Motives RS: Talk 13 The motivic Euler char. of Calabi-Yau var., the motivic Göttsche formula and the arithmetic Yau-Zaslow formula

Introduction

k field of char. 0

X K3 surface (smooth proj. surface over k s.t. $\Omega^2_X \simeq \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$)
(i.e. canonical bundle $K_X \simeq \mathcal{O}_X$)

Hodge diamond:

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
\qquad
\begin{array}{l}
\leftarrow h^{2,2} = b_4 \\
\leftarrow h^{2,1} \text{ and } h^{1,2},\ \Sigma = b_3 = 0 \\
\leftarrow h^{2,0}, h^{1,1} \text{ and } h^{0,2},\ \Sigma = b_2 = 22 \\
\leftarrow h^{1,0} \text{ and } h^{0,1},\ \Sigma = b_1 = 0 \\
\leftarrow h^{0,0} = b_0
\end{array}
$$

(compactly supported) Euler characteristic

$$e(X) = \sum_{i=0}^{4} (-1)^i b_i = 24$$

$\mathcal{E}$: complete g-dim. linear system of geometrically integral curves of genus g in X

no need (We assume $NS(X) \xrightarrow{\sim} \mathbb{Z}$, $[C] \mapsto 1$ ← Néron-Severi group) and: $\forall C \in \mathcal{E}$, if $\hat{C}$ is rational then C is nodal (generic case by Chen's thm)

$\pi : \mathcal{E} \xrightarrow{\text{proj.}} \mathbb{P}^g_k$.

We denote by $N_\mathcal{E}$ the number of rational curves in $\mathcal{E}$.

Yau-Zaslow formula ($k = \mathbb{C}$): $N_\mathcal{E} = e(g)$ ← denoted N_g since it only depends on g, with $\sum_{g \geq 0} e(g) t^g = \prod_{m \geq 1} (1 - t^m)^{-24} = \frac{t}{\Delta(t)}$ with Δ the modular form $q \mapsto q \prod_{m \geq 1} (1 - q^m)^{24}$.

Beauville's proof: (i) $N_\mathcal{E} = e(\overline{J}^g(\mathcal{E}))$ ← relative compactified Jacobian

(ii) $e(\overline{J}^g(\mathcal{E})) = e(X^{[g]})$

(iii) $e(X^{[g]}) = e(g)$ (Göttsche formula)

$G(X) = \{ q : \mathrm{Spec}(K(q)) \to \mathbb{P}^g_k,\ C_q := \pi^{-1}(q) \text{ is of geometric genus } 0 \}$ $\longleftrightarrow$ $(C_q)_{\bar{k}}$ is rational

For $q \in G(X)$, $S(C_q)$ is the set of singularities of C_q and: $\forall p \in S(C_q)$ $\Phi_p = 1 + N_{K(p)/k} (\langle -2 \rangle - \langle 2 \alpha_p \rangle) \in \widehat{W}(k)$

$$\mathbb{B}^{mot}_\mathcal{E}(X) = \sum_{q \in G(X)} \mathrm{Tr}_{K(q)/k} \Big(\prod_{p \in S(C_q)} \Phi_p \Big) \in \widehat{W}(k)$$

$\mathrm{rank}(\mathbb{B}^{mot}_\mathcal{E}(X)) = \sum_{q \in G(X)} [K(q) : k]$ is the number of rational curves in $\mathcal{E}$ after base changing to the algebraic closure $\bar{k}$ of k

done last talk (hopefully true) → Analogue of (i): $\mathbb{B}^{mot}_\mathcal{E}(X) = \chi^{mot}(\overline{\mathrm{Pic}}^g(\mathcal{E}))$ ← relative compactified Picard variety, in $\widehat{W}(k) / J_g$ $(J_g = (I^2 + \Delta_g) \cap T)$ ← ker(rank) ∩ ker(disc$_I$), I ← ker(rank) ∩ ker(sign), T ← torsion subgroup

Analogue of (ii): part I of this talk

Analogue of (iii): part II of this talk (motivic Göttsche formula)

Consequences: part III of this talk

I The motivic Euler characteristic of Calabi-Yau varieties

Recall that: $\forall g \geq 0$ $\overline{\mathrm{Pic}}^g(E)$ and $X^{[g]}$ are birationally equivalent Calabi-Yau varieties, (Mukai) (Beauville)

(*) so that by Batyrev's / Kontsevich's theorem, they have the same (compactly supp.) Euler char. over $\bar{k}$ (Betti numbers (Batyrev)), $\mathbb{C}$ (Hodge numbers (Kontsevich)).

We want to prove: $\forall g \geq 0$ $\chi^{mot}(\overline{\mathrm{Pic}}^g(E)) = \chi^{mot}(X^{[g]})$ in $\widehat{W}(k)/J$

where $J = I^2 \cap T$ with $I = \ker(\mathrm{rank})$ the fundamental ideal of $\widehat{W}(k)$ and T the torsion subgroup of $\widehat{W}(k)$. In Pfister's 1966 article, he proved:

1. $I/I^2 \xrightarrow[\sim]{\mathrm{disc}} k^*/(k^*)^2$, i.e. $I^2 = \ker(\mathrm{rank}) \cap \ker(\mathrm{disc})$;

2. local-global principle (for the Witt ring, but also works for $\widehat{W}(k)$, see Lam's book):

torsion subgroup of $W(k)$ $= \ker(\mathrm{sign})$

$$T = \ker(\mathrm{rank}) \cap \ker(\mathrm{sign}) \text{ where } \mathrm{sign}: \begin{cases} \widehat{W}(k) \longrightarrow (\mathrm{Sper}(k) \to \mathbb{Z}) \\ \alpha \longmapsto (< \longmapsto \mathrm{sign}(\alpha_{k_<}) = m - n \end{cases}$$ is the signature

("real spectrum"; $W(k)$; ordering; base change to the real closure)

(by Sylvester's law of inertia, $\exists! (m,n) \in \mathbb{Z}^2$ $\alpha_{k_<} = m\langle 1 \rangle + n\langle -1 \rangle$)

Thus, $J = \ker(\mathrm{rank}) \cap \ker(\mathrm{sign}) \cap \ker(\mathrm{disc})$.

Since $\forall Y \in \mathrm{Sch}_k$ $\mathrm{rank}(\chi^{mot}(Y)) = e(Y)$ (the compactly supp. Euler characteristic of $Y(\bar{k})$), (see Def. 2.9 and Thm 2.10 in Pajwani-Pál's paper)

by Kontsevich's theorem (combined with Beauville's theorem and Mukai's theorems, see (*)):

$$\mathrm{rank}(\chi^{mot}(\overline{\mathrm{Pic}}^g(E))) = \mathrm{rank}(\chi^{mot}(X^{[g]})).$$

(see Thm 2.16 in Pajwani-Pál's paper and the four def. before it) ($\mathrm{Sper}(k)$: real spectrum of k)

Since: $\forall Y \in \mathrm{Sch}_k$ $\mathrm{sign}(\chi^{mot}(Y)) = \epsilon(Y) : \begin{cases} \mathrm{Sper}(k) \to \mathbb{Z} \\ < \mapsto \chi^{rc}_{k_<}(Y(k_<)) \end{cases}$ (comp. supp. real closed Euler char.)

and $\epsilon(\overline{\mathrm{Pic}}^g(E)) = \epsilon(X^{[g]})$ (Lemma 7.4 in Pajwani-Pál's paper), we have:

$$\mathrm{sign}(\chi^{mot}(\overline{\mathrm{Pic}}^g(E))) = \mathrm{sign}(\chi^{mot}(X^{[g]})).$$

See ($\square$) ← on page 5

(Talk 8) (Thm 2.27 in Pajwani-Pál's paper)

In a previous talk, we saw: $\forall \ell$ prime number $\forall Y \in \mathrm{Sch}_k$: $\mathrm{disc}(\chi^{mot}(Y)) = \det_\ell(Y)\, \mathbb{Q}_\ell(w(Y))(-1)^{w(Y)}$ in $\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^*)$

Since $w(\overline{\mathrm{Pic}}^g(E)) = w(X^{[g]})$ (since bir. equiv. var. have the same dimension)

$$\frac{\dim(\,) \times e(\,)}{2} \qquad \frac{\dim(\,) \times e(\,)}{2}$$ (already said it's the same)

and $\forall \ell$ prime $\det_\ell(\overline{\mathrm{Pic}}^g(E)) = \det_\ell(X^{[g]})$ (since these are bir. equiv. (smooth proj.) C-Y var.) (Lemma 7.5 in Pajwani-Pál's paper):

$$\mathrm{disc}(\chi^{mot}(\overline{\mathrm{Pic}}^g(E))) = \mathrm{disc}(\chi^{mot}(X^{[g]})).$$

Therefore: $\chi^{mot}(\overline{\mathrm{Pic}}^g(E)) = \chi^{mot}(X^{[g]})$ in $\widehat{W}(k)/J$.

$24 = e(X), \epsilon(X), \mathrm{disc}(\chi^{mot}(X))$ and those from $\chi^{mot}(X) \in \widehat{W(k)}/J$

II The motivic Göttsche formula

We want to get $\forall g \geq 0$ $\chi^{mot}(X^{[g]}) \in \widehat{W(k)}/J$ from $\{\chi^{mot}(X^{(n)}), n \in \mathbb{N}_0\} \subset \widehat{W(k)}/J$, i.e.

we want to get $\{\underbrace{\mathrm{rank}}_{= e(X^{[g]})}(\chi^{mot}(X^{[g]})), \underbrace{\mathrm{sign}}_{= \epsilon(X^{[g]})}(\chi^{mot}(X^{[g]})), \mathrm{disc}(\chi^{mot}(X^{[g]})), g \in \mathbb{N}_0\}$ from $\mathrm{rk}, \mathrm{sign}, \mathrm{disc}$ ($\mathrm{rk} = e(X^{(n)})$, $\mathrm{sign} = \epsilon(X^{(n)})$)

As before: $\forall \ell$ prime $\forall g, n \in \mathbb{N}_0$

$$\mathrm{disc}(\chi^{mot}(X^{[g]})) = \det_\ell(X^{[g]})\,\mathbb{Q}_\ell(w(X^{[g]}))\,(-1)^{w(X^{[g]})} = \det_\ell(X^{[g]})\,\mathbb{Q}_\ell(g\,e(X^{[g]}))\,(-1)^{g\,e(X^{[g]})} \quad (\Delta)$$

in $\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^*)$ (since $\dim(X^{[g]}) = 2g$)

and

$$\det_\ell(X^{(n)}) = \mathrm{disc}(\chi^{mot}(X^{(n)}))\,\mathbb{Q}_\ell(-w(X^{(n)}))\,(-1)^{w(X^{(n)})} = \mathrm{disc}(\chi^{mot}(X^{(n)}))\,\mathbb{Q}_\ell(-n\,e(X^{(n)}))\,(-1)^{n\,e(X^{(n)})}$$

in $\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^*)$ (since $\dim(X^{(n)}) = 2n$)

$\chi_w : \begin{cases} K_0(\mathrm{Var}_k) \to E(\mathbb{Z}) \\ [X] \mapsto (e(X), w(X)) \end{cases}$ ring morphism

$\chi_\ell : \begin{cases} K_0(\mathrm{Var}_k) \to E(\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^*)) \\ [X] \mapsto (e(X), \det_\ell(X)) \end{cases}$ ring morphism

see Def. 2.34 and Prop. 2.35 in Pajwani–Pál's paper

$e, \epsilon, \chi_\ell, \chi_w$ factor through the ring $\mathrm{Chow}(k)$ of rational Chow motives over k

(since $e(Y) = \sum_{i \geq 0} (-1)^i \dim_k H^i_{dR}(Y/k)$ for any $Y \in \mathrm{Sch}_k$, etc. for ϵ, χ_ℓ and χ_w)

Hence, as a consequence of de Cataldo–Migliorini's theorem, which states that as soon as X is a smooth geom. irreducible projective surface over k, for all $g \geq 0$,

$$\underbrace{M(X^{[g]})}_{(X^{[g]}, \Delta_{X^{[g]}})} \simeq \bigoplus_{\alpha \in P(g),\ \alpha = (1^{\alpha_1}, \dots, g^{\alpha_g})} M(X^{(\alpha_1)}) \cdots M(X^{(\alpha_g)}) \text{ in } \mathrm{Chow}(k)$$

($\checkmark$ for our K3 surface X)

$$e(X^{[g]}) = \sum_{\alpha \in P(g)} e(X^{(\alpha_1)}) \cdots e(X^{(\alpha_g)}) \text{ in } \mathbb{Z}$$

$$\Rightarrow \sum_{g \geq 0} e(X^{[g]}) t^g = \prod_{m \geq 1} \Big(\sum_{n \geq 0} e(X^{(n)}) t^{nm}\Big) \underset{\text{see below}}{=} \prod_{m \geq 1} (1 - t^m)^{-e(X)} = \prod_{m \geq 1} (1 - t^m)^{-24}$$

$$\epsilon(X^{[g]}) = \sum_{\alpha \in P(g)} \epsilon(X^{(\alpha_1)}) \cdots \epsilon(X^{(\alpha_g)}) \text{ in } \mathbb{Z}^{\mathrm{Spr}(k)} \text{ (with pointwise addition)}$$

$$\Rightarrow \sum_{g \geq 0} \epsilon(X^{[g]}) t^g = \prod_{m \geq 1} \Big(\sum_{n \geq 0} \epsilon(X^{(n)}) t^{nm}\Big)$$

$$\chi_\ell(X^{[g]}) = \sum_{\alpha \in P(g)} \chi_\ell(X^{(\alpha_1)}) \cdots \chi_\ell(X^{(\alpha_g)}) \text{ in } E(\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^*))$$

$$\Rightarrow \sum_{g \geq 0} \chi_\ell(X^{[g]}) t^g = \prod_{m \geq 1} \Big(\sum_{n \geq 0} \chi_\ell(X^{(n)}) t^{nm}\Big)$$

with (Δ) above we get $\mathrm{disc}(\chi^{mot}(X^{[g]}))$ from $\mathrm{disc}(\chi^{mot}(X^{(n)}))$ and $e(X^{(n)})$

$$\chi_w(X^{[g]}) = \sum_{\alpha \in P(g)} \chi_w(X^{(\alpha_1)}) \cdots \chi_w(X^{(\alpha_g)}) \text{ in } E(\mathbb{Z})$$

$$\Rightarrow \sum_{g \geq 0} \chi_w(X^{[g]}) t^g = \prod_{m \geq 1} \Big(\sum_{n \geq 0} \chi_w(X^{(n)}) t^{nm}\Big)$$

Note that: $\forall n \geq 0$ $e(X^{(n)}) = \binom{e(X) + n - 1}{n} = \binom{n + 23}{n}$ (Macdonald's formula (1962))

$$\forall n \geq 0\ \forall < \in \mathrm{Spr}(k)\quad \epsilon(X^{(n)})(<) = (-1)^{n\,\mathbb{1}[\epsilon(X)(<) < 0]} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n - 2i - 1 + |\epsilon(X)(<)|}{n - 2i} \binom{i - 1 + \frac{e(X) - |\epsilon(X)(<)|}{2}}{i}$$

$\frac{e(X) - |\epsilon(X)(<)|}{2} = 12 - \frac{|\epsilon(X)(<)|}{2}$

$\mathbb{1}[\ldots]$: 1 if $[\ldots]$ is verified, 0 otherwise

(Lemma 8.6 in P-P)

$$\forall n \geq 0\ \forall \ell \text{ prime}\quad \det_\ell(X^{(n)}) = \det_\ell(X)^{\binom{n + e(X) - 1}{n - 1}} = \det_\ell(X)^{\binom{n + 23}{n - 1}} \quad \text{(Lemma 8.7 in P-P)}$$

$$\forall n \geq 0\quad w(X^{(n)}) = n\,e(X^{(n)}) = n \binom{n + e(X) - 1}{n} \underset{\dim(X) = 2}{=} n \binom{n + w(X) - 1}{n} = n \binom{n + 23}{n}$$

Thus we get $(\underbrace{e(X^{[g]})}_{= \mathrm{rank}(\chi^{mot}(X^{[g]}))}, \underbrace{\epsilon(X^{[g]})}_{= \mathrm{sign}(\chi^{mot}(X^{[g]}))}, \mathrm{disc}(\chi^{mot}(X^{[g]})))$ (see (Δ) at the top of the page) from $(e(X) = \mathrm{rank}(\chi^{mot}(X)), \epsilon(X) = \mathrm{sign}(\chi^{mot}(X)), \mathrm{disc}(\chi^{mot}(X)))$, i.e. $\chi^{mot}(X^{[g]})$ modulo J from $\chi^{mot}(X)$ modulo J. For a nice formula, stop at the intermediate step with the $\chi^{mot}(X^{(n)})$, you probably get:

$$\sum_{g \geq 0} \chi^{mot}(X^{[g]}) t^g = \prod_{m \geq 1} \Big(\sum_{n \geq 0} \chi^{mot}(X^{(n)}) t^{nm}\Big) \text{ in } \widehat{W(k)}[[t]]/J$$

III The arithmetic Yau-Zaslow formula

__Not:__ $\mathcal{Y} = \{\sum_{g\geq 0} a_g t^g,\ \forall g \geq 0\ a_g \in \mathcal{J}_g\}$ (subgroup of $\widehat{W}(k)[[t]]$ ✓ ideal: $\forall \sum_{g\geq 0} a_g t^g \in \mathcal{Y}\ \forall \sum_{k\geq 0} b_k t^k \in \widehat{W}(k)[[t]]$ $\sum_{f\geq 0} (\underbrace{\sum_{g+k=f} a_g b_k}_{\in \mathcal{J}_f}) t^f \in \mathcal{Y}$ ✓)

since $a_g b_k \in \mathcal{J}_g \subset \mathcal{J}_f$ since $0 \leq g \leq f$ (since $0 \leq k$)

Last talk: $\forall g \geq 0$ $\mathbb{B}_\mathcal{E}^{mot}(X) = \chi^{mot}(\overline{\mathrm{Pic}}^g(\mathcal{E}))$ in $\widehat{W}(k)/\mathcal{J}_g$ hence:

$$\sum_{g\geq 0} \mathbb{B}_\mathcal{E}^{mot}(X) t^g = \sum_{g\geq 0} \chi^{mot}(\overline{\mathrm{Pic}}^g(\mathcal{E})) t^g \text{ in } \widehat{W}(k)[[t]]/\mathcal{Y}$$

I of this talk: $\forall g \geq 0$ $\chi^{mot}(\overline{\mathrm{Pic}}^g(\mathcal{E})) = \chi^{mot}(X^{[g]})$ in $\widehat{W}(k)/\mathcal{J}$ hence in $\widehat{W}(k)/\mathcal{J}_g$ $(\mathcal{J} \subset \mathcal{J}_g)$

hence: $$\sum_{g\geq 0} \chi^{mot}(\overline{\mathrm{Pic}}^g(\mathcal{E})) t^g = \sum_{g\geq 0} \chi^{mot}(X^{[g]}) t^g \text{ in } \widehat{W}(k)/\mathcal{J}[[t]] \text{ hence in } \widehat{W}(k)[[t]]/\mathcal{Y}$$

II of this talk: Motivic Göttsche formula:

Probably: $\sum_{g\geq 0} \chi^{mot}(X^{[g]}) t^g = \prod_{m\geq 1} (\sum_{n\geq 0} \chi^{mot}(X^{(n)}) t^{nm})$ in $\widehat{W}(k)/\mathcal{J}[[t]]$ hence in $\widehat{W}(k)[[t]]/\mathcal{Y}$

which we can get from $\chi^{mot}(X)$ in $\widehat{W}(k)/\mathcal{J}$

In any case: the corresponding results for $e, \epsilon, \chi_{ij}\chi_{w}$.

Arithmetic Yau-Zaslow formula:

Probably: $\sum_{g\geq 0} \mathbb{B}_\mathcal{E}^{mot}(X) t^g = \prod_{m\geq 1} (\sum_{n\geq 0} \chi^{mot}(X^{(n)}) t^{nm})$ in $\widehat{W}(k)[[t]]/\mathcal{Y}$

In any case (by considering e) (ok since $\mathcal{J}_g \subset \ker(\mathrm{rank})$ $\Leftarrow$ I)

$\mathrm{rank}(\mathbb{B}_\mathcal{E}^{mot}(X)) = \sum_{q\in G(X)} [k(q) : \mathbb{C}] = |G(X)|$ ($|G(X_{\bar{k}})|$ generally)

Yau-Zaslow formula (or more gen. k of char. 0) $\rightarrow$ __$k = \mathbb{C}$:__ Rank on both sides: $\sum_{g\geq 0} N_g t^g = \prod_{m\geq 1} (\sum_{n\geq 0} e(X^{(n)}) t^{nm})$ in $\mathbb{Z}[[t]]$

i.e. $\forall g \geq 0$ $N_g = \sum_{(1^{a_1},\dots,g^{a_g})\in P(g)} e(X^{(a_1)})\cdots e(X^{(a_g)})$

In any case (by cons. e, ϵ) (and with MacDonald's formula: $\sum_{n\geq 0} e(X^{(n)}) t^n = (1-t)^{-e(X)} = (1-t)^{-24}$ ($e(X^{(n)}) = \binom{n+e(X)-1}{n}$)

hence: $\sum_{g\geq 0} N_g t^g = \prod_{m\geq 1} (1-t^m)^{-e(X)} = \prod_{m\geq 1} (1-t^m)^{-24}$) ✓

already done by Kharlamov and Răsdeaconu

__k real closed (e.g. $\mathbb{R}$):__ Signature on both sides (ok since $\mathcal{J}_g \subset \ker(\mathrm{sign})$): $\sum_{g\geq 1} (\sum_{q\in G(X)} (-1)^{c_q}) t^g = \prod_{m\geq 1} (\sum_{n\geq 0} \epsilon(X^{(n)}) t^{nm})$

$c_q = |\{p \in S(C_q),\ p$ and the tangent spaces at p are def. over $k\}|$

$\underset{(**)}{(-1)^{c_q}} = (-1)^g \underbrace{(-1)^{s_q}}_{W(C_q) \text{ Welschinger number of } C_q}$ since $g = c_q + s_q + 2n_{C_q}$ ($s_q = |\{p \in S(C_q),\ p$ def. over k and whose tangent spaces are not def. over $k\}|$ and n_q = number of $\mathbb{C}$ points in $S(C_q)$)

(s_{C_q} = number of 0-dim. semi-alg. connected comp. of $C_q(k)$ (or at least same parity))

$\mathcal{W}(X, \mathcal{E}) := \sum_{q\in G(X)} W(C_q)$ and $\mathrm{sign}(\mathbb{B}_\mathcal{E}^{mot}(X)) = (-1)^g \mathcal{W}(X, \mathcal{E})$ (by (**)) hence in $\mathbb{Z}[[t]]$:

$$\sum_{g\geq 0} \mathcal{W}(X, \mathcal{E}) t^g = \prod_{m\geq 1} (\sum_{n\geq 0} \epsilon(X^{(n)}) (-t)^{nm})$$
$$= \prod_{m\geq 1} \Big(\sum_{n\geq 0} \Big((-1)^{n \mathbb{1}_{\{\epsilon(X)<0\}}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-2i-1+|\epsilon(X)|}{n-2i} \binom{i-1+\frac{e(X)-|\epsilon(X)|}{2}}{i}\Big) (-t)^{nm}\Big)$$

with $\mathbb{1}_{\{\epsilon(X)<0\}} = \begin{cases} 1 & \text{if } \epsilon(X) < 0 \\ 0 & \text{otherwise} \end{cases}$

k field of char. 0: $\quad r_g := \operatorname{rank}(\mathbb{B}_E^{mot}(X)) = \sum_{q \in G(X)} [K(q):k]$ ← only depends on g and X by the Yau–Zaslow formula (on page 4)

(in any case, by considering e, χ_ℓ, χ_{mot})

$$\mathbb{D}_E^{mot}(X) := (-1)^{g r_g} \operatorname{disc}(\mathbb{B}_E^{mot}(X)) \in k^* / (k^*)^2$$

$$k^*/(k^*)^2 \simeq \operatorname{Hom}(\mathrm{Gal}_k, \mathbb{Z}/2\mathbb{Z}) \quad (\leftarrow \text{group law: } (\varphi+\psi)(\sigma) = \varphi(\sigma)+\psi(\sigma))$$

(Kummer) ↑ Galois group of $k \subset \bar{k}$ ← also works otherwise, with closed subgroups of Gal_k for the Krull topology

(at least when $k \subset \bar{k}$ is a finite extension)

$[1] \longmapsto k \longmapsto \mathrm{Gal}(k, \bar{k}) \longmapsto (\mathrm{Gal}_k \twoheadrightarrow \mathrm{Gal}_k / \mathrm{Gal}_k = 0 \hookrightarrow \mathbb{Z}/2\mathbb{Z})$ ← Galois corresp. ← ker

$[a] \longmapsto k[\sqrt{a}] \longmapsto \mathrm{Gal}(k[\sqrt{a}], \bar{k}) \longmapsto (\mathrm{Gal}_k \twoheadrightarrow \mathrm{Gal}_k / \mathrm{Gal}_{k[\sqrt{a}]} \simeq \mathbb{Z}/2\mathbb{Z})$

nonsquare ← ext. of k ← Galois correspondence ← ker (closed for the Krull top. ✓) — its order $(|\cdot|)$ is $[k[\sqrt{a}]:k] = 2$ — only one iso. since $0 \mapsto 0$

(weakest top. such that $\left\{\begin{array}{l}\mathrm{Gal}_k \to \mathrm{Gal}(k, L) \\ \sigma \mapsto \sigma_{|L}\end{array}\right.$ is continuous $(\forall L)$, discrete top., finite ext. of k)

Part $\oplus$ ↑ ($\square$)

(and $\simeq$ as groups since:
$\mathrm{Gal}(k[\sqrt{ab}], \bar{k})$
$= (\mathrm{Gal}_{k[\sqrt{a}]} \cap \mathrm{Gal}_{k[\sqrt{b}]}) \cup$
$(\mathrm{Gal}_k \setminus \mathrm{Gal}_{k[\sqrt{a}]} \cap \mathrm{Gal}_k \setminus \mathrm{Gal}_{k[\sqrt{b}]})$

$\left(\begin{array}{l}\forall \sigma \in \mathrm{Gal}_k \\ \sigma(\sqrt{a}) \in \{-\sqrt{a}, \sqrt{a}\} \\ \sigma(\sqrt{b}) \in \{-\sqrt{b}, \sqrt{b}\}\end{array}\right)$ since $\sqrt{a}\sqrt{b} = \sqrt{ab} = (-\sqrt{a})(-\sqrt{b})$
and $(-\sqrt{a})\sqrt{b} \neq \sqrt{ab} \neq \sqrt{a}(-\sqrt{b})$)

Since $\left\{\begin{array}{l}\mathbb{Z}/2\mathbb{Z} \to \{-1,1\} \\ i \mapsto (-1)^i\end{array}\right.$ is an iso. of groups, with $\{-1,1\} \subset \mathbb{Q}_\ell^*$ (subgroup),

$\operatorname{Hom}(\mathrm{Gal}_k, \mathbb{Z}/2\mathbb{Z})$ is a subgroup of $\operatorname{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^*)$ (ℓ prime number).

$$\operatorname{disc}(\mathbb{B}_E^{mot}(X)) = \prod_{q \in G(X)} \underbrace{(\mathrm{Tr}_{K(q)/k}(\langle 1 \rangle)}_{\operatorname{disc}(K(q)/k)} (-1)^{[K(q):k]} N_{K(q)/k}(\operatorname{disc}(\prod_{p \in S(q)} \Phi_p)))$$

$$= \prod_{q \in G(X)} (\mathrm{Tr}_{K(q)/k}(\langle 1 \rangle)(-1)^{[K(q):k]} \prod_{p \in S(q)} N_{K(p)/k}(\alpha_p))$$

thus $\mathbb{D}_E^{mot}(X) = (-1)^{(g-1) r_g} \prod_{q \in G(X)} (\mathrm{Tr}_{K(q)/k}(\langle 1 \rangle) \prod_{p \in S(q)} N_{K(p)/k}(\alpha_p))$.

In a previous talk, we saw ✓: $\operatorname{disc}(\chi^{mot}(Z)) = \det_\ell(Z)\, \mathbb{Q}_\ell(w(Z))\, (-1)^{w(Z)}$ with ℓ prime nb

In part.: $\det_\ell(\mathrm{Pic}^g(E)) = \operatorname{disc}(\chi^{mot}(\mathrm{Pic}^g(E)))\, \mathbb{Q}_\ell(-g r_g)(-1)^{g r_g}$ in $\operatorname{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^*)$

since $e(\mathrm{Pic}^g(E)) = e(X^{[g]}) = r_g$ and $\dim(\mathrm{Pic}^g(E)) = 2g$.

Hence, in $\operatorname{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^*)/\Delta_g$: $\det_\ell(\mathrm{Pic}^g(E)) = \operatorname{disc}(\mathbb{B}_E^{mot}(X))\, \mathbb{Q}_\ell(-g r_g)\, (-1)^{g r_g}$
$= \mathbb{D}_E^{mot}(X)\, \mathbb{Q}_\ell(-g r_g)$

hence in $E(\operatorname{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^*)/\Delta_g)$: $\chi_\ell(\mathrm{Pic}^g(E)) = (r_g, \mathbb{D}_E^{mot}(X)\, \mathbb{Q}_\ell(-g r_g))$, hence only depends on g and X

$E(\ldots) = \mathbb{Z} \oplus \operatorname{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^*)/\Delta_g$ with $(a,i)(b,j) = (ab, i^b j^a)$

and $\chi_\ell(X^{[g]}) = (r_g, \mathbb{D}_E^{mot}(X)\, \mathbb{Q}_\ell(-g r_g))$, with $r_g = e(X^{[g]})$ and $\mathbb{D}_E^{mot}(X)\,\mathbb{Q}_\ell(-g r_g) = \det_\ell(X^{[g]})$ $(\dim(X^{[g]}) = 2g)$

In $E[[t]]$: $\displaystyle\sum_{g \geq 0} (r_g, \mathbb{D}_g^{mot}(X)\, \mathbb{Q}_\ell(-g r_g))\, t^g = \sum_{g \geq 0} \chi_\ell(\mathrm{Pic}^g(E))\, t^g = \sum_{g \geq 0} \chi_\ell(X^{[g]})\, t^g$

$$= \prod_{m \geq 1} \left(\sum_{n \geq 0} \chi_\ell(X^{(n)})\, t^{nm}\right)$$

$$= \prod_{m \geq 1} \left(\sum_{n \geq 0} \left(\binom{e(X)+n-1}{n}, \det_\ell(X)^{\binom{e(X)+n-1}{n-1}}\right) t^{nm}\right)$$

with $\det_\ell(X) = \operatorname{disc}(\chi^{mot}(X))\, \mathbb{Q}_\ell(-e(X))(-1)^{e(X)}$. ↖ Yau–Zaslow formula for discriminants modulo Δ_g.