The η-inverted sphere

Recall $SH(k)[\rho^{-1}] \simeq SH_{ret}(k)$

& $D_{\mathbb{A}^1}(k)[\rho^{-1}] \simeq D(Sh^{Nis}_{ret}/k)\ (D(k_{ret}))$

Note ret dim = Krull dim

We use these results to make computation in $SH(k)^-$

Recall that the switch $\tau : \mathbb{P}^1 \wedge \mathbb{P}^1 \circlearrowleft$ gives

$$1 = \frac{1}{2}(1 + \tau) + \frac{1}{2}(1 - \tau) \quad \text{in } SH(k)[1/2]$$

and $\left[\frac{1}{2}(1 \pm \tau)\right]^2 = \frac{1}{4}(1 + \tau^2 \pm 2\tau) = \frac{1}{2}(1 \pm \tau)$

$$(1 + \tau)(1 - \tau) = 0$$

so we have 2 decomposition using orthogonal idempotents

$\leadsto SH(k)[1/2] = SH(k)^+ \times SH(k)^-$

$\qquad\qquad\qquad\qquad \uparrow \tau = id \qquad\quad \uparrow \tau = -id$

To explain in detail: We recall Morel's theorem and make computations in $K^{MW}_*(k)$

For $a \in k^\times$ we have $i_a : S^0 \to \mathbb{G}_m \leftrightarrow a \in \mathbb{G}_m(k)$

$S^0 = \operatorname{Spec} k_+$ $\rightsquigarrow$ $i_a : \mathbb{1}_k \to \mathbb{1}_k \wedge \mathbb{G}_m$

and we have $\eta : \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$ $\rightsquigarrow$ $\eta : \mathbb{1}_k \to \mathbb{1}_k \wedge \mathbb{G}_m^{-1}$

Recall $K^{MW}_*(k)$ has generators $[a]$ in degree 1, $a \in k^\times$, η in degree -1

and relations: $\eta[a] = [a]\eta$, $[a][1-a] = 0$,
$[ab] = [a] + [b] + \eta[a][b]$, $\eta(2 + \eta[-1]) = 0$

<u>Thm</u> (Morel) Sending $[a]$ to i_a, η to η gives an iso

$$K^{MW}_*(k) \xrightarrow{\sim} [\mathbb{1}_k, \mathbb{1}_k \wedge \mathbb{G}_m^{*}] = \pi_0(k)_*$$

of rings

and extends to an iso of Nisnevich sheaves

$$\underline{K}^{MW}_* \xrightarrow{\sim} \underline{\pi}_0$$

<u>Note</u>

Let $\langle a \rangle = 1 + \eta[a]$. Then $\langle ab \rangle = \langle a \rangle \langle b \rangle$

$[ab] = [a] + \langle a \rangle [b]$
$= [b] + \langle b \rangle [a]$

$[1]\eta(2 + \eta[-1]) = 0 \Rightarrow (\langle 1 \rangle - 1)(\langle -1 \rangle + 1) = 0$

$\left(\langle 1 \rangle - 1\right)\left(\langle -1 \rangle + 1\right) = \langle -1 \rangle - \langle -1 \rangle + \langle 1 \rangle - 1$

$\Rightarrow \langle 1 \rangle = 1 \Rightarrow \eta[1] = 0$

Then $[1] = [1 \cdot 1] = 2[1] + \eta[1][1] = 2[1] \Rightarrow [1] = 0$.

By definition $\rho = -[-1]$. One can show that $\langle a \rangle \in \pi_0(k)_0$

is induced by $\mathbb{P}^1 \circlearrowleft [x_0 : x_1] \to [x_0, -x_1]$, and τ is the map $\mathbb{A}^2/(\mathbb{A}^2 \setminus \{0\})$ induced by $\begin{pmatrix}0&1\\1&0\end{pmatrix} \sim \begin{pmatrix}1&0\\0&-1\end{pmatrix} \Rightarrow \tau = \langle -1 \rangle = 1 + \eta[-1]$
$= 1 - \eta\rho$

Write $K_*^{MW}[1/2] = K^+ \times K^-$ $\quad \tau = \pm id$ on $K^\pm$

Then $\eta\rho = 0$ on K^+, $\eta\rho = 2$ on $K^- \Rightarrow \eta\rho$ is invertible on K^-

Let $h = 1 + \tau = 2 + \eta[-1]$, so $\eta h = 0$

$h = 2$ on K^+ $\quad \eta = 0$ on $K^+ \Rightarrow K^+ = \ker(\times\eta : K_*^{MW}[1/2] \circlearrowleft)$

$\rho^2 h = 2[-1]^2 + \eta[-1]^3$

$0 = [1] = [(-1)(-1)] = 2[-1] + \eta[-1]^2 \Rightarrow \eta[-1]^3 = -2[-1]^2$

so $\rho^2 h = 2[-1]^2 - 2[-1]^2 = 0$. Restricting to K^+

$h = 2$, so $0 = \rho^2 h = 2\rho^2 \Rightarrow \rho^2 = 0$ on K^+

This yields

Lemma 39 $\quad K_*^{MW}[1/2, 1/\rho] = K^- = K_*^{MW}[1/2, 1/\eta]$

$K^+ = \ker(\times\eta : K_*^{MW}[1/2] \circlearrowleft)$

Since $K_*^{MW} = \mathrm{End}(1\!\!1_k, 1\!\!1_k \wedge \mathbb{G}_m^{\wedge *})$, this decomposes $SH(k)[1/2]$ as $SH(k)[1/2] = SH(k)^+ \times SH(k)^-$

with $SH(k)^- = SH(k)[1/2, 1/\rho] = SH(k)[1/2, 1/\eta]$

$$SH(k)^+ = SH(k)[1/2]_{\eta=0}$$

and similarly for $D_{\mathbb{A}^1}(k)$.

<u>Prop 40</u> (a result of Röndigs) Let k be a perfect field. Then $\underline{\pi}_i(1\!\!1_k[1/2, 1/\eta]) = 0$ for $i = 1, 2$

Pf Lemma 39 $\Rightarrow$ $\underline{\pi}_i(1\!\!1_k[1/2, 1/\eta]) = \underline{\pi}_i(1\!\!1_k[1/2, 1/\rho])$

To show $\underline{\pi}_i(1\!\!1_k[1/2, 1/\rho]) = 0$ $i = 1, 2$, $\underline{\pi}_i$ is an unramified sheaf $\Rightarrow$ need only check on fields K, so need only check on k (change notation)

By Theorem 25 $\underline{\pi}_i(\ )[1/2, 1/\rho]$ are ret sheaves, so for $k \to k^{rc}$ the map on $\underline{\pi}_i$ is injective, so can assume k real closed (this handles char $k > 0$)

But then $SH(k)[1/2, 1/p] \simeq SH(k_{ret})[1/2]$

$\simeq SH[1/2]$

and $\pi_i(1\!\!1_k[1/2, 1/p]) \to \pi_i^s[1/2] = 0$ for $i = 1, 2$

since $\pi_i^s = \mathbb{Z}/2$ for $i = 1, 2$

We now look at some results of Ananyevskiy-L.-Panin

1. $\pi_i(1\!\!1[1/\eta])_{\mathbb{Q}} = 0$ for $i \neq 0$
2. We have the homotopy module $W_\bullet$ & $EM(W_\bullet)$ a Witt ring spectrum. Then $SH(k)^-_{\mathbb{Q}} \simeq EM(W_\bullet)_{\mathbb{Q}}$-Mod

Bachmann improves these results

We have the free-forget adjunction

$\mathbb{Z}_{tr}: SH(k) \rightleftarrows DM_{\mathbb{A}^1}(k) : EM$

Let $H_{\mathbb{A}^1}\mathbb{Z} := EM(\mathbb{Z}_{\mathbb{A}^1})$, A homotopy module is an object in $D_{\mathbb{A}^1}(k)^{\heartsuit}$

($\sim$ unit in $D_{\mathbb{A}^1}(k)$)

and the unit map $\mathbb{Z}_{\mathbb{A}^1} \to W_\bullet$ induces the map of ring spectra $H_{\mathbb{A}^1}\mathbb{Z} \to EM(W_\bullet)$ in $SH(k)$, $DM_W(k, \mathbb{Z}[1/2])$

We have $W_\bullet = \pi_0(1\!\!1_k[1/\eta])_*$ so $W_\bullet[1/2] = \pi_0(1\!\!1_k[1/2, 1/p])_*$

Let $DM_W(k,\Lambda) = \text{Mod-}E_\infty(W_\bullet \otimes \Lambda)$

Prop 41
- $\underline{\pi}_i(H_{\mathbb{A}^1}\mathbb{Z}[1/p]) = 0$ for $i \neq 0$, so

  $\underline{\pi}_i(H_{\mathbb{A}^1}\mathbb{Z}[1/2, 1/p]) = 0$ for $i \neq 0$
- $\underline{\pi}_i(1_{\mathbb{Q}}[1/p]) = \underline{\pi}_i(1_{\mathbb{Q}}[1/p]) = 0$ for $i \neq 0$

  (A.-L.-P.)
- We have equivalences

$$D_{\mathbb{A}^1}(k, \mathbb{Z}[1/2]) \simeq DM_W(k, \mathbb{Z}[1/2]) \simeq D(\mathrm{Spec}\,k_{ret}, \mathbb{Z}[1/2])$$

$$SH(k)^-_{\mathbb{Q}} \simeq DM_W(k, \mathbb{Q}) \simeq D(\mathrm{Spec}(k)_{ret}, \mathbb{Q})$$

(A.-L.-P.)

Pf As done we have

$$D_{\mathbb{A}^1}(k)[1/2] = D_{\mathbb{A}^1}(k)^+ \times D_{\mathbb{A}^1}(k)^-$$

$$D_{\mathbb{A}^1}(k)^- \simeq D_{\mathbb{A}^1}(k)[1/2, 1/\eta] = D_{\mathbb{A}^1}(k)[1/2, 1/\rho] \quad \text{(Thm 35)}$$

$$\simeq D(\mathrm{Spec}\,k_{ret}, \mathbb{Z}[1/2])$$

$$SH(k)_{\mathbb{Q}}^{-} \simeq SH(k)_{\mathbb{Q}}[1/\eta] \simeq SH(k)_{\mathbb{Q}}[1/\rho] \simeq SH(\operatorname{Spec} k_{ret})_{\mathbb{Q}}$$

By classical ret'l stable homotopy theory

$$SH(\operatorname{Spec} k_{ret})_{\mathbb{Q}} \simeq D(\operatorname{Spec} k_{ret}, \mathbb{Q})$$

Similarly $SH(k)[1/\rho] = SH(\operatorname{Spec} k_{ret})$

Thus for $k \hookrightarrow K$

$$\underline{\pi}_n(H_{\mathbb{A}^1}\mathbb{Z}[1/\rho])(K) = [\mathbb{1}[1/\rho]_K, H\mathbb{Z}_{\mathbb{A}^1}[1/\rho]_K[-n]]_{SH(k)[1/\rho]}$$

$$= [\mathbb{Z}[1/\rho]_{\mathbb{A}^1,K}, \mathbb{Z}_{\mathbb{A}^1}[1/\rho][-n]]_{D_{\mathbb{A}^1}(K)[1/\rho]} = [\mathbb{Z}_{ret}, \mathbb{Z}_{ret}[-n]]_{D(\operatorname{Spec} K_{ret})}$$

$$= H^{-n}_{ret}(\operatorname{Spec} K, \underline{\mathbb{Z}})$$

But $\text{retcd} \leq \text{Krull dim}$ (Scheiderer) $\Rightarrow H^{-n}_{ret}(\operatorname{Spec} K, \underline{\mathbb{Z}}) = 0$ for $n \neq 0$

$\Rightarrow \underline{\pi}_n(H_{\mathbb{A}^1}[1/\rho]) = 0$ for $n \neq 0 \Rightarrow \underline{\pi}_n(H_{\mathbb{A}^1}[1/2, 1/\eta]) = 0$ for $n \neq 0$

In addition, a similar computation for $n=0$

$\Rightarrow \underline{\pi}_0(H_{A'}\mathbb{Z}[1/p]) = \underline{H}^0(\underline{\mathbb{Z}}_{ret}) = \underline{\mathbb{Z}}_{ret}$

But $\underline{W}[1/2]_{ret} = \underline{W}[1/2, 1/p] = \underline{K}_*^{MW}[1/\eta][1/2, 1/p]$

$= K_*^{MW}[1/2, 1/p]$

$= \underline{\mathbb{Z}}_{ret}[1/2]$

Thus $H_{A'}\mathbb{Z}[1/2, 1/p] = H_{A'}\mathbb{Z}[1/2, 1/\eta]$ by Jacobson $\longrightarrow EM(W_0[1/2])$

giving the equivalence

$$D_{A'}(k)[1/2, 1/\eta] \simeq DM_W(k, \mathbb{Z}[1/2])$$

$\Rightarrow$

$$SH(k)^-_{\mathbb{Q}} \simeq D(\mathrm{Sp\acute{e}t}\, k_{ret}, \mathbb{Q}) \simeq D_{A'}(k)_{\mathbb{Q}}[1/p]$$

$$\simeq D_{A'}(k)_{\mathbb{Q}}[1/\eta] \simeq DM_{W_{\mathbb{Q}}}(k)$$

Cor 92 For k real closed or k a number field with exactly 1 real embedding

$\underline{\pi}_*(\mathbb{1}_k[1/p])(k) = \pi_*^S$ and

$\underline{\pi}_*(\mathbb{1}_k[1/p, 1/2])(k) = \pi_*^S[1/2]$.

Pf $SH(k)[1/p] = SH(\mathrm{Spec}(k)_{ret}) = SH$

since $Shv(\mathrm{Spec}\, k_{ret}) = Set$ for

k real closed or if k has exactly 1 real embedding

(e.g. $k = \mathbb{Q}$, but also $k = \mathbb{Q}[x]/(f)$ where

$f \in \mathbb{Q}[x]$ is irreducible with exactly one real root).