Motives RS: Talk 4
Elementine Lemarié--Rieusset

Göttsche's formula: recovering the Betti numbers of $S^{[n]}$ from the Betti numbers of S

① Göttsche's formula

$(*)$ quasi (is ok for Macdonald's formula) $p(X,-1) =: e(X)$ (compactly supported) Euler char. of X

Let S be a smooth projective surface over $\mathbb{C}$ and $n \in \mathbb{N}_0$.
(variety $(**)$ with $2\dim(S)$ rather than 4)

$S^n / \mathfrak{S}_n =: S^{(n)}$ ($\mathfrak{S}_n$ ← n^{th} symmetric group)

and $p(X,z) = \sum_{i=0}^{2\dim(X)} b_i(X) z^i$ Poincaré polynomial of X (X: $\mathbb{C}$-scheme of finite dim), $b_i(X)$: i^{th} Betti number of X

Macdonald's formula (1962):

$$\sum_{n\geq 0} p(S^{(n)},z)\,t^n = \prod_{i=0}^{4} \left(1+(-1)^{i+1} z^i t\right)^{(-1)^{i+1} b_i(S)}$$

one can recover the Betti numbers of $S^{(n)}$ from this

In part.: $$\sum_{n\geq 0} e(S^{(n)})\,t^n = \prod_{i=0}^{4} (1-t)^{(-1)^{i+1} b_i(S)} = (1-t)^{-e(S)} = \sum_{\ell\geq 0} \binom{\ell+e(S)-1}{\ell} t^\ell$$

$e(S^{(n)}) = (-1)^n \binom{-e(S)}{n} = \binom{n+e(S)-1}{n}$ if $e(S)\leq 0$; $\binom{n+e(S)-1}{n}$ if $e(S)\geq 1$ ("negative binomial formula")

$S^{[n]} := \mathrm{Hilb}^n(S)$ and Hilbert-Chow morphism $\omega_n : S^{[n]} \to S^{(n)}$

On $\mathbb{C}$-points, $\omega_n(x_1 \amalg \dots \amalg x_r) = (\underbrace{x_1,\dots,x_1}_{\ell_1},\dots,\underbrace{x_r,\dots,x_r}_{\ell_r})$.
$\underbrace{\mathrm{length}(x_1)}_{=\ell_1}+\dots+\underbrace{\mathrm{length}(x_r)}_{=\ell_r} = n$

proof: use the Weil conj. and first prove analogous formulas for $\mathbb{F}_q$, q prime power ($q=p^m$, p prime)

In 1990, Göttsche gave formulas to recover the Poincaré polynomial of $S^{[n]}$ from the Poincaré polynomial of S (i.e. to recover the Betti numbers of $S^{[n]}$ from the Betti numbers of S) and special case: to recover the Euler characteristic of $S^{[n]}$ from the Euler characteristic of S:
$$\sum_{n\geq 0} e(S^{[n]})\,t^n = \prod_{m\geq 1} \underbrace{(1-t^m)^{-e(S)}}_{\sum_{\ell\geq 0}\binom{\ell+e(S)-1}{\ell} t^{m\ell}}.$$
$e(S^{[0]}) = 1$, $e(S^{[1]}) = e(S)$, $e(S^{[2]}) = e(S) + \binom{e(S)+1}{2}$, …

In 2001, Göttsche published another article in which he proved an equality in $K_0(\mathrm{Var}_{\mathbb{C}})$ (actually $K_0(\mathrm{Var}_k)$ with k alg. closed of char. 0) from which we can deduce:
$$p(S^{[n]},z) = \sum_{\substack{\alpha=(1^{a_1},\dots,n^{a_n})\\ \text{partition of } n\ (\alpha\in P(n))}} p(S^{(a_1)},z)\cdots p(S^{(a_n)},z)\, z^{2(n-|\alpha|)}$$
(i.e. $\sum_{i=1}^{n} a_i\, i = n$), $a_i \in \mathbb{N}_0$ (nb of occurrences of i in α), $|\alpha| = \sum_{i=1}^{n} a_i$ (= number of integers in the partition)

Also:
$$e(S^{[n]}) = \sum_{\substack{\alpha\in P(n)\\ (1^{a_1},\dots,n^{a_n})}} e(S^{(a_1)})\cdots e(S^{(a_n)})$$
$= (-1)^{a_1}\binom{-e(S)}{a_1}\cdots(-1)^{a_n}\binom{-e(S)}{a_n}$ if $e(S)\leq 0$; $\binom{a_1+e(S)-1}{a_1}\cdots\binom{a_n+e(S)-1}{a_n}$ if $e(S)\geq 1$ (and in general)

Together with Macdonald's formula, this formula allows us to recover the (compactly supported) Betti numbers of $S^{[n]}$ from the Betti nb of S. (same everywhere)

Ex: Let's compute the Euler characteristic of $\mathbb{P}^{2\,[n]}$

$\mathbb{P}^2$

$1 = b_0(\mathbb{P}^2) = b_2(\mathbb{P}^2) = b_4(\mathbb{P}^2)$, $0 = b_1(\mathbb{P}^2) = b_3(\mathbb{P}^2)$ $\quad$ $(e(\mathbb{P}^2) = 3)$ $= b_0(\mathbb{P}^2) - b_1(\mathbb{P}^2) + b_2(\mathbb{P}^2) - b_3(\mathbb{P}^2) + b_4(\mathbb{P}^2)$

($\mathbb{P}^2_{\mathbb{C}}$ here so: $\mathbb{A}^0_{\mathbb{C}} \cup \mathbb{A}^1_{\mathbb{C}} \cup \mathbb{A}^2_{\mathbb{C}}$, $\dim_{\mathbb{R}}=0$, $\dim_{\mathbb{R}}=2$, $\dim_{\mathbb{R}}=4$)

Macdonald's formula: $$\sum_{n\geq 0} e(\mathbb{P}^{2(n)})\,t^n = \frac{1}{(1-t)^3}$$
hence $e(\mathbb{P}^{2(n)}) = \binom{n+2}{n} = \frac{(n+1)(n+2)}{2}$

Göttsche's formula: $$e(\mathbb{P}^{2\,[n]}) = \sum_{\substack{\alpha\in P(n)\\ =(1^{a_1},\dots,n^{a_n})}} \frac{1}{2^{r}}(a_1+1)\cdots(a_n+1)(a_1+2)\cdots(a_n+2)$$

In part. ($e(\mathbb{P}^{2[0]})=1$) $e(\mathbb{P}^{2[1]}) = \underset{1=1}{3}$), $e(\mathbb{P}^{2[2]}) = \underset{1+1=2,\ 2=2}{6+3=9}$, $e(\mathbb{P}^{2[3]}) = \underset{1+1+1=3,\ 1+2=3,\ 3=3}{10+3\times 3+3=22}$

Let us discuss the proof of Göttsche's formula (the one which follows from his 2001 article).

III The proof of Göttsche's formula

1) Göttsche's theorem in $K_0(\mathrm{Var}_k)$

Thm (Göttsche, 2001): If S is a smooth quasiproj. variety over an alg. closed field k of char. 0, and if $n \in \mathbb{N}_0$ then in $K_0(\mathrm{Var}_k)$, $[S^{[n]}] = \sum_{\alpha\in P(n)} [S^{(\alpha)} \times \mathbb{A}^{n-|\alpha|}]$

where $S^{(\alpha)} := S^{(a_1)}\times\dots\times S^{(a_n)}$ if $\alpha = (1^{a_1},\dots,n^{a_n})$ (and $|\alpha| = \sum_{i=1}^{n} a_i$).

Equiv., $[S^{[n]}] = [\coprod_{\alpha\in P(n)} S^{(\alpha)}\times \mathbb{A}^{n-|\alpha|}]$ (with the scissors relations).

(↑ disjoint union as a scheme for the topological as well) (↑ easy special case)

Recall that the class in $K_0(\mathrm{Var}_k)$ is the universal motivic invariant and that the Poincaré polynomial is a motivic invariant. $b_0(\mathbb{A}^1)=0,\ b_1(\mathbb{A}^1)=0,\ b_2(\mathbb{A}^1)=1$

Cor. ($k=\mathbb{C}$) $p(S^{[n]}, z) = p(\coprod_{\alpha\in P(n)} S^{(\alpha)}\times\mathbb{A}^{n-|\alpha|}, z)$

Equiv., $p(S^{[n]}, z) = \sum_{\alpha\in P(n) = (1^{a_1},\dots,n^{a_n})} p(S^{(a_1)}, z)\cdots p(S^{(a_n)}, z) \underbrace{\overbrace{p(\mathbb{A}^1, z)}^{z^2}\cdots\overbrace{p(\mathbb{A}^1, z)}^{z^2}}_{n-|\alpha|\text{ terms}}$.

2) Why are there partitions? ⚠ Do a) Stratification first! ← following sheet

b) Cellular decomposition

In their 1987 and 1988 articles, Ellingsrud and Strømme gave a cell decomposition (a cellular) of $\mathrm{Hilb}^n(\mathbb{P}^2)$ ($=\mathbb{P}^{2[n]}$) for each $n\in\mathbb{N}_0$ (in order to understand its homology), and by the way gave a cell decomp. of $\mathrm{Hilb}^n(\mathbb{A}^2)$ ($=\mathbb{A}^{2[n]}$), one of $\mathrm{Hilb}^n(\mathbb{A}^2, \mathbb{A}^1)$ (the closed subscheme of $\mathrm{Hilb}^n(\mathbb{A}^2)$ whose points (which are closed subschemes of $\mathbb{A}^2$) have their support in a given line of $\mathbb{A}^2$) and one of $\underbrace{\mathrm{Hilb}^n(\mathbb{A}^2, \mathbb{A}^0)}_{\text{we are interested in this}}$ (the closed subscheme " in a given point of $\mathbb{A}^2$).

Cell decomposition of X: $X = X_\lambda \supset X_{\lambda-1}\supset\dots\supset X_0\supset X_{-1} = \emptyset$ closed subschemes s.t. $X_i\setminus X_{i-1} = \coprod$ subschemes $\cong \mathbb{A}^{q_i}$ (↑ scheme-theoretically) (affine spaces of some dimension (= cells))

$T := \mathbb{G}_{m,\mathbb{C}}^2$ (↑ 2-dim torus) $\curvearrowright \mathbb{C}[x,y]$ via $t.x = \lambda(t)x$ and $t.y = \mu(t)y$, λ,μ lin. indep. characters of $\mathbb{G}^2_{m,\mathbb{C}}$ ($\mathbb{G}^2_{m,\mathbb{C}}\to\mathbb{G}_{m,\mathbb{C}}$)

(this is tech.) technical → those of colength n induce a cell decomposition of $\mathrm{Hilb}^n(\mathbb{A}^2, \mathbb{A}^0)$

T-invariant ideals of $\mathbb{C}[x,y] \leftrightarrow$ partitions of an $n\in\mathbb{N}_0$

↑ such an ideal is generated by monomials

colength of the ideal I ($\dim_{\mathbb{C}}(\mathbb{C}[x,y]/I)$)

better than U: cellular decomp. (change the roles of x and y)

$\mathrm{Hilb}^n(\mathbb{A}^2, \mathbb{A}^0) = \bigcup_{\substack{\alpha\in P(n)\\ =(\pi_0,\pi_1,\dots,\pi_{\ell-1})\\ |\alpha| = \ell}} \mathbb{A}^{n-\pi_0} = \bigcup_{\substack{\alpha\in P(n)\\ |\alpha|=\ell}} \mathbb{A}^{n-\ell}$

I of colength n

$\forall i\in\mathbb{N}_0$ $\pi_i := \min\{j\in\mathbb{N}_0,\ x^i y^j\in I\}$ (one can reconstruct I from the π_i)

↑ only finitely many are nonzero, they sum up to n (i.e. are a partition of n) and they form a non-increasing sequence (one can construct I from a partition: the biggest is π_0, etc.)

↑ if $x^i y^j\in I$ then $x^{i+1}y^j \in I$ ideal

$1, y, \dots, y^{\pi_0-1}$
$x, xy, \dots, xy^{\pi_1-1}$
$\vdots$
$x^{\ell-1}, x^{\ell-1}y, \dots, x^{\ell-1}y^{\pi_{\ell-1}-1}$

$\pi_\ell = 0$ → earliest one

better: cellular decomposition

CCL of E-S '87: $\mathrm{Hilb}^n(\mathbb{A}^2, \mathbb{A}^0) = \bigcup_{\substack{\alpha \in P(n) \\ \text{"}(\alpha_0 \geq \alpha_1 \geq \dots \geq \alpha_r)\text{"}}} \mathbb{A}^{n-\alpha_0}$

Ferrers diagram: (or Young diagram with squares)

$\cdots\cdots\cdot \}\ \alpha_0$ points

$\cdots\cdots\cdot \}\ \alpha_1$ points

$\vdots$

$\cdots \}\ \alpha_r$ points

"If you read the diagram: $\downarrow$ then $\downarrow$ then $\downarrow$ instead of $\xrightarrow{\text{then}}$ $\xrightarrow{\text{then}}$" called conjugation ($\leftrightarrow$ bijection on partitions)

then the biggest number of the new partition is the number of integers of the old partition

better: cellular decomp.

= number of integers of α (nb of parts in the partition α)

Thus, $\mathrm{Hilb}^n(\mathbb{A}^2, \mathbb{A}^0) = \bigcup_{\alpha \in P(n)} \mathbb{A}^{n-|\alpha|}$.

In $K_0(\mathrm{Var}_{\mathbb{C}})$: $[\mathrm{Hilb}^n(\mathbb{A}^2, \mathbb{A}^0)] = \sum_{\alpha \in P(n)} [\mathbb{A}^{n-|\alpha|}]$ (scissors relations)

$= [\coprod_{\alpha \in P(n)} \mathbb{A}^{n-|\alpha|}]$

or k, since E and S said it should also work for any field k (in part. for our k)

"$X = \underbrace{X_r \setminus X_{r-1}}_{\coprod \mathbb{A}^{\cdots}} \amalg \underbrace{X_{r-1}}_{\underbrace{X_{r-1} \setminus X_{r-2}}_{\coprod \mathbb{A}^{\cdots}} \amalg X_{r-2}}$" ← closed, closed

etc

$\underbrace{X_1 \setminus X_0}_{\coprod \mathbb{A}^{\cdots}} \amalg \underbrace{X_0}_{\coprod \mathbb{A}^{\cdots}}$ ← closed

a) <u>Stratification</u> (Here, there are partitions but they will actually disappear)

Let P be a partially ordered set (poset). A stratification of a scheme X is:

$(Z_i)_{i \in P}$ a family of closed subschemes of X such that:

- $X = \bigcup_{i \in P} Z_i$ (set-theoretically)
- $\forall x \in X\ \exists U$ open in X, $x \in U$ and $U \cap Z_i = \emptyset$ for all but a finite number of Z_i ← in our case, P will be finite ✓
- $Z_i \cap Z_j = \bigcup_{i,j \leq k} Z_k$ (<u>scheme</u>-theoretically)

$X_i := Z_i \setminus \bigcup_{i<j} Z_j$ verify $X = \coprod_{i \in P} X_i$ set-theoretically (⚠ not necessarily topologically: the X_i don't have to be open in X (and they don't have to be closed in X))

the "strata" of the stratification of X

<u>not</u> a $\amalg$ scheme-theoretically

P finite here

In $K_0(\mathrm{Var}_k)$, $[X] = \sum_{\substack{p \in P \\ p \text{ biggest} \\ \text{for } \leq}} [\underbrace{Z_p}_{X_p}] + [X \setminus \coprod_{\substack{p \in P \\ p \text{ biggest} \\ \text{for } \leq}} Z_p]$ ← $Z_{p_1} \cap Z_{p_2} = \emptyset$ if p_1 and p_2 biggest for $\leq$

$= \sum_{i \in P} [X_i]$ (do the same with the biggest for $\leq$ in $P \setminus \{\text{biggest for } \leq\}$ and so on).

$S^{(n)}$ has a stratification: $P = P(n)$ ← finite, the set of partitions of n with $(\alpha_1, \dots, \alpha_r) \leq (\beta_1, \dots, \beta_s)$ iff $\exists J_1, \dots, J_s \subset \{1, \dots, r\}$, $\{1, \dots, r\} = \coprod_{i=1}^{s} J_i$ and $\forall i\ \beta_i = \sum_{j \in J_i} \alpha_j$ and ($\alpha_1 + \dots + \alpha_r = n = \beta_1 + \dots + \beta_s$, $\alpha_i, \beta_i \in \mathbb{N}$)

$Z_{(\alpha_1, \dots, \alpha_r)} = \Phi(\Delta_{\alpha_1} \times \dots \times \Delta_{\alpha_r})$

$Z_{(\alpha_1,\dots,\alpha_n)\in P(n)} = \Phi(\Delta_{\alpha_1}\times\dots\times\Delta_{\alpha_n})$ with $\Phi : S^n \to S^{(n)}$ the quotient morphism and Δ_i the diagonal of S^i $\left(= \{(\underbrace{s,\dots,s}_{i \text{ times}}), s\in S\}\right)$

↑ denoted $\underline{S}^{(n)}_{\alpha}$ in the article

(and $X_{(\alpha_1,\dots,\alpha_n)}$ is denoted $S^{(n)}_{\alpha}$ in the article)

Z_α closed subscheme in $S^{(n)}$, $Z_{(1,\dots,1)} = \Phi(\underset{S}{\underset{\parallel}{\Delta_1}}\times\dots\times\underset{S}{\underset{\parallel}{\Delta_1}}) = \Phi(S^n) = S^{(n)}$,

$Z_\alpha \cap Z_\beta = \bigcup_{\alpha,\beta\in\gamma} Z_\gamma$.

In $K_0(Var_k)$, $[S^{(n)}] = \sum_{\beta\in P(n)} [S^{(n)}_\beta]$.

This stratification gives a stratification $S^{[n]}_\beta := \omega_n^{-1}(S^{(n)}_\beta)$ (with ω_n the Hilbert-Chow morphism of $S^{[n]}$)

$$[S^{[n]}] = \sum_{\beta\in P(n)} [S^{[n]}_\beta]$$

and a stratification $g^{-1}(S^{(n)}_\beta)$ of $\coprod_{\alpha\in P(n)} S^{(\alpha)}\times \mathbb{A}^{n-|\alpha|}$ ← nb of integers in α (nb of parts in α) (with $g = \coprod_{\alpha\in P(n)} g_\alpha : S^{(\alpha)}\times\mathbb{A}^{n-|\alpha|}\to S^{(n)}$

and $g_\alpha = h_\alpha \circ proj_{S^{(\alpha)}}$, with $\forall h_\alpha : S^{(\alpha)} \to S^{(n)}$ the morphism $(\xi_1,\dots,\xi_n)\mapsto(1.\xi_1,\dots,n.\xi_n)$)

$S^{(\alpha)} = S^{(a_1)}\times\dots\times S^{(a_n)}$ (a_i nb of occurrences of i in α)

the points in ξ_1 with multiplicity 1, …, the points in ξ_n with mult. n

$$\left[\coprod_{\alpha\in P(n)} S^{(\alpha)}\times\mathbb{A}^{n-|\alpha|}\right] = \sum_{\beta\in P(n)} g^{-1}(S^{(n)}_\beta)$$

To prove Göttsche's Thm of 2001, it suffices to prove: $\forall \beta\in P(n)$ $[S^{[n]}_\beta] = [g^{-1}(S^{(n)}_\beta)]$.

$S^{[n]}_{(n)}$ is a locally trivial fibre bundle over S with fibre $Hilb^n(\mathbb{A}^2, \mathbb{A}^0)$ (def…)

($n = n$) hence, in $K_0(Var_k)$, $[S^{[n]}_{(n)}] = [S][Hilb^n(\mathbb{A}^2,\mathbb{A}^0)]$.

(b) Cellular decomposition ⚠ Go to the preceding sheet.

3 Proving Göttsche's theorem

From what we did, $[S^{[n]}_{(n)}] = [S][\coprod_{\alpha\in P(n)} \mathbb{A}^{n-|\alpha|}]$. Not: $Y \xrightarrow{f} X^{(m)}$ ($\to S^{(n)}$ for $\to$) $Y_* := f^{-1}(\{m \text{ distinct points}\})$

Also: $\forall \beta\in P(n)$ $(1^{b_1},\dots,n^{b_n})$ $\left(\prod_{i=1}^n (S^{[i]}_{(i)})^{(b_i)}\right)_* \to S^{[n]}_\beta$ is a bijection on k-points

$(Z_1,\dots,Z_n) \mapsto \coprod_{i=1} Z_i$

length 1 length n

Hence $[S^{[n]}_\beta] = \left[\left(\prod_{i=1}^n (S^{[i]}_{(i)})^{(b_i)}\right)_*\right] = \left[\left(\prod_{i=1}^n \left(S\times\coprod_{\alpha_i\in P(i)}\mathbb{A}^{i-|\alpha_i|}\right)^{(b_i)}\right)_*\right]$

$= \left[\left(\prod_{i=1}^n \coprod_{\alpha_i\in P(i)} S\times\mathbb{A}^{i-|\alpha_i|}\right)^{(b_i)}_*\right]$

$= \left[\left(\prod_{i=1}^n \coprod_{\substack{f_i : P(i)\to\mathbb{N}_0 \\ \sum_{\gamma\in P(i)} f_i(\gamma) = b_i}} \prod_{\alpha_i\in P(i)} (S\times\mathbb{A}^{i-|\alpha_i|})^{(f_i(\alpha_i))}\right)_*\right]$

$$= \left[\coprod_{\substack{f : \coprod_{m\in\mathbb{N}} P(m)\to\mathbb{N}_0 \\ \sum_{\gamma\in P(i)} f(\gamma) = b_i\ \forall i\in\{1,\dots,n\}}} \left(\prod_{\alpha\in\coprod P(m)} S^{(f(\alpha))}\right)_* \times \mathbb{A}^{n-\sum_{\alpha\in\coprod_{m\in\mathbb{N}} P(m)} f(\alpha)|\alpha|}\right]$$

↑ this is $[g^{-1}(S^{(n)}_\beta)]$ since this scheme is in bijection (as a set) with $g^{-1}(S^{(n)}_\beta)$

Let us prove this!

$$g^{-1}(S_\beta^{(n)}) = \coprod_{\gamma \in P(n)} g_\gamma^{-1}(S_\beta^{(n)}) = \coprod_{\gamma \in P(n)} h_\gamma^{-1}(S_\beta^{(n)}) \times \mathbb{A}^{n-|\gamma|}$$

It suffices to show that $h_\gamma^{-1}(S_\beta^{(n)})$ is in bijection with $\coprod_{f : \coprod_{m\in\mathbb{N}} P(m) \to \mathbb{N}_0} \left(\prod_{\alpha \in \coprod P(m)} S^{(f(\alpha))}\right)_*$.

$$\sum_{\delta \in P(i)} f(\delta) = b_i \ \forall i \qquad (\square)$$

since $=\gamma$, f must verify $\rightarrow$
$$\sum_{\delta \in \coprod_{m\in\mathbb{N}} P(m)} \underbrace{f(\delta)\delta}_{(1^{f(\delta)d_1},\dots)} = \gamma \qquad (\Delta)$$

$f(\delta)=0$ for all but finitely many δ

$\delta = (1^{d_1},\dots) \leftarrow i^{d_i}$; $(1^{f(\delta)d_1},\dots) \leftarrow i^{f(\delta)d_i}$

$\exists m \in \mathbb{N}\ \forall M > m\ d_M = 0$ (and $\delta \in P(\sum_i d_i i)$)

$\delta + \varepsilon = (1^{d_1+e_1},\dots) \leftarrow i^{d_i+e_i}$, $\varepsilon = (1^{e_1},\dots) \leftarrow i^{e_i}$

Let $\xi = (\xi_1,\dots,\xi_n) \in h_\gamma^{-1}(S_\beta^{(n)})$, $\xi_1 \in S^{(e_1)}, \dots, \xi_n \in S^{(e_n)}$, $\gamma = (1^{e_1},\dots,n^{e_n})$.

$\forall x \in S\ \forall i\ \ m_x(\xi_i) :=$ multiplicity of x in ξ_i (0 if x is not in ξ_i)

$\forall x \in S\ \ \gamma_x(\xi) := (1^{m_x(\xi_1)},\dots,n^{m_x(\xi_n)})$

$$f_\xi : \begin{cases} \coprod_{m\in\mathbb{N}} P(m) \to \mathbb{N}_0 \\ \delta \mapsto |\{x \in S, \gamma_x(\xi) = \delta\}| \end{cases}$$ verifies: $\sum_{\delta\in P(i)} f_\xi(\delta) = b_i\ \forall i$ and $\sum_{\delta \in \coprod_{m\in\mathbb{N}} P(m)} f_\xi(\delta)\delta = \gamma$.

finite since there are a finite number of points in each ξ_i, $i \in \{1,\dots,n\}$

$$\Phi(\xi) := \Big(\sum_{\substack{x\in S \\ \gamma_x(\xi)=\alpha}} x\Big)_{\alpha \in \coprod_{m\in\mathbb{N}} P(m)} \in \Big(\prod_{\alpha\in\coprod P(m)} S^{(f_\xi(\alpha))}\Big)_*$$

Now let us construct the inverse Ψ of Φ.

Let $f : \coprod_{m\in\mathbb{N}} P(m) \to \mathbb{N}_0$ be such that $(\square)$ and (Δ).

Let $\theta = (\theta_\alpha)_{\alpha \in \coprod_{m\in\mathbb{N}} P(m)}$ with $\theta_\alpha \in S^{(f(\alpha))}$, such that $\theta \in (\prod_\alpha S^{(f(\alpha))})_*$.

$$\Psi(\theta) := \Big(\sum_{\alpha\in\coprod_{m\in\mathbb{N}} P(m)} a_1\theta_\alpha, \dots, \sum_{\alpha\in\coprod_{m\in\mathbb{N}} P(m)} a_n\theta_\alpha\Big) \in h_\gamma^{-1}(S_\beta^{(n)})$$

$\alpha = (1^{a_1},\dots)$, first component $\in S^{(e_1)}$, last component $\in S^{(e_n)}$

We have our bijection, hence $[S_\beta^{[n]}] = [g^{-1}(S_\beta^{(n)})]$, hence $[S^{[n]}] = [\coprod_{\alpha\in P(n)} S^{(\alpha)} \times \mathbb{A}^{n-|\alpha|}]$

–