Motivic and real étale stable homotopy theory - overview

Main goal: to understand the ρ-localization of $SH(S)$

Background Let S be a noetherian scheme of finite Krull dimension. $SH(S)$ is constructed by starting with the category $Spc(S)$ of "spaces over S", i.e. presheaves of spaces (simplicial sets) on Sm/S. One then localizes to impose Nisnevich descent and $\mathbb{A}^1$-homotopy invariance

$$\begin{tikzcd}
 & Spc \arrow[d, "c"] & \\
Sm/S \arrow[r, "h"'] & Spc(S) \arrow[r, bend left, "L_{mot}"] & Spc_{Nis,\mathbb{A}^1}(S) \arrow[l, hook, bend left, "i"]
\end{tikzcd}$$

in $Spc_{Nis,\mathbb{A}^1}(S)$ one has $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$

Form $SH(S)$ by inverting $-\wedge\mathbb{P}^1$. As model objects are $\mathbb{P}^1$-spectra: $E = \{(E_0, E_1, \dots), \varepsilon_n : E_n \wedge \mathbb{P}^1 \to E_{n+1}\}$

E_i: pointed obj in $Spc_{Nis,\mathbb{A}^1}(S)$

Since $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$ one can also 1st form $SH^{S^1}(S)$: S^1-spectra, by inverting $- \wedge S^1$, then take $\mathbb{G}_m$-spectra in S^1-spectra, inverting $- \wedge \mathbb{G}_m$

One can also restrict to the purely "topological" version, just take the Nisnevich localization of S^1-spectra

$$Spc_{Nis}(S), \quad SH^{S^1}_{Nis}(S)$$

$\sim$ Nisnevich sheaves of spaces

$\sim$ Nisnevich sheaves of spectra

These latter are easier to understand, but our main interest is $SH(S)$.

Realization Functor Suppose $S = \operatorname{Spec} k$, k a field. If we have an embedding $\sigma : k \hookrightarrow \mathbb{C}$, we get a functor $Re_{\sigma,\mathbb{C}} : SH(k) \to SH$ sending $\Sigma^\infty_{\mathbb{P}^1} X_+$ for $X \in Sm(k)$ to the suspension spectrum $\Sigma^\infty X(\mathbb{C})_+$

Similarly, given an embedding $\beta : k \subset \mathbb{R}$, we set $Re_{\beta,\mathbb{R}} : SH(k) \to SH$ sending $\Sigma^{\infty}_{\mathbb{P}^1} X_+$ to $\Sigma^{\infty} X(\mathbb{R})_+$

These give some information on $SH(k)$, e.g. if $\mathcal{E} \xrightarrow{f} y$ is an equiv then $Re_{\alpha,\mathbb{C}}(f)$ and $Re_{\beta,\mathbb{R}}(f)$ are equivalences and these give splittings to $SH \xrightarrow[c_{\mathbb{C}}]{} SH(\mathbb{C})$ resp $SH \xrightarrow[c_{\mathbb{R}}]{} SH(\mathbb{R})$ for $k = \mathbb{C}, \mathbb{R}$

$Re_{\alpha,\mathbb{C}}$ sends $\mathbb{G}_m$ to $\mathbb{C}^* \simeq S^1$ and $Re_{\beta,\mathbb{R}}$ sends $\mathbb{G}_m$ to $\mathbb{R}^{\times} \simeq S^0$. Note that there is a map $\rho' : S^0 \to \mathbb{G}_m$ in $Spc(k)$ with $Re_{\beta,\mathbb{R}}(\rho')$ an equivalence $S^0 \to \mathbb{R}^{\times}$, namely $\rho'(bp) = +1 \in \mathbb{G}_m$
$\rho'(\text{non-}bp) = -1 \in \mathbb{G}_m$

There is no non-trivial map $S^1 \to \mathbb{G}_m$ in $Spc(k)$

Any: $Re_{\beta,\mathbb{R}} : SH(k) \to SH$ factors through the ρ-localized ($\rho = -\rho'$)

$$\begin{tikzcd}
SH(k) \arrow[rr, "Re_{B,\mathbb{R}}"] \arrow[d, phantom, "\cup"] \arrow[dr, shift left] & & SH \\
\rho\text{-local } SH(k) & SH(k)[\rho^{-1}] \arrow[ul, hook, shift left] \arrow[ur, "\overline{Re}_{B,\mathbb{R}}"'] &
\end{tikzcd}$$

Thm (for $k = \mathbb{R}$) $SH(k)[\rho^{-1}] \to SH$ is an equivalence, precisely $Re_{\mathbb{R}}$ has a right adjoint $R^*: SH \to SH(\mathbb{R})$ and R^* is fully faithful with essential image the ρ-stable $E \in SH(\mathbb{R})$: $\mathrm{Map}_*(X_+ \wedge \mathbb{G}_m, E) \overset{\sim}{\underset{\rho^*}{\to}} \mathrm{Map}(X, E)$

for $X \in \mathrm{Sm}/\mathbb{R}$

Other base schemes What about a general S? we still have the ρ-localization $SH(S)[\rho^{-1}]$. This also has an "elementary" description via the real spectrum S_r of S

Def The real spectrum $R(S)$ of S is a topological space. The points of $R(S)$ are pairs $(x, <)$ where x is a point of S and $<$ is an ordering on the residue field $k(x)$. A basis of open neighborhoods of $(x, <)$ is given as follows: let $U = \operatorname{Spec}(A)$ be an open nbd of x in S. For $y \in U$, $a \in A$ let $\tilde{a}^y \in k(y)$ be the image of a in $k(y)$ and let

$$D(a) = \{ (y, <) \in U_r \mid \tilde{a}^y > 0 \text{ in } k(y) \}$$

Then $\left\{ D(a) \;\middle|\; \begin{matrix} a \in A \\ \tilde{a}^x > 0 \text{ in } k(x) \end{matrix} \right\}$ is a basis of open nbds of $(x, <)$ in S_r. For $S = \operatorname{Spec} A$, write $\operatorname{sper} A$ for $R(S)$

Ex $S = \mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec} \mathbb{R}[T]$. One has $(x, <_{\mathbb{R}})$ for $x \in \mathbb{A}^1_{\mathbb{R}}(\mathbb{R})$, $<_{\mathbb{R}}$ the usual ordering $k(x) = \mathbb{R}$

and for η the generic point, one has 4 types of order on $\mathbb{R}(\eta) = \mathbb{R}(T)$

$D^{\pm}$) $<_{x,\pm}$ ~~x~~ $x \in \mathbb{R}$ for $f \in \mathbb{R}(T)$

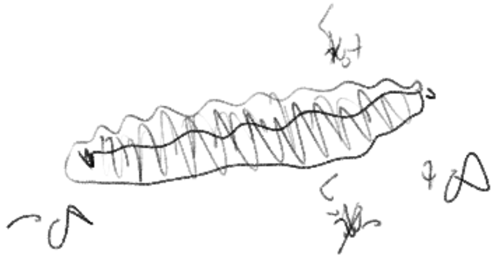
$0 <_{x,+} f$ iff $f(x+\varepsilon) > 0$ for $0 < \varepsilon << 0$

$0 <_{x,-} f$ iff $f(x-\varepsilon) > 0$ for $0 < \varepsilon << 0$

$\pm\infty$) $<_{+\infty}$ $0 <_{+\infty} f$ iff $\exists\, x_0 \in \mathbb{R}$ s.t.
$f(x) > 0 \ \forall x > x_0$

$<_{-\infty}$ $0 <_{-\infty} f$ iff $\exists\, x_0 \in \mathbb{R}$ s.t.
$f(x) > 0 \ \forall x < x_0$

Sper $\mathbb{R}[T]$:



Ex $(\eta, <_{\infty+}) \in D(f)$ iff $f(x) > 0 \ \forall x > x_0$ some x_0

$\Leftrightarrow (\eta, <_{x,+}) \in D(f) \ \forall x > x_0$

$\Rightarrow (\eta, <_{\infty+}) \in \overline{\{(x, <_x) / x \in \mathbb{R}, x > x_0\}}$

Given a topological space T one has a model category
local

~~that~~ $\mathrm{Shv}(T)$ and $\mathrm{SH}(\mathrm{Shv}(T))$: presheaves of spectra on T satisfying "hyperdescent"

sheaves of spaces

"sheaves of spectra" for short

<u>Thm 2</u>

There is a "realization" functor

$$\begin{tikzcd}
\mathrm{SH}(S) \arrow[rr, "\mathrm{Re}_{S,r}"] \arrow[dr] & & \mathrm{SH}(\mathrm{Shv}(R/S)) \\
& \mathrm{SH}(S)[\rho^{-1}] \arrow[ur, "\mathrm{Re}_{r,\rho}"'] &
\end{tikzcd}$$

factoring through

and $\mathrm{Re}_{r,\rho}$ is an equivalence

<u>Note</u> for $S = \mathrm{Spec}(\mathbb{R})$, $S_r = \{*\}$, $\mathrm{SH}_{loc}(S_r) = \mathrm{SH}$

and $\mathrm{Re}_{S,r} = \mathrm{Re}_{\mathbb{R}}$ so Thm 1 is a special case of

Thm 2

<u>The real étale topology</u> The proof of Thm 2

uses a reinterpretation of $SH(Shv(RS))$ as the "real étale" $SH(-)$, i.e. replace Nisnevich localization with localization with respect to the real étale topology

Def The small real étale site S_{ret} of S has objects

$U \to S$ finite type étale

A family $\{U_\alpha \xrightarrow[f_\alpha]{} U\}$ in S_{ret} is a covering family

if $\quad U_r = \bigcup f_\alpha(U_{\alpha,r})$

The big real étale site $Sm(S)_{ret}$ has objects $U \to S$ in $Sm(S)$ and covering families as above

Thm 3 In the seqn $SH(Shv(RS)) \xrightarrow{a} SH(S_{ret})$

$$\xhookrightarrow{b} SH(Sm(S)_{ret}) \xrightarrow[c]{} SH^{S^1}(S)^{ret}[\rho^{-1}] \xrightarrow{d} SH(S)^{ret}[\rho^{-1}]$$

a is an equivalence, b, c, and d are fully faithful and dcb a is an equivalence

Thm 2 is proven as

$$\begin{tikzcd}
SH(S) \arrow[r] \arrow[dr, bend right] & SH(S)^{ret}[\rho^{-1}] \arrow[d, phantom, "\cong"] \\
 & SH(Shv(RS))
\end{tikzcd}$$

to define $Re_{S,r}$ and

then show the above defines an equivalence

$$\begin{tikzcd}
SH(S) \arrow[rr] \arrow[dr] & & SH(S)^{ret}[\rho^{-1}] \\
 & SH(S)[\rho^{-1}] \arrow[ur, "\sim"'] &
\end{tikzcd}$$

Idea of proof ① a is an equivalence by results of Scheiderer

$SH(S_{ret}) \xrightarrow[b]{} SH(Sm(S)_{ret})$ is fully faithful by applying i_S^* for $i_S : S_{ret} \to Sm(S)_{ret}$ the inclusion

$SH(Sm(S)_{ret}) \xrightarrow{c} SH^{S^1}(S)^{ret}[\rho^{-1}]$

is fully faithful by using

the fact that $H^*_{ret}(U \times \mathbb{A}^1, F) \cong H^0_{ret}(U, F)$)

② show that $SH^{S^1}(S)^{r\acute{e}t}[p^{-1}] \xrightarrow{d} SH(S)^{r\acute{e}t}[p^{-1}]$

is fully faithful

by using $\rightarrow H^*_{r\acute{e}t}(U \times \mathbb{G}_m, F) \cong H^*_{r\acute{e}t}(U, F)$

③ show that $SH(S_{r\acute{e}t}) \xrightarrow{dcba} SH(S)^{r\acute{e}t}[p^{-1}]$

is essentially surjective. This

uses an argument of Cisinski-Déglise to

reduce to show that $S \mapsto SH(S_{r\acute{e}t})$

$\searrow SH(S)^{r\acute{e}t}[p^{-1}]$

satisfy proper base change for $SH(-)^{r\acute{e}t}[p^{-1}]$

This follows from 6-functors for $SH(-)$

↳ showing $SH(-)^{r\acute{e}t}[p^{-1}] \simeq SH(-)[p^{-1}]$

For $SH((-)_{r\acute{e}t})$ this follows from proper

base-change for rét cohomology

Finally, to show that

$$SH(-)[\rho^{-1}] \xrightarrow{\sim} SH(-)^{\text{rét}}[\rho^{-1}]$$

Bachmann uses work of Jacobson:

$$\overset{\text{Morel}}{\pi_0(1)_* = \underline{K}^{MW}_*} \Rightarrow \pi_0(1)_*[\rho^{-1}] = \operatorname{colim}_n (\underline{K}^{MW}_n \xrightarrow{\rho} \underline{K}^{MW}_{n+1})$$

$$\cong \underline{K}^{MW}_*[\rho^{-1}] \qquad \cong \operatorname{colim}_n (\underline{I}^n \xrightarrow{\cdot\rho} \underline{I}^{n+1})$$

$$\cong$$

Then improve this to show $\qquad a_{\text{rét}}\mathbb{Z}$
for F_* a homotopy module $F_*[\rho^{-1}]$ is a sheaf on $Sm(k)_{\text{rét}}$ and then use this to show

$$H^*_{Nis}(X, F_*[\rho^{-1}]) \xrightarrow{\sim} H^*_{\text{rét}}(X, F_*[\rho^{-1}])$$

This shows that $SH(k)[\rho^{-1}] \xrightarrow{\sim} SH(k)^{\text{rét}}[\rho^{-1}]$

Then use localization to pass from k to general S

Bachmann also shows that the sequence

$$\begin{tikzcd}
SH(\mathbb{R}) \arrow[r] & SH(\mathbb{R})[\rho^{-1}] \arrow[r] & SH(\mathrm{Spec}(\mathbb{R})_{r\acute{e}t}) \arrow[d, "\wr"] \\
 & & SH(\mathbb{R}(\mathrm{Spec}\,\mathbb{R})) \arrow[d, phantom, "\wr"] \\
\text{is} & \text{the } \mathrm{Re}_{\mathbb{R}} \arrow[r] & SH
\end{tikzcd}$$

Some applications

1. $SH(\mathbb{R})[\eta^{-1}, 1/2] \cong SH[1/2]$

The map $\eta : \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$ is the algebraic Hopf map

one has (in $SH(k)$) $\quad \eta : \mathbb{G}_m \to S^0$

and $\quad \rho\eta = 2\,\mathrm{id}_{\mathbb{G}_m} \Rightarrow SH(\ )[\rho^{-1}, 1/2]$

$\quad\quad\quad \eta\rho = 2\,\mathrm{id}_{S^0} \quad\quad \overset{\cong}{} \; SH(\ )[\eta^{-1}, 1/2]$

so $\quad SH(S)[\eta^{-1}, 1/2] \cong SH(Sm(RS))[1/2]$

In particular $\quad SH(\mathbb{R})[\eta^{-1}, 1/2] \cong SH[1/2]$

<u>Ex</u> Under this equiv $\quad W^{[1/2]} = \mathrm{colim}(\times\eta : K^{MW}_n \to K^{MW}_{n-1})[1/2]$

$$= \mathrm{colim}(\times\rho : K^{MW}_n \to K^{MW}_{n+1})[1/2]$$

$$\underset{\text{ret}}{\cong} \mathbb{Z}[1/2]$$

(Jacobson)

$\Rightarrow$ for $X \in Sm(\mathbb{R})$, we have

$$H^*(X, W[1/2]) \cong H^*(X(\mathbb{R}), \mathbb{Z}[1/2])$$

This also gives an explicit description of the ρ localiz

2. Comparison for $\mathbb{1}[\eta^{-1}]$: we know

$$\mathbb{1}[\eta^{-1}, 1/2] = \mathbb{1}[\rho^{-1}, 1/2]$$

Then compare with $\pi_i(\mathbb{1})$ in SH

$$\pi_1(\mathbb{1}) = \mathbb{Z}/2 = \pi_2(\mathbb{1})$$

to show that $\underline{\pi}_i(\mathbb{1}_{\mathbb{R}})[\eta^{-1}, 1/2] = 0$ for $i = 1, 2$

(recovering results of Röndigs)

As in (1) one shows

$$D_{\mathbb{A}^1}(k, \mathbb{Z})[1/\rho] = D(\operatorname{Spec} k_{ret})$$

$$\Rightarrow DM_W(k, \mathbb{Z}[1/2]) = D_{\mathbb{A}^1}(k, \mathbb{Z}[1/2])[\rho^{-1}]$$

$$= D_{\mathbb{A}^1}(k, \mathbb{Z}[1/2])^{-}$$

since $SH(k)^-_{\mathbb{Q}} \simeq D_{\mathbb{A}^1}(k,\mathbb{Q})^-$

this strengthens results of Ananyevskiy-Levine-Panin

3. <u>Rigidity</u>: for $E \in SH(k)[\rho^{-1}] \simeq SH(k_{ret})$

$\Rightarrow$ $\underline{\pi}_i(E)$ are extended from sheaves on

k_{ret}

Consequence: e = exp character then

$F = \underline{\pi}_i(\mathbb{1})_0[1/e]$ is <u>rigid</u>

$F(X) = F(x)$ for X essentially smooth local henselian with closed point x

Schedule

Lecture 2 (Andrew) local homotopy theory

Lecture 3 (Marco) Real étale cohomology

Lecture 4 (Clémentine) Motivic homotopy theory

Lecture 5 (Christian) pre-motivic categories and monoidal Bousfield localization

Lecture 6 The theorem of Jacobson and ρ-stable homotopy modules

Lecture 7 Preliminaries for the proof of main theorem

Lecture 8 Main theorem

Lecture 9 Identifying the real realization

Lecture 10 The η-inverted sphere

Lecture 11 Rigidity