened construction of the Rost motive.* https://sites.ualberta.ca/ karpenko/publ/rost.pdf

[14] T. Y. Lam, **The algebraic theory of quadratic forms**, Reading, MA: The Benjamin/Cummings Publ., 1973.

[15] M. Levine, *3 Lectures on motivic cohomology* 3 Lectures-Link

[16] C. Mazza, V. Voevodsky, C. Weibel, **Lecture notes on motivic cohomology**. Clay Mathematics Monographs, 2. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2006.

[17] A.S. Merkurjev, A.A. Suslin, *K-cohomology of Severi-Brauer varieties and the norm residue homomorphism.* 1983 Math. USSR Izv. **21** 307.

[18] J. Milnor, *Algebraic K-theory and quadratic forms.* Invent. Math. 9 (1969/70), 318–344.

[19] F. Morel, *Suite spectral d'Adams et invariants cohomologiques des formes quadratiques*, C.R. Acad. Sci. Ser. 1 Math. 328 (1999), no. 11, 963–968.

[20] F. Morel and V. Voevodsky, **$\mathbb{A}^1$-homotopy theory of schemes**, Publ. Math. IHES, (90) (1999), 45–143.

[21] Y.P. Nesterenko, A.A. Suslin, *Homology of the general linear group over a local ring, and Milnor's K-theory.* Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 1, 121–146; translation in Math. USSR-Izv. 34 (1990), no. 1, 121–145.

[22] D. Orlov, A. Vishik, V. Voevodsky, *An exact sequence for KM?/2 with applications to quadratic forms.* Ann. of Math. (2) 165 (2007), no. 1, 1–13.

[23] A. Pfister, *Quadratische Formen in beliebigen Körpern.* Inventiones math. 1, 116 - 132 (1966).

[24] M. Rost, *On the spinor norm and $A_0(X, K_1)$ for quadrics*, 1988.

[25] M. Rost, *Some new results on the Chow groups of quadrics*, 1990.

[26] A.A. Suslin, *Homology of GL_n, characteristic classes and Milnor K-theory.* Algebraic K-theory, number theory, geometry and analysis (Bielefeld, 1982), 357–375, Lecture Notes in Math., 1046, Springer, Berlin, 1984.

[27] A. Suslin, V. Voevodsky, *Bloch-Kato conjecture and motivic cohomology with finite coefficients.* The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), 117–189, NATO Sci. Ser. C Math. Phys. Sci., 548, Kluwer Acad. Publ., Dordrecht, 2000.

[28] J. Tate, *Relations between K_2 and Galois cohomology.* Invent. Math. 36 (1976), 257–274.

[29] B. Totaro, *Milnor K-theory is the simplest part of algebraic K-theory.* K-Theory 6 (1992), no. 2, 177–189.

[30] V. Voevodsky, *Motivic cohomology with $\mathbb{Z}/2$-coefficients* Publ. Math. IHES

[31] V. Voevodsky *Reduced power operations in motivic cohomology*, Publ. Math. IHES

[32] V. Voevodsky, *Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic*, Int. Math. Res. Not., (7) (2002), 351–355.

[33] V. Voevodsky,*Cancellation theorem.* Doc. Math. 2010, Extra vol.: Andrei A. Suslin sixtieth birthday, 671–685.

[34] V. Voevodsky, E. M. Friedlander and A. Suslin, **Cycles, transfers and motivic homology theories**, Princeton: Princeton University Press, 2000.