GAP BETWEEN LYAPUNOV EXPONENTS FOR HITCHIN REPRESENTATIONS

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Abstract. We study Lyapunov exponents for flat bundles over hyperbolic curves defined via parallel transport over the geodesic flow. We consider them as invariants on the space of Hitchin representations and show that there is a gap between any two consecutive Lyapunov exponents. Moreover we characterize the uniformizing representation of the Riemann surface as the one with the extremal gaps.

The strategy of the proof is to relate Lyapunov exponents in the case of Anosov representations to other invariants, where the gap result is already available or where we can directly show it. In particular, firstly we relate Lyapunov exponents to a foliated Lyapunov exponent associated to a foliation Hölder isomorphic to the unstable foliation on the unitary tangent bundle of a Riemann surface. Secondly, we relate them to the renormalized intersection product in the setting of the thermodynamic formalism developed by Bridgeman, Canary, Labourie and Sambarino.

Contents

1. Introduction

Lyapunov exponents are characteristic numbers associated to the dynamics of trajectories of a dynamical system. The interest for these invariants in this paper's context originated from the study of the dynamical properties of billiard trajectories on polygonal billiards and trajectories of wind-tree models for the diffusion of gas molecules. Both these settings can be studied by restating the problem in terms of properties of the geodesic flow on a flat surface, i.e. a topological surface equipped with a flat metric with finitely many conical singularities (see the survey [28]).

It turned out that, in order to study properties of a special flat surface, it is convenient to study properties of the associated family given by the deforming the flat surface. The Lyapunov exponents of the original problem for a special flat surface can be identified with the ones associated to the flat cohomology bundle over the flat surface associated family. It is at this point that the work of Eskin-Kontsevich-Zorich [12] allowed to compute the sum of the positive Lyapunov exponents in this setting by relating it to the algebraic degree of a holomorphic vector bundle.

Consequent works generalized the relation between Lyapunov exponents and degrees of holomorphic bundles in the case of special flat bundles coming from families of curves over ball quotients [20], family of K3 surfaces [13], and more generally in the case of any flat bundle over a Riemann surface [11]. Daniel-Deroin [10] generalized even more this relation to the case of flat bundles over Kähler manifolds. The first author [9] refined the relation of [11] and proposed to study the Lyapunov exponents as functions on representation varieties and conjectured an inequality for the gap between Lyapunov exponents for Hitchin representations.

Hitchin representations are the central objects of study in *Higher Teichmüller Theory*, which generalizes the theory of Fuchsian and Kleinian groups to Lie groups of rank ≥ 2 (see Subsection 4.2). They were introduced by Hitchin [17] as representations in the connected components of character varieties containing an embedding of the Teichmüller space. Labourie [21] showed that Hitchin representations possess a rich dynamical structure and in particular they are faithful and discrete.

Generalizing the rank one case, various asymptotic quantities describing the geometry of a representation have been studied: the orbital counting problem [25], critical exponent and entropies [23], Hausdorff dimension of limit sets [24] [14]. A defining feature of representations in higher rank is that the relevant notion of "size" of a matrix is not just a number, the norm, but the collection of all the singular values, which form a vector called the *Cartan projection* which lives in the Cartan subspace. The asymptotic geometry of the Cartan projection of the image of the representation is a central subject of investigations. In the present work we study the Lyapunov exponents of a Hitchin representation with respect to a hyperbolic metric on a surface. These characteristic numbers are quantities that measure the asymptotic growth of the norm of vectors under parallel transport in the flat bundle associated to the representation, where the parallel transport happens over the geodesic flow defined by the hyperbolic metric. The Lyapunov exponents define a vector in the Cartan subspace which reflects asymptotic properties of the representation with respect to the hyperbolic metric.

Let S be a compact surface of genus $g \geq 2$ and X be a structure of Riemann surface on S. Given a representation $\rho : \pi_1(S) \to \mathrm{SL}(d, \mathbb{R})$, we can define the Lyapunov exponents $\lambda_1(X, \rho) \geq \cdots \geq \lambda_d(X, \rho)$ of ρ with respect to X (see Subsection 4.1).

Computer experiments made by the first author hinted of a gap between the first and the second Lyapunov exponents for Hitchin representations. In this work, we can show the following more general statement.

Theorem 1.1. *Let X be a structure of Riemann surface on a compact surface S of genus $g \geq 2$ and $\rho : \pi_1(X) \to \mathrm{SL}(d, \mathbb{R})$ be a Hitchin representation. Then it holds*

$$\lambda_i(X, \rho) - \lambda_{i+1}(X, \rho) \geq 1$$

for every $i = 1, \ldots, d-1$.

Moreover the bound is attained for every $i = 1, \ldots, d-1$ if and only if ρ is conjugated to the image of the Fuchsian representation uniformizing X under the irreducible representation $\mathrm{SL}(2, \mathbb{R}) \to \mathrm{SL}(d, \mathbb{R})$.

The previous result follows from a more general statement regarding a specific class of Anosov representations, the (1,1,2)-hyperconvex Anosov representations. They are Anosov representations whose action on their limit set exhibit a form of asymptotic conformality (see Subsection 4.2).

Theorem 1.2. *Let X be a structure of Riemann surface on a compact surface S of genus $g \geq 2$ and let $\rho : \pi_1(X) \to \mathrm{SL}(d, \mathbb{R})$ be a (1,1,2)-hyperconvex Anosov representation. Then it holds*

$$\lambda_1(X, \rho) - \lambda_2(X, \rho) \geq 1.$$

We give two proofs of the previous result, both being a consequence of the relation we show between Lyapunov exponents and other characteristic numbers (see Theorem 5.1).

The first proof relies on the study of a deformation of the weak unstable foliation of the geodesic flow on the unitary tangent bundle of X and it is a consequence of a more general statement about the transverse Lyapunov exponent of this foliation (see Section 2). The second proof is more concise but it uses the machinery of the thermodynamic formalism for Anosov representations developed in [5] and [6] (see Section 3). It is however only in this context where we can show that the equalities of Theorem 1.1 characterizes the uniformizing representation.

These two approaches fit naturally in the perspective that Tholozan developed in his note [26]. He explains that there is a correspondence between *Anosov actions on the circle*, *deformations of the weak unstable foliation of the geodesic flow*, and *reparametrizations of the geodesic flow*. Furthermore this correspondence preserves the (appropriately defined) periods of each object. The two proofs we provide here consist of considering the Anosov action on the circle given by an Anosov representation and interpreting it in one case as a deformation of the weak unstable foliation and in the other case as a reparametrization of the geodesic flow. It is however interesting to notice that the two proofs are not a simple translation of each other. It is probable that there exists a third proof using directly the Anosov action on the circle which would use a random walk discretizing the geodesic flow and an adapation of Ledrappier formula [22] in this context.

Organization. In Section 2 we define the transverse Lyapunov exponent associated to a foliation Hölder isomorphic to the unstable foliation on T^1X and prove the main bound for this quantity.

In Section 3 we recall the setting of the thermodynamic formalism developed by Bridgeman, Canary, Labourie and Sambarino in [5] and [6] (see also [8]), in particular the main bound about the renormalized intersection.

In Section 4 we recall the main definitions of Lyapunov exponents via Oseledets Theorem and the properties of Anosov and Hitchin representations. We also show some implications that being Anosov has on the Lyapunov exponents.

In Section 5 we show the relation between Lyapunov exponents, the foliated Lyapunov exponent and the theormodynamic formalism. We finally prove the main results Theorem 1.1 and Theorem 1.2.

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2. Transverse Lyapunov exponent

In this section we provide a general setting to study a new invariant measuring the growth of the holonomy of a reparametrization of the weak unstable foliation of the geodesic flow on a Riemann surface. We call this invariant the *transverse Lyapunov exponent* and we will relate it to the usual Lyapunov exponents in the case where the reparametrization of the foliation is induced by a $(1,1,2)$-hyperconvex Anosov representation.

2.1. **Setup and notation.** Let S be a compact surface and X be a Riemann surface structure on S. We denote by T^1X the unitary tangent bundle of X, by $\pi : T^1X \to X$ the projection and by (Ψ_t) the geodesic flow on T^1X. We also consider the the weak unstable foliation $\mathcal{F}_u$ of the geodesic flow on T^1X. We finally equip T^1X with the Liouville volume form v_L, normalized to be a probability measure.

From now on we say that a function is $C^{k+\text{Hölder}}$ if it has continuous derivatives up through order k and such that the k-th partial derivatives are Hölder continuous for some exponent β, where $0 < \beta \leq 1$.

Let now M be a $C^{1+\text{Hölder}}$-manifold which is Hölder isomorphic to T^1X. Let us call $\Xi : T^1X \overset{\cong}{\to} M$ the isomorphism and assume that Ξ is C^1 along the leaves of $\mathcal{F}_u$. Denote by $\mathcal{F}_u^M$ the foliation on M induced by Ξ and assume that this foliation is $C^{1+\text{Hölder}}$. Denote also by $\Psi_t^M = \Xi \circ \Psi_t \circ \Xi^{-1}$ the flow induced on M. Note that $\mathcal{F}_u^M$ is by definition the weak unstable foliation of Ψ_t^M, and this flow is Hölder and C^1 along the leaves of $\mathcal{F}_u^M$.

Remark 2.1. This setting may seem arbitrary but it is exactly the setting we are in when we deform the conformal class of the weak unstable foliation of Ψ_t on T^1X, c.f. [26].

Let us finally define a volume form ν on M in the following way. Since the foliation is transversly oriented, we can fix an arbitrary 1-form α on M which defines the foliation, meaning that its kernel defines the tangent bundle of $\mathcal{F}_u^M$. The form α induces a metric on the normal bundle of $\mathcal{F}_u^M$ and we define a volume form by $\nu = \alpha \wedge \pi^{M,*}\omega_P$, where ω_P is the Poincaré metric on the surface X and $\pi^M : M \to X$ the projection $\pi^M := \pi \circ \Xi^{-1}$. Up to normalizing α we can assume that ν induces a probability measure.

Remark 2.2. Note that that T^1X is equipped with the Liouville measure v_L and M with the volume form ν. The measures ν and $\Xi_* v_L$ are in general mutually singular, but the disintegrations of ν and $\Xi_* v_L$ along the leaves of $\mathcal{F}_u^M$ are both equal to the Poincaré leafwise measure $\pi^{M,*}\omega_P$.

2.2. **The definition of the transverse Lyapunov exponent.** We will now define an invariant of the flow Ψ_t^M and the foliation $\mathcal{F}_u^M$, which we call the *transverse Lyapunov exponent*, which measures the asymptotic growth of the norm transverse to $\mathcal{F}_u^M$ of vectors under the flow.

In order to define these characteristic numbers, we would like to apply directly ergodic theory machinery, but the subtlety here is that the measure ν is not preserved by the flow. We hence use the Liouville measure as an accessory, and then show that we can compute the foliated Lyapunov exponent for almost every point with respect to ν.

Given a path $c : [0, t] \to M$ contained in a leaf, we define $|D\text{hol}(c)|_\alpha$ as the norm of the derivative of the holonomy of this path along the foliation $\mathcal{F}_u^M$. More precisely, the derivative of the holonomy induces a linear map between the one-dimensional fibers of the normal bundle $N_{\mathcal{F}_u^M, c(0)}$ and $N_{\mathcal{F}_u^M, c(t)}$, and we define $|D\text{hol}(c)|_\alpha$ as the norm of this linear map for the metric on $N_{\mathcal{F}_u^M}$ induced by α. Note that $|D\text{hol}(c)|_\alpha$ depends on our choice of α.

The local expression of $|D\text{hol}(c)|_\alpha$ can be described in the following way. Suppose that the image of c is contained in a chart V which trivializes the foliation, i.e. $V \simeq U \times I$ where U is a disk in $\mathbb{H}^2$ and I is an interval. In this chart the measure ν can be written as $f_\alpha(z, x)\omega_P(dz)dx$, so

$$|D\text{hol}(c)|_\alpha = f_\alpha(c(t))/f_\alpha(c(0)). \tag{1}$$

Let $\Psi_{[0,t]}^M(x)$ be the path $s \mapsto \Psi_s^M(x)$ defined on $[0, t]$.

Theorem 2.3. *There exists a number λ_T such that for every leaf L of $\mathcal{F}_u^M$ and for $\pi^{M,*}\omega_P$ almost every $x \in L$ we have*

$$\lim_{t \to +\infty} \frac{1}{t} \log \left| D\text{hol}\left(\Psi_{[0,t]}^M(x)\right)\right|_\alpha = \lambda_T.$$

We call the number λ_T the *transverse Lyapunov exponent.*

Remark 2.4. The transverse Lyapunov exponent λ_T is independent of the transverse form α. Indeed, since M is compact, any two norms are uniformly bounded away from each other.

Proof. For any $x \in T^1X$, we set $H_t(x) = \log|D\mathrm{hol}(\Psi^M_{[0,t]}(\Xi(x)))|_\alpha$. This is a cocycle over the geodesic flow on T^1X. As the geodesic flow on T^1X is ergodic with respect to the Liouville measure, there exists a number λ_T such that, for v_L-almost every $x \in T^1X$, we have

$$\lim_{t\to+\infty} \frac{1}{t} H_t(x) = \lambda_T.$$

Let us denote by G the set of "good points", that is those for which the above equality holds. We know that it is a set of full v_L measure, and it is invariant under the flow. We are going to show that for *every* leaf L of $\mathcal{F}_u$, the set $G \cap L$ is of full measure. This implies the main statement, as Ξ maps leaves of $\mathcal{F}_u$ to leaves of $\mathcal{F}^M_u$ and by Remark 2.2 it maps full measure subsets to full measure subsets.

First, it is clear that there exists one leaf L_0 for which this is true. Then we are going to show that G is a set foliated by the *strong stable* foliation $\mathcal{F}_{ss}$, that is the 1-dimensional foliation such that a leaf through a point x is the set of y's such that $d(\Psi_t(x), \Psi_t(y))$ goes to 0 exponentially fast. This will imply the result. Indeed, starting from the leaf L_0 such that $G \cap L_0$ is of full measure, consider another leaf L of $\mathcal{F}_u$. Consider the first return map of the horocyclic flow $L_0 \to L$. Because the strong stable and unstable foliation are transverse to each other, this is a smooth map and it is surjective. It sends the set of full measure $L_0 \cap G$ to a set of full measure which is included in $L \cap G$.

Let's finally show that the set G is foliated by the strong stable foliation. The holonomy of the foliation $\mathcal{F}^M_u$ is $C^{1+\text{Hölder}}$. In particular, the derivative of the holonomy along a geodesic of length 1 is *uniformly* β-Hölder, for some $\beta > 0$. Let us denote by $\|\cdot\|_{C^\beta}$ the β-Hölder norm

$$\|f\|_{C^\beta} := \sup_{x,y\in T^1X} \frac{|f(x) - f(y)|}{d(x,y)^\beta}$$

where $f : T^1X \to \mathbb{R}$ and $d(\cdot,\cdot)$ is a distance in T^1X (any two distances are equivalent since T^1X is compact). Then, by compactness of M, there exists a constant $C > 0$ such that

$$\|x \mapsto H_1(x)\|_{C^\beta} \leq C.$$

Let x and y be in the same leaf of the strong stable foliation. We show now that x and y belong to G. We denote $z_k = \Psi_k(z)$. We have then

$$\|H_n(x) - H_n(y)\|_{C^\beta} \leq \sum_{k=0}^{n-1} \|H_1(x_k) - H_1(y_k)\|_{C^\beta} \leq C \sum_{k=0}^{n-1} d(x_k, y_k)^\beta \leq C \sum_{k=0}^{\infty} d(x_k, y_k)^\beta,$$

and the last sum is convergent because x_k and y_k become exponentially close. This implies that $\lim_{n\to\infty} H_n(x)/n = \lim_{n\to\infty} H_n(y)/n$, which is what we wanted to show. □

The conclusion of the theorem implies that the limit defining λ_T exists for ν-almost every $x \in M$, because $\Xi_* v_L$ and ν are absolutely continuous along the leaves.

The manifold M is a foliated fiber bundle $\pi^M : M \to X$, where $\pi^M = \pi \circ \Xi^{-1}$. Hence, the transverse Lyapunov exponent can be expressed using the fibered structure. The flow Ψ^M_t induces a projective transformation between fibers $\Psi^M_{t,x} : M_{\pi^M(x)} \to M_{\pi^M\left(\Psi^M_t(x)\right)}$ for each $x \in M$. For each $\zeta \in M_{\pi^M(x)}$, we denote by $D_\zeta \Psi^M_{t,x}$ the derivative of $\Psi^M_{t,x}$ at the point ζ. Then we have

$$\lambda_T = \lim_{t\to+\infty} \frac{1}{t} \log\|D_\zeta \Psi^M_{t,x}\| \tag{2}$$

for almost every x in every leaf, where ζ is the point in $M_{\pi^M(x)}$ corresponding to x and for any choice of norm on the tangent bundle to the fibers. This follows from the the fact that the holonomy along paths of the form $\Psi^M_{[0,t]}(x)$ is given by the maps $\Psi^M_{t,x}$ between fibers and the tangent bundle to the fibers are identified with the normal bundle of the foliation.

2.3. **Bound on the transverse Lyapunov exponent.** We will now show the main estimate for the transverse Lyapunov exponent.

Theorem 2.5. *The transverse Lyapunov exponent satisfies $\lambda_T \leq -1$.*

The previous theorem is a consequence of the following lemma which explains how the measure ν is transformed by the flow.

Lemma 2.6. *The transformation of the measure ν under the flow on M is given by*

$$(\Psi^M_{-t})_*\nu(dx) = e^t |D\mathrm{hol}\Psi^M_{[0,t]}(x)|\nu(dx)$$

for any positive t.

Proof. First of all note that, since Ψ^M_t is not smooth, we cannot work directly with forms and we can only argue with measures. Let now A be a set contained in a foliation chart $V \simeq U \times I$ stable under Ψ^M_t and such that in this chart $A \simeq U' \times I'$. To prove the result it is enough to compute $(\Psi^M_{-t})_*\nu(A)$ for all such A's, since such sets cover M.

On every plaque $U \times \{\zeta\}$, the flow Ψ^M_t induces a flow $\Psi^{M,\zeta}_t$ and we have $\Psi^M_t(U' \times I') = \left\{(\Psi^{M,\zeta}_t(z), \zeta); (z,\zeta) \in U' \times I'\right\}$. Recall that the measure ν in the chart V can be written as $f_\alpha(z,\zeta)\omega_P(dz)d\zeta$. Because Ψ_t is conjugated to the geodesic flow on T^1X by Ξ, and Ξ is smooth and measure preserving along the leaves, we have

$$(\Psi^{M,\zeta}_t)^*\omega_P = e^t \omega_P.$$

We can finally compute

$$\begin{aligned}
(\Psi^M_{-t})_*\nu(A) = \nu(\Psi^M_t(A)) &= \int_{\Psi^M_t(U'\times I')} f_\alpha(z,\zeta)\omega_P(dz)d\zeta \\
&= \int_{I'}\int_{\Psi^{M,\zeta}_t(U')} f_\alpha(z,\zeta)\omega_P(dz)d\zeta \\
&= \int_{I'}\int_{U'} f_\alpha(\Psi^{M,\zeta}_t(z),\zeta) e^t \omega_P(dz)d\zeta \\
&= \int_{I'}\int_{U'} \frac{f_\alpha(\Psi^{M,\zeta}_t(z),\zeta)}{f_\alpha(z,\zeta)} e^t f_\alpha(z,\zeta)\omega_P(dz)d\zeta \\
&= \int_{U'\times I'} e^t |D\mathrm{hol}\Psi^M_{[0,t]}(z,\zeta)| f_\alpha(z,\zeta)\omega_P(dz)d\zeta,
\end{aligned}$$

where the last equality follows from the local expression (1) of the norm of the holonomy. This is exactly what we wanted to prove. $\square$

We can prove now prove Theorem 2.5 about the main estimate for the transverse Lyapunov exponent. The main idea is that the flow Ψ^M_t is expanding along the leaves of $\mathcal{F}^M_u$ while the total volume of ν remains constant.

Proof of theorem 2.5. Using Lemma 2.6 we obtain

$$1 = \int_M \nu = \int_M (\Psi^M_{-t})_* \nu = \int_M e^t |D\mathrm{hol}\Psi^M_{[0,t]}(x)| \nu(dx).$$

Applying the logarithm function and using Jensen's inequality we get

$$\int_M (t + \log|D\mathrm{hol}\Psi^M_{[0,t]}(x)|)\nu(dx) \leq 0.$$

Finally we can divide by t and obtain

$$1 + \int_M \frac{1}{t} \log|D\mathrm{hol}\Psi^M_{[0,t]}(x)|\nu(dx) \leq 0.$$

Now, by theorem 2.3 the integrand converges to λ_T for ν-almost every x and it is bounded uniformly in t, by continuity of the holonomy. By the dominated convergence theorem, the integral converges then to λ_T and so we have the desired estimate

$$1 + \lambda_T \leq 0.$$

□

3. Thermodynamic formalism

In this section, we recall the setting of the thermodynamic formalism developed by Bridgeman, Canary, Labourie and Sambarino in [6] and [5] (see also [8]). In the next section, we will relate it to Lyapunov exponents and use it to prove Theorem 1.1.

This thermodynamic formalism is a machinery which allows to encode quantities associated to Anosov representations using reparametrisations of the geodesic flow.

We recall now the notions and results we will need for our purposes, and refer to [6] for all the precise definitions and proofs.

Let as above X be a Riemann surface, T^1X be its unit tangent bundle and Ψ_t be the geodesic flow on it (note that in [6] and [5], they work with the more general geodesic flow of the group $\pi_1(X)$). We denote by O the set of periodic orbits of Ψ_t and, for any $a \in O$, we denote by $p(a)$ the period of a.

Given a positive Hölder continuous function f on T^1X, there exists a reparametrisation Ψ^f of the flow Ψ such that, for every periodic orbit $a \in O$, the period of a for the flow Ψ^f is given by

$$p_f(a) = \int_0^{p(a)} f(\Psi_s(x))ds,$$

where x is any point in a. The flow Ψ^f is only Hölder continuous but it is a Metric Anosov flow (see [6, sec. 3.2]).

Denote by $R_T(f)$ the set $\{a \in O \mid p_f(a) \leq T\}$. The topological entropy of the flow Ψ^f is given by

$$h_f = \lim_{T\to+\infty} \frac{1}{T} \log \#R_T(f),$$

and it is finite and positive. The unique probability measure of maximal entropy μ_f for Ψ^f is given by

$$\mu_f = \lim_{T\to+\infty} \frac{1}{\#R_T(f)} \sum_{a\in R_T(f)} \delta^f_a,$$

where δ^f_a is the probality measure supported on a and invariant by the flow Ψ^f.

Given two positive Hölder continuous functions f and g on T^1X, we define their intersection $I(f,g)$ by

$$I(f,g) = \int_{T^1X} \frac{g}{f} d\mu_f,$$

and their renormalized intersection $J(f,g)$ by

$$J(f,g) = \frac{h_g}{h_f} I(f,g).$$

The Hessian of the renormalized intersection was used in [6] to define a pressure form on the set of pressure zero Hölder functions, generalizing the Weil-Peterson form.

We finally recall an important estimate of the renormalized intersection. Recall that f and g are Livsic cohomologous when the flows Ψ^f and Ψ^g are Hölder conjugate. In that case these two flows have the same periods.

Proposition 3.1. *[6, Prop. 3.8])] The renormalized intersection satifisfies the lower bound:*

$$J(f,g) \geq 1,$$

with equality if and only if $h_f f$ and $h_g g$ are Livsic cohomologous.

4. Lyapunov exponents and Anosov and Hitchin representations

In this section we recall the definition of Lyapunov exponents associated to a representation of the fundamental group of a Riemann surface and investigate the special cases of Anosov and, more specially, Hitchin representations.

4.1. **Oseledets theorem and Lyapunov exponents.** Let X be a compact Riemann surface of genus greater than one and T^1X be its unit cotangent bundle equipped with the Liouville probability measure. Let moreover $\rho : \pi_1(X) \to \mathrm{SL}(d,\mathbb{R})$ be a representation and E_ρ be the associated flat vector bundle over T^1X, i.e. the quotient of $T^1\tilde{X} \times \mathbb{R}^d$ by the diagonal action of $\pi_1(X)$, acting on the second factor by ρ. The geodesic flow Ψ_t on T^1X induces a flow $\widetilde{\Psi}_t$ on E_ρ by parallel transport. We finally equip E_ρ with an arbitrary measurable norm $\|\cdot\|$ (here measurable, and in particular defined up to measure zero sets, is enough since, similarly to Remark 2.4, we can use the Poincaré recurrence Theorem to work on a compact subset where the norm is defined).

We define now the Lyapunov exponents of ρ with respect to X by applying the theorem of Oseledets (see e.g. [1]) to the linear flow $\widetilde{\Psi}_t$ lying over the ergodic flow Ψ_t.

Theorem 4.1 (Oseledets)**.** *There exist real constants $\tilde{\lambda}_1 > \cdots > \tilde{\lambda}_r$ and a decomposition*

$$E_\rho = \bigoplus_{i=1}^{r} E_\rho^i$$

by measurable real vector bundles such that for a.e. $x \in T^1X$ and all $v \in (E_\rho^i)_x \setminus \{0\}$, it holds

$$\tilde{\lambda}_i = \pm \lim_{t\to+\infty} \frac{1}{t} \log\|\widetilde{\Psi}_{\pm t} v\|.$$

Moreover, for all $i \neq j$, we have

$$\lim_{t\to+\infty} \frac{1}{t} \log d((E_\rho^i)_{\Psi_t x}, (E_\rho^j)_{\Psi_t x}) = 0, \tag{3}$$

where d is the Hausdorff distance on the compact subset of the projective space.

The set of values $\lambda_i = \lambda_i(X, \rho)$, for $i = 1, \dots, d$, obtained by considering the values $\tilde{\lambda}_i$ *repeated with multiplicity* $\dim E^i_\rho$, is called the set of *Lyapunov exponents* or Lyapunov spectrum of (X, ρ). Note that the Lyapunov exponents are independent of the choice of the norm function on E_ρ.

Remark 4.2. Lyapunov exponents can be equivalently defined as

$$\lambda_i = \lim_{t \to +\infty} \frac{1}{t} \log \sigma_i(\widetilde{\Psi}_t),$$

where σ_i are the singular values (defined in Subsection 4.2). We will use this characterization when we will relate Lyapunov exponents to other quantities in Section 5.

We finally recall that we can associate two flags to the decomposition of E_ρ, *the forward flag* $F^1_\rho \subset F^2_\rho \subset \cdots \subset F^r_\rho$ and the *backward flag* $B^1_\rho \subset B^2_\rho \subset \cdots \subset B^r_\rho$ defined by

- $F^i_\rho = E^{r+1-i}_\rho \oplus \cdots \oplus E^r_\rho$,
- $B^i_\rho = E^1_\rho \oplus \cdots \oplus E^i_\rho$.

These measurable bundles satisfy the following properties. For almost any $x \in T^1X$ it holds:

- $(B^i_\rho)_x \cap (F^{d+1-i}_\rho)_x = (E^i_\rho)_x$,
- $\lim_{t\to+\infty} \frac{1}{t} \log\|\widetilde{\Psi}_t v\| = \tilde{\lambda}_{r+1-i}$ if and only if $v \in (F^i_\rho)_x \setminus (F^{i-1}_\rho)_x$,
- $\lim_{t\to+\infty} \frac{1}{t} \log\|\widetilde{\Psi}_{-t} v\| = -\tilde{\lambda}_i$ if and only if $v \in (B^i_\rho)_x \setminus (B^{i-1}_\rho)_x$.

Note that all the measurable bundles E^i_ρ, B^i anf F^i are equivariant with respect to the action of $\widetilde{\Psi}_t$. We say that a point $x \in T^1X$ is called *regular* is it is a point for which the previous properties hold.

4.2. **Anosov and Hitchin representations.** We consider now special representations $\rho : \pi_1(X) \to \mathrm{SL}(d, \mathbb{R})$. The notion of Anosov representations of fundamental groups of hyperbolic manifolds has been introduced by Labourie in [21]. Since then, it has been generalized to general hyperbolic groups and different equivalent definitions have been found [16], [15], [18], [19], [4]. Here we state a definition adapted to what is needed in the following.

Let $|\cdot|$ be word metric on $\pi_1(X)$ associated to the choice of an arbitrary symmetric generating set. Fix a Euclidean norm $\|\cdot\|$ on $\mathrm{SL}(d, \mathbb{R})$. For a matrix $g \in \mathrm{SL}(d, \mathbb{R})$, denote by $\sigma_1(g) \geq \cdots \geq \sigma_d(g)$ its singular values, i.e. the eigenvalues of $\sqrt{g^*g}$ defined using this norm. Remark that $\|g\| = \sigma_1(g)$ and $\|\wedge^k g\| = \sigma_1(g) + \cdots + \sigma_k(g)$, where $\wedge^k g$ is the automorphism of the exterior power $\wedge^k \mathbb{R}^d$ induced by g.

The following definition is from [4]. For any $p = 1, \dots, d$, we say that a representation $\rho : \pi_1(X) \to \mathrm{SL}(d, \mathbb{R})$ is *p-Anosov* if there exist constants $C, \lambda > 0$ such that

$$\frac{\sigma_{p+1}(\rho(\gamma))}{\sigma_p(\rho(\gamma))} \leq Ce^{-\lambda|\gamma|},$$

for every γ in $\pi_1(X)$. This property is independent of the word metric chosen. Note that by substituting γ^{-1} to γ in the previous expression, it is easy to show that a representation is p-Anosov if and only if it is $(d-p)$-Anosov and that $\wedge^p \rho$ is 1-Anosov if and only if ρ is p-Anosov.

Recall that the boundary $\partial_\infty \pi_1(X)$ of the group $\pi_1(X)$ is a topological circle with a Hölder structure. We recall now one important property of p-Anosov representations .

Theorem 4.3 ([4, Prop. 4.9])**.** *Let $\rho : \pi_1(X) \to \mathrm{SL}(d, \mathbb{R})$ be a p-Anosov representation. There exist two ρ-equivariant Hölder continuous maps $\xi_p : \partial_\infty \pi_1(X) \to G_p$ and ξ_{d-p} :*

$\partial_\infty \pi_1(X) \to G_{d-p}$, where G_i is the Grassmanian of i-planes in $\mathbb{R}^d$, satisfying

$$\xi_p(x) \oplus \xi_{d-p}(y) = \mathbb{R}^d,$$

for every $x \neq y$ in $\partial_\infty \pi_1(X)$.

The maps ξ_i are called the *boundary maps* of ρ.

We are now able to recall the definition of hyperconvex representations following [24]. These representations are the ones for which the bound of Theorem 1.2 holds.

Definition 4.4. Let $\rho : \pi_1(X) \to \mathrm{SL}(d, \mathbb{R})$ be both 1-Anosov and 2-Anosov. We say that ρ is $(1, 1, 2)$-hyperconvex if for every pairwise distinct x, y, z in $\partial_\infty \pi_1(X)$ it holds

$$(\xi_1(x) \oplus \xi_1(y)) \cap \xi_{d-2}(z) = \{0\},$$

Remark that we always have $\xi_1(x) \subset \xi_2(x)$ when ρ is 1-Anosov and 2-Anosov [16]. The main property of $(1, 1, 2)$-hyperconvex that we will use in order to relate the Lyapunov exponents given by Oseldets theorem to the transverse Lyapunov exponents and to the thermodynamic formalism is the following.

Theorem 4.5 ([24, Prop. 7.4])**.** *Let $\rho : \pi_1(X) \to \mathrm{SL}(d, \mathbb{R})$ be a $(1, 1, 2)$-hyperconvex representation. Then the image $\xi_1(\partial_\infty \pi_1(X))$ is a $C^{1+\text{Hölder}}$ manifold and its tangent space at $\xi_1(x)$ can be described as*

$$T_{\xi_1(x)} \xi_1(\partial_\infty \pi_1(X)) = T_{\xi_1(x)} \mathbb{P} \xi_2(x).$$

Note that even if the image of ξ_1 is $C^{1+\text{Hölder}}$, the map ξ_1 itself is only Hölder.

We recall now the definition and properties of Hitchin representations, which are a special instance of Anosov representations. Consider the connected component of the character variety of representations $\pi_1(X) \to \mathrm{PSL}(d, \mathbb{R})$ containing the Fuchsian representations (via the irreducible embeding $\mathrm{PSL}(2, \mathbb{R}) \to \mathrm{PSL}(d, \mathbb{R})$). This component is called Hitchin component and the representations parametrized by this component are called Hitchin representations. They have been introduced by Hitchin in [17] and are central in the study of higher Teichmüller theory (see e.g. [27]). A Hitchin representation $\pi_1(X) \to \mathrm{PSL}(d, \mathbb{R})$ can be lifted to a representation $\rho : \pi_1(X) \to \mathrm{SL}(d, \mathbb{R})$ and the properties we are interested in are independent of the lift, so in the following we will only work with representation with values in $\mathrm{SL}(d, \mathbb{R})$.

We recall now the main properties of Hitchin representation that we will use.

Theorem 4.6. *Let $\rho : \pi_1(X) \to \mathrm{SL}(d, \mathbb{R})$ be a Hitchin representation. Then*

- *ρ is i-Anosov, for every $i = 1, \dots, d-1$ (see [21]);*
- *$\wedge^k \rho : \pi_1(X) \to \mathrm{SL}(\wedge^k \mathbb{R}^d)$ is $(1, 1, 2)$-hyperconvex for any $k = 1, \dots, d-1$ (see [24, sec. 9.2]).*

4.3. **Oseledets and Anosov representations.** Assume now that $\rho : \pi_1(X) \to \mathrm{SL}(d, \mathbb{R})$ is a p-Anosov representation. We explain some consequences that this property has on the Lyapunov exponents given by the Theorem of Oseledets. First we explain the relationship between the notion of dominated splitting and the Anosov property.

Consider a continous flow (φ_t) on a compact space B. Suppose that this flow lifts to a flow $(\widetilde{\varphi}_t)$ on a bundle $E \to B$ over B . A splitting $E = U \oplus S$ of E is said to be dominated for $(\widetilde{\varphi}_t)$ if there exists constants $C, a > 0$ such that:

$$\frac{\|\widetilde{\varphi}_t v\|}{\|v\|} \leq C \frac{\|\widetilde{\varphi}_t w\|}{\|w\|} e^{-at},$$

for every $v \in S$ and $w \in U$, see [3]. In this case we say that $(\widetilde{\varphi}_t)$ admits a dominated splitting of index $\dim U$.

When ρ is p-Anosov, we can construct such a splitting of the bundle E_ρ. First we construct a splitting of the bundle $T^1\widetilde{X} \times \mathbb{R}^d$: to a point $x \in T^1\widetilde{X}$ we associated the splitting

$$\xi_p(x_{-\infty}) \oplus \xi_{d-p}(x_{+\infty})$$

of the fiber $\mathbb{R}^d$. By equivariance of ξ_p and ξ_{d-p}, this splitting descends to a splitting of E_ρ that we denote by $U \oplus S$. Observe that this splitting is invariant under the lift $\widetilde{\Psi}_t$ of the geodesic flow on T^1X. According to [4, prop. 4.6, prop. 4.9], ρ being p-Anosov is equivalent to this splitting being dominated for $(\widetilde{\Psi}_t)$. In particular $(\widetilde{\Psi}_t)$ is dominated of index p.

We are now able to show how the property of the dominated splitting of E_ρ implies inequalities between Lyapunov exponents.

Proposition 4.7. *Let $\rho : \pi_1(X) \to \mathrm{SL}(d, \mathbb{R})$ be p-Anosov. Then*

$$\lambda_i > \lambda_{i+1}$$

for $i = p, d-p$. Equivalently

$$\dim E_\rho^p = \dim E_\rho^{d-p} = 1.$$

Proof. As explained above, the flow $(\widetilde{\Psi_t})$ admits a dominated splitting of index p because ρ is p-Anosov. By [3, th. A], this implies that $\frac{\sigma_{p+1}}{\sigma_p}(\widetilde{\Psi_t})$ uniformly decreases to 0 exponentially fast.

Since by Remark 4.2 we have $\lambda_p = \lim \frac{1}{t} \log \sigma_p(\widetilde{\Psi_t})$, this implies that $\lambda_p > \lambda_{p+1}$. As p-Anosov implies $d-p$-Anosov, we also have $\lambda_{d-p} > \lambda_{d-p+1}$. □

We finally relate the boundary maps $\xi_i : \partial_\infty \pi_1(X) \to G_i$ associated to a i-Anosov representation ρ given by Theorem 4.3 and the forward and backward flags F_ρ^i and B_ρ^i given by the theorem of Oseledets (see Subsection 4.1). Note that we can identify $\mathbb{S}^1$ and $\partial_\infty \pi_1(X)$ using the Fuchsian representation $j_X : \pi_1(X) \to \mathrm{SL}(2, \mathbb{R})$ inducing the complex structure X on a surface S. Let now $x \in T^1X$ be a regular point, i.e., a point for which the Oseledets decomposition and the Oseledets flags are defined, and let $x_{+\infty} \in \mathbb{S}^1$ and $x_{-\infty} \in \mathbb{S}^1$ be the boundary points in the future and in the past of the geodesic defined by x.

Proposition 4.8. *Let $\rho : \pi_1(X) \to \mathrm{SL}(d, \mathbb{R})$ be p-Anosov. Then we have the relations*

$$\xi_i(x_{+\infty}) = (F_\rho^i)_x \quad \text{and} \quad \xi_i(x_{-\infty}) = (B_\rho^i)_x$$

for $i = p, d-p$ and for any regular point $x \in T^1X$.

Proof. Consider the dominated splitting $E_\rho = U \oplus S$ associated to the the representation ρ and recall that $U_x = \xi_p(x_{-\infty})$ and $S_x = \xi_{d-p}(x_{+\infty})$.

In the proof of [3, Th. A] it is shown that, when $E_\rho = U \oplus S$ is dominated for $(\widetilde{\Psi}_t)$, then for a generic x we have

$$U_x = (B_\rho^p)_x \text{ and } S_x = (F_\rho^{d-p})_x,$$

where B_ρ and F_ρ are the backward and forward flags given by Oseledets theorem.

Combining these two facts we have that

$$\xi_p(x_{-\infty}) = (B_\rho^p)_x \text{ and } \xi_{d-p}(x_{+\infty}) = (F_\rho^{d-p})_x$$

for any regular point $x \in T^1X$. Finally, since if ρ is p-Anosov it is also $d-p$-Anosov, we also have

$$\xi_{d-p}(x_{-\infty}) = (B_\rho^{d-p})_x \text{ and } \xi_p(x_{+\infty}) = (F_\rho^p)_x.$$

□

5. Relationships and proofs of main gap theorem

In this section we prove the results about the gaps between Lyapunov exponents stated in Theorem 1.1 and Theorem 1.2 in two independent ways. The main idea is to relate the Lyapunov exponents given by Oseledets Theorem in the case of $(1,1,2)$-hyperconvex representations to other invariants for which we can prove an inequality statement. The first approach relates the Lyapunov exponents to the transverse Lyapunov exponent defined in Section 2 and uses the inequality obtained in Theorem 2.5, while in the second approach we relate them to the thermodynamic formalism described in Section 3 and we use the inequality of Proposition 3.1.

Let X be a Riemann surface and $\rho : \pi_1(X) \to \mathrm{SL}(d,\mathbb{R})$ be a (1,1,2)-hyperconvex Anosov representation. Let first $\lambda_i(X,\rho)$ be the Lyapunov exponents given by Oseledets Theorem as in Subsection 4.1. Let then $\lambda_T(M_\rho)$ be the transverse Lyapunov exponent, as in Section 2, associated to the manifold M_ρ defined below in Subsection 5.1. Let finally $J(\mathbf{1}, f_\rho)$ be the renormalized intersection as defined in Section 3 of the constant function $\mathbf{1}$ and the function f_ρ asssociated to ρ as defined below in Subsection 5.2. The main relations we want to show in this section are the following.

Theorem 5.1. *Let $\rho : \pi_1(X) \to \mathrm{SL}(d,\mathbb{R})$ be a (1,1,2)-hyperconvex Anosov representation. Then we have the relation*

$$\lambda_1(X,\rho) - \lambda_2(X,\rho) = -\lambda_T(M_\rho) = J(\mathbf{1}, f_\rho).$$

Using the previous Theorem 5.1 we are now able to prove the gap statement of Theorem 1.2

Proof of Theorem 1.2. If $\rho : \pi_1(X) \to \mathrm{SL}(d,\mathbb{R})$ is a (1,1,2)-hyperconvex Anosov representation, the inequality $\lambda_1(X,\rho) - \lambda_2(X,\rho) \geq 1$ follows from the relations expressed in Theorem 5.1 together with either the bound of Theorem 2.5 for the transverse Lyapunov exponent or the bound of Proposition 3.1 for the renormalized intersection. □

The proof of Theorem 1.1 is the content of Subsection 5.3.

5.1. **Relationship with transverse Lyapunov exponent.** As usual, let S be a compact surface and X be a Riemann surface structure on S. We denote by $j_X : \pi_1(X) \to \mathrm{SL}(2,\mathbb{R})$ the Fuchsian representation defining the Riemann surface structure. The Fuchsian representation j_X induces an identifications between the universal cover $\widetilde{S} = \widetilde{X}$ and the hyperbolic plane $\mathbb{H}^2$, and between the boundary $\partial_\infty \pi_1(X)$ of the fundamental group of X and the boundary at infinity $\mathbb{S}^1$ of $\mathbb{H}^2$. We will consider the isomorphism

$$f_u : T^1 X \xrightarrow{\cong} (\mathbb{H}^2 \times \mathbb{S}^1)/\pi_1(X)$$

where $\pi_1(X)$ acts diagonally by j_X. The isomorphism is given by associating to $(\widetilde{x}, v)$ in $T^1\widetilde{X}$ the point $(\widetilde{z}, \zeta) \in \mathbb{H}^2 \times \mathbb{S}^1$, where $\widetilde{z}$ is given by the uniformization map and ζ is the point in the boundary $\mathbb{S}^1$ reached by the geodesic defined by (x, v) when times goes to negative infinity. With this identification, the *horizontal foliation of this flat bundle is identified with the weak unstable foliation of the geodesic flow.*

Let now $\rho : \pi_1(X) \to \mathrm{SL}(2,\mathbb{R})$ be a $(1,1,2)$-hyperconvex representation. Then by Theorem 4.5 we know that the image $S_\rho := \xi_1(\partial_\infty \pi_1(X))$ is a $C^{1+\text{Hölder}}$-submanifold of $\mathbb{P}^{d-1}(\mathbb{R})$. Using j_X, we identify $\partial_\infty \pi_1(X)$ and $\mathbb{S}^1$ and so we obtain a map

$$\xi_1^X : \mathbb{S}^1 \longrightarrow S_\rho \subset \mathbb{P}^{d-1}(\mathbb{R}).$$

We finally construct the $C^{1+\text{Hölder}}$-manifold $M_\rho = (\mathbb{H}^2 \times S_\rho)/\pi_1(X)$ where $\pi_1(X)$ acts diagonally by $j_X \times \rho$.

The manifold M_ρ is Hölder-homeomorphic to T^1X via the map Ξ induced by the composition of f_u and the map $(\tilde{z}, \zeta) \mapsto (\tilde{z}, \xi_1^X(\zeta))$. We will denote by $\pi^\rho : M_\rho \to X$ the projection. The image $\mathcal{F}_u^{M_\rho}$ of the unstable foliation under Ξ is $C^{1+\text{Hölder}}$ and Ξ is C^1 along the leaves of the foliation. We are then in the same setting as Subsection 2.1 and we can define the transverse Lyapunov exponent associated to M_ρ.

Now that we have constructed M_ρ, we can now prove the relation stated in Theorem 5.1 between the Oseledets Lyapunov exponents associated to X and ρ and the transverse Lyapunov exponent associated to M_ρ.

Proposition 5.2. *If $\rho : \pi_1(X) \to \mathrm{SL}(d, \mathbb{R})$ is a (1,1,2)-hyperconvex Anosov representation, then $\lambda_T(M_\rho) = \lambda_2(X, \rho) - \lambda_1(X, \rho)$.*

Proof. We will use the notation of Section 2. We consider the definition of the transverse Lyapunov exponent via the expression (2), i.e.

$$\lambda_T = \lim_{t \to +\infty} \frac{1}{t} \log \| D_\zeta \Psi_{t,x}^{M_\rho} \|$$

for almost any $x \in M_\rho$ in every leaf, where recall that $\Psi_{t,x}^{M_\rho}$ is the projective linear transformation induced by the flow $\Psi_t^{M_\rho}$ on M_ρ between the vertical fibers of π^ρ and ζ is the projection of x to the vertical fiber. Fix a point $x \in M_\rho$ for which the previous limit holds. The point $x \in M_\rho$ corresponds to the point $x' = \Xi^{-1}(x)$ in T^1X.

Consider now the linear map $\widetilde{\Psi}_{t,x'} : (E_\rho)_{x'} \to (E_\rho)_{\Psi_t(x')}$, where E_ρ is the flat bundle associated to ρ as defined in Subsection 4.1 and $\widetilde{\Psi}_t$ is the lift of the geodesic flow to E_ρ.

The important remark here is that $\Psi_{t,x}^{M_\rho}$ is the projective linear transformation between the fibers $(M_\rho)_{\pi^\rho(x)}$ and $(M_\rho)_{\pi^\rho(\Psi_t^M(x))}$ induced by the restriction to S_ρ of the projectivication of $\widetilde{\Psi}_{t,x'}$, i.e.

$$\mathbb{P}\widetilde{\Psi}_{t,x'}{}_{|S_\rho} = \Psi_{t,x}^{M_\rho}.$$

Moreover, by definition, ζ is the point in $(M_\rho)_{\pi^\rho(x)}$ corresponding to x. By construction of M_ρ, we have then

$$\zeta = \xi_1^X(x'_{-\infty}),$$

where $x'_{-\infty} \in \mathbb{S}^1$ is the boundary point in the past of the geodesic defined by x'.

Recall finally that, since ρ is $(1,1,2)$-hyperconvex, by Proposition 4.7 and Proposition 4.8, we have

$$\xi_1^X(x'_{-\infty}) = (E_\rho^1)_{x'} \quad \text{and} \quad \xi_2^X(x'_{-\infty}) = (B_\rho^2)_{x'} = (E_\rho^1)_{x'} \oplus (E_\rho^2)_{x'}$$

and that moreover, by Theorem 4.5, we have

$$T_{\xi_1^X(s)} S_\rho = T_{\xi_1^X(s)} \mathbb{P}\xi_2^X(s).$$

Putting together the previous displayed expressions, we find

$$D_\zeta \Psi_{t,x}^{M_\rho} = D_{\xi_1^X(x'_{-\infty})} \Psi_{t,x}^{M_\rho} = D_{\mathbb{P}(E_\rho^1)_{x'}} \left(\mathbb{P}\widetilde{\Psi}_{t,x'}{}_{|S_\rho} \right) = D_{\mathbb{P}(E_\rho^1)_{x'}} \left(\mathbb{P}\widetilde{\Psi}_{t,x'} \right)_{|T_{\mathbb{P}(E_\rho^1)_{x'}} \mathbb{P}(B_\rho^2)_{x'}}$$

We choose $u \in (E_\rho^1)_{x'}$ and $v \in (E_\rho^2)_{x'}$ two non-zero vectors. Since we can equip $E_\rho = T^1\tilde{X} \times \mathbb{R}^d/\pi_1(X)$ with the measurable norm given by the constant norm on $\mathbb{R}^d$ (which is well-defined only up to a measurable zero set of discontinuity), we can apply the formula shown below in Lemma 5.3 and obtain

$$\| D_\zeta \Psi_{t,x}^{M_\rho} \| = \frac{\| \widetilde{\Psi}_{t,x'} u \wedge \widetilde{\Psi}_{t,x'} v \|}{\| u \wedge v \|} \left(\frac{\| \widetilde{\Psi}_{t,x'} u \|}{\| u \|} \right)^{-2}.$$

Remark that $\widetilde{\Psi}_{t,x'}u$ is in $(E^1_\rho)_{\Psi_t(x')}$ and $\widetilde{\Psi}_{t,x'}v$ is in $(E^2_\rho)_{\Psi_t(x')}$. Recall the estimate (3) on the angle between $(E^1_\rho)_{\Psi_t(x')}$ and $(E^2_\rho)_{\Psi_t(x')}$ given by Oseledets theorem 4.1:

$$\lim_{t\to+\infty}\frac{1}{t}\log d((E^1_\rho)_{\Psi_t(x')},(E^2_\rho)_{\Psi_t(x')})=0.$$

The previous expression implies

$$\log\|\widetilde{\Psi}_{t,x'}u\wedge\widetilde{\Psi}_{t,x'}v\|=\log(\|\widetilde{\Psi}_{t,x'}u\|\|\widetilde{\Psi}_{t,x'}v\|)+o(t),$$

since $d(L_1,L_2)=\|u\wedge v\|/\|u\|\|v\|$ for any two lines L_1,L_2 and non-zero vectors $u\in L_1$ and $v\in L_2$. We can then compute

$$\begin{aligned}\log\|D_\zeta\Psi^{M_\rho}_{t,x}\|&=\log\|\widetilde{\Psi}_{t,x'}u\|+\log\|\widetilde{\Psi}_{t,x'}v\|-2\log\|\widetilde{\Psi}_{t,x'}u\|+o(t)\\&=\log\|\widetilde{\Psi}_{t,x'}v\|-\log\|\widetilde{\Psi}_{t,x'}u\|+o(t).\end{aligned}$$

Taking the limit for $t\to\infty$ of the previous expression and using that

$$\frac{1}{t}\log\|\widetilde{\Psi}_{t,x'}u\|\to\lambda_1(X,\rho)\quad\text{and}\quad\frac{1}{t}\log\|\widetilde{\Psi}_{t,x'}v\|\to\lambda_2(X,\rho)$$

since $u\in(E^1_\rho)_{x'}\setminus\{0\}$ and $v\in(E^2_\rho)_{x'}\setminus\{0\}$, we have proved the desired relation. □

We finally give a self-contained proof of the formula used previously to compute the norm of the derivative of a projective linear transformation.

Lemma 5.3. *For a linear transformation $g:\mathbb{R}^d\to\mathbb{R}^d$, the operator norm of the restriction of the derivative of $\mathbb{P}g$ to a line $\mathbb{P}(\langle u,v\rangle)\cong\mathbb{P}^1$ at the point $\mathbb{P}(\langle u\rangle)$ is*

$$\|D_{\mathbb{P}(\langle u\rangle)}\mathbb{P}g_{|\mathbb{P}\langle u,v\rangle}\|=\frac{\|gu\wedge gv\|}{\|u\wedge v\|}\left(\frac{\|gu\|}{\|u\|}\right)^{-2}$$

where on the left hand side we consider the operator norm and on the right hand side the euclidean norm on $\mathbb{R}^d$.

Proof. For any point $L\in\mathbb{P}\mathbb{R}^d$ (identified with a line in $\mathbb{R}^d$), the tangent space of $\mathbb{P}\mathbb{R}^d$ at L is canonically identified with the space of linear maps $\mathrm{Hom}(L,\mathbb{R}^d/L)$.

The derivative $D_L\mathbb{P}g:T_L\mathbb{P}\mathbb{R}^d\to\mathbb{T}_{gL}P\mathbb{R}^d$ of this map at the point L is canonically identified with the map

$$\begin{aligned}\mathrm{Hom}(L,\mathbb{R}^d/L)&\to\mathrm{Hom}(gL,\mathbb{R}^d/gL)\\\varphi&\mapsto g\circ\varphi\circ g^{-1}.\end{aligned}$$

A choice of Euclidian norm $\|\cdot\|$ on $\mathbb{R}^d$ induces norms on $L,\mathbb{R}^d/L$ and $\mathrm{Hom}(L,\mathbb{R}^d/L)$, and this defines a metric on $T\mathbb{P}\mathbb{R}^d$.

A choice of vector $u\in L$ defines an isometry

$$\begin{aligned}\mathrm{Hom}(L,\mathbb{R}^d/L)&\to L\wedge\mathbb{R}^d\\\varphi&\mapsto\frac{u}{\|u\|}\wedge\varphi(u),\end{aligned}$$

where $L\wedge\mathbb{R}^d\subset\wedge^2\mathbb{R}^d$ is equiped with the norm on $\wedge^2\mathbb{R}^d$ induced by the norm on $\mathbb{R}^d$.

For $\varphi\in T_L\mathbb{P}\mathbb{R}^d=\mathrm{Hom}(L,\mathbb{R}^d/L)$ and any $u\in L\setminus\{0\}$ we have:

$$\|\varphi\|_{\mathrm{Hom}(L,\mathbb{R}^d/L)}=\frac{\|\varphi(u)\|_{\mathbb{R}^d/L}}{\|u\|}=\frac{\|u\wedge\varphi(u)\|_{\wedge^2\mathbb{R}^d}}{\|u\|^2},$$

and

$$\|D_L\mathbb{P}g(\varphi)\|_{\mathrm{Hom}(gL,\mathbb{R}^d/gL)}=\frac{\|g\varphi g^{-1}(gu)\|_{\mathbb{R}^d/gL}}{\|gu\|}=\frac{\|gu\wedge g\varphi(u)\|_{\wedge^2\mathbb{R}^d}}{\|gu\|^2}.$$

Now, given two non-collinear vectors $u, v \in \mathbb{R}^d$, the tangent space at $L = \mathbb{P}\langle u\rangle$ of $P = \mathbb{P}\langle u, v\rangle$ is generated by $\varphi : u \mapsto v \in \mathrm{Hom}(L, \mathbb{R}^d/L)$. By the previous equalities we have:

$$\|D_L \mathbb{P} g_{|P}\| = \frac{\|D_L \mathbb{P} g_{|P}(\varphi)\|}{\|\varphi\|} = \frac{\|gu \wedge gv\|}{\|u \wedge v\|}\left(\frac{\|gu\|}{\|u\|}\right)^{-2}.$$

□

5.2. **Relationship with thermodynamic formalism.** We show now the relation of Oseledets Lyapunov exponents to the thermodynamic formalism introduced in Section 3.

For a matrix $g \in \mathrm{SL}(d, \mathbb{R})$, denote by $\lambda_1(g) \geq \ldots \lambda_d(g)$ the modulus of the eigenvalues of g and define the weight $\varphi(g) = \log \frac{\lambda_1(g)}{\lambda_2(g)}$.

Remark 5.4. Note that one can consider different weight functions, e.g. in [6, Cor. 1.5] they consider the weight $\varphi(g) = \log \lambda_1(g)$.

Now we recall how to associate a Hölder continuous function on T^1X to an Anosov representation of $\pi_1(X)$. Recall that the periodic orbits of the geodesic flow are in bijection with the conjugacy classes of elements of $\pi_1(X)$.

Theorem 5.5 ([24], Prop. B7)**.** *Let $\rho : \pi_1(X) \to \mathrm{SL}(d, \mathbb{R})$ be a $(1, 2)$-Anosov representation. Then there exists a positive Hölder continuous function $f_\rho : T^1X \to \mathbb{R}$ associated to ρ and the weight φ such that for all Ψ_t-periodic orbits $a \in O$ and every $\gamma \in \pi_1(X)$ associated to a we can write the period of a for the reparametrized flow Ψ^{f_ρ} as*

$$p_{f_\rho}(a) = \varphi(\rho(\gamma)).$$

Remark 5.6. Note that the constant function $\mathbf{1}$ on T^1X is the same as f_{j_X}, i.e. the function given by the Theorem 5.5 from the uniformizing representation j_X of X.

Now consider a $(1, 1, 2)$-hyperconvex representation $\rho : \pi_1(X) \to \mathrm{SL}(d, \mathbb{R})$. Since it is crucial for us to know the entropy of f_ρ, we recall the following fact.

Theorem 5.7 ([24], Cor. 9.1)**.** *Let $\rho : \pi_1(X) \to \mathrm{SL}(d, \mathbb{R})$ be a $(1, 1, 2)$-hyperconvex representation. The entropy h_{f_ρ} of the associated flow Ψ^{f_ρ} satisfies $h_{f_\rho} = 1$.*

We can now prove the relation stated in Theorem 5.1 between the Oseledets Lyapunov exponents associated to X and ρ and the renormalized intersection $J(\mathbf{1}, f_\rho)$ (see the definition in Section 3) of the constant function $\mathbf{1} : T^1X \to \{1\}$ and f_ρ.

Proposition 5.8. *If $\rho : \pi_1(X) \to \mathrm{SL}(d, \mathbb{R})$ is a (1,1,2)-hyperconvex Anosov representation, then*

$$J(\mathbf{1}, f_\rho) = \lambda_1(X, \rho) - \lambda_2(X, \rho).$$

Proof. Note that by definition the constant function $\mathbf{1}$ is the function associated to the geodesic flow Ψ_t. It is classical that the entropy of the geodesic flow is one $h_{\mathbf{1}} = 1$, and that the measure of maximal entropy is the Liouville measure $\mu_{\mathbf{1}} = v_L$.

Since by Theorem 5.7 we have $h_{\mathbf{1}} = h_{f_\rho} = 1$, the renormalized intersection is the same as the intersection, i.e.

$$J(\mathbf{1}, f_\rho) = I(\mathbf{1}, f_\rho) = \int_{T^1X} f_\rho dv_L.$$

Applying Birkhoff erdogic theorem, we obtain that for v_L-almost any $x \in T^1X$ it holds

$$\int_{T^1X} f_\rho dv_L = \lim_{T \to +\infty} \frac{1}{T} \int_0^T f_\rho(\Psi_t(x)) dt.$$

For every $T > 0$, we approximate the geodesic segment $\Psi_{[0,T]}(x)$ by a closed geodesic $a_T(x) \in O$ of length $T + O(1)$ by closing it by a short geodesic arc and we have

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T f_\rho(\Psi_t(x))dx = \lim_{T\to+\infty} \frac{1}{T} p_{f_\rho}(a_T(x)).$$

Moreover, if we denote by $\gamma_T(x) \in \pi_1(X)$ the element corresponding to $a_T(x)$, we can rewrite the previous expression using the characterizing property of f_ρ given Theorem 5.5 as

$$\begin{aligned}\lim_{T\to+\infty} \frac{1}{T} p_{f_\rho}(a_T(x)) &= \lim_{T\to+\infty} \frac{1}{T}\varphi(\rho(\gamma_T(x)))\\ &= \lim_{T\to+\infty} \frac{1}{T}\left(\log\lambda_1(\rho(\gamma_T(x))) - \log\lambda_2(\rho(\gamma_T(x)))\right).\end{aligned}$$

Summarizing, we have show that

$$J(\mathbf{1}, f_\rho) = \lim_{T\to+\infty} \frac{1}{T}(\log\lambda_1(\rho(\gamma_T(x))) - \log\lambda_2(\rho(\gamma_T(x))))$$

for almost any $x \in T^1X$.

Since $(\gamma_T(x))$ is a quasi-geodesic asymptotic to the geodesic ray $\Psi_t(x)$ and $\wedge^2\rho$ is 1-Anosov if ρ is 2-Anosov, we can apply Lemma 5.9 and Lemma 5.10 below to ρ and $\wedge^2\rho$ and obtain

$$\begin{aligned}&\lim_{T\to+\infty}\left(\frac{1}{T}\log\lambda_1(\rho(\gamma_T(x))) - \frac{1}{T}\log\lambda_2(\rho(\gamma_T(x)))\right)\\ =&\lim_{T\to+\infty}\left(\frac{1}{T}\log\sigma_1(\rho(\gamma_T(x))) - \frac{1}{T}\log\sigma_2(\rho(\gamma_T(x)))\right)\\ =&\ \lambda_1(X,\rho) - \lambda_2(X,\rho).\end{aligned}$$

where in the last line we have used that $\lambda_1(X, \wedge^2\rho) = \lambda_1(X,\rho) + \lambda_2(X,\rho)$. □

We give here self-contained proofs of the two results, which are probably well known, needed at the end of the previous proof to conclude. Recall that a sequence (γ_n) in $\pi_1(X)$ is a quasi-geodesic if

$$A^{-1}n - C \leq |\gamma| \leq An + C,$$

for some constants $A > 0$ and $C \in \mathbb{R}$. By definition a quasi-geodesic (γ_n) is asymptotic to a unique γ_∞ in $\partial_\infty\pi_1(X)$, its limit point.

Lemma 5.9. *Let $\rho : \pi_1(X) \to \mathrm{SL}(d,\mathbb{R})$ be a 1-Anosov representation and let (γ_n) be a quasi-geodesic in $\pi_1(X)$. Then*

$$\lim_{n\to+\infty} \frac{1}{n}\log\lambda_1(\rho(\gamma_n)) = \lim_{n\to+\infty} \frac{1}{n}\log\sigma_1(\rho(\gamma_n)).$$

Proof. First observe that the two limits exists, the first by a similar reasoning as in the first part of the proof of Proposition 5.8 and the second by a subadditive argument. In particular it is enough to prove the equality for a subsequence.

For a matrix $g \in \mathrm{SL}(n,\mathbb{R})$ with $\sigma_2(g) < \sigma_1(g)$, let $U^1(g)$ be the eigenspace associated to the greatest eigenvalue of gg^* and $S_{d-1}(g)$ the sum of the eigenspaces associated to the $d-1$ lowest eigenvalues of g^*g. Let $\delta(g)$ be half the projective distance between $U^1(g)$ and $S_{d-1}(g)$. According to [2, Lemma 14.14] if

$$\frac{\sigma_2(g)}{\sigma_1(g)} < \delta(g)^2$$

then $\delta(g)\|g\| \leq \lambda_1(g)$.

Now if ρ is 1-Anosov and (γ_n) is a quasi-geodesic, by definition there exist constants $C, a > 0$ such that

$$\frac{\sigma_2(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))} \leq Ce^{-an}.$$

Denoting by ξ_1 and ξ_{d-1} the limit maps of ρ and by γ_∞ the limit point in $\partial_\infty \pi_1(X)$ associated to (γ_n), by [4, Lemma 4.7] we have

$$\xi_1(\gamma_\infty) = \lim_{n\to\infty} U^1(\rho(\gamma_n)),$$

and

$$\xi_{d-1}(\gamma_\infty) = \lim_{n\to\infty} S_{d-1}(\rho(\gamma_n)^{-1}).$$

On the other hand, up to taking a subsequence, (γ_n^{-1}) converges to a point $\gamma_\infty^{-1} \neq \gamma_\infty$, because $\pi_1(X)$ acts on its boundary as a uniform convergence group. By Theorem 4.3, i.e. by the transversality of the limit maps, we have hence that $\xi_1(\gamma_\infty) \notin \xi_{d-1}(\gamma_\infty^{-1})$. Moreover

$$\xi_{d-1}(\gamma_\infty^{-1}) = \lim_{n\to\infty} S_{d-1}(\rho(\gamma_n)).$$

In particular, there exists $\delta > 0$ such that for n large enough, the distance between $U^1(\rho(\gamma_n))$ and $S_{d-1}(\rho(\gamma_n))$ is greater than 2δ. As

$$\frac{\sigma_2(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))} \to 0,$$

for n large enough, by the general fact recalled above we have

$$\delta \|\rho(\gamma_n)\| \leq \lambda_1(\rho(\gamma_n)).$$

Since $\sigma_1 = \|\cdot\|$, then $\lambda_1(\rho(\gamma_n)) \leq \|\rho(\gamma_n)\|$ and so this implies that

$$\lim_{n\to+\infty} \frac{1}{n} \log \lambda_1(\rho(\gamma_n)) = \lim_{n\to+\infty} \frac{1}{n} \log \sigma_1(\rho(\gamma_n)).$$

□

The next lemma is classical.

Lemma 5.10. *Let $\rho : \pi_1(X) \to \mathrm{SL}(d, \mathbb{R})$ be a 1-Anosov representation and let (γ_T) in $\pi_1(X)$ be the sequence constructed above associated to the geodesic ray $\Psi_T(x)$ for a generic $x \in T^1X$. Then*

$$\lim_{T\to+\infty} \frac{1}{T} \log \sigma_1(\rho(\gamma_T)) = \lambda_1(X, \rho)$$

Proof. Pick a $x \in T^1X$ generic both for the Birkhoff theorem and the Oseledets theorem. Since the operator norm of $\widetilde{\Psi}_t$ is by definition the same as its first singular value, by Remark 4.2 we have

$$\lambda_1(\rho) = \lim_{T\to+\infty} \frac{1}{T} \log\|\widetilde{\Psi}_t\|_{(E_\rho)_x},$$

where recall that $\widetilde{\Psi}_t$ is the lifted geodesic flow to the linear bundle E_ρ associated to ρ above T^1X and $\|\widetilde{\Psi}_t\|_{(E_\rho)_x}$ is the operator norm of $\widetilde{\Psi}_t$ between the fibers $(E_\rho)_x$ and $(E_\rho)_{\Psi_t(x)}$.

The operators $(\widetilde{\Psi}_t)_{(E_\rho)_x}$ between $(E_\rho)_x$ and $(E_\rho)_{\Psi_t(x)}$ and the automorphism $(\widetilde{\Psi}_t)_{\gamma_T}$ of a fiber above the loop γ_T differ by a composition by a bounded operator because (γ_T) is constructed by closing by a short geodesic arc the path $\Psi_{[0,T]}(x)$, so we have:

$$\lambda_1(\rho) = \lim_{T\to+\infty} \frac{1}{T} \log\|\widetilde{\Psi}_t\|_{(E_\rho)_x} = \lim_{T\to+\infty} \frac{1}{T} \log\|\widetilde{\Psi}_t\|_{\gamma_T}.$$

Since parallel transport in E_ρ over γ_T is given by the image of the monodromy $\rho(\gamma_T)$, we can finally conclude

$$\lambda_1(\rho) = \lim_{T\to+\infty} \frac{1}{T} \log \|\widetilde{\Psi_t}\|_{\gamma_T} = \lim_{T\to+\infty} \frac{1}{T} \log \|\rho(\gamma_T)\|$$

which proves what we want since by definition $\|\rho(\gamma_T)\| = \sigma_1(\rho(\gamma_T))$. □

5.3. **Proof of Theorem 1.1.** In this section we can finally explain how to deduce the inequality statement of Theorem 1.1 from Theorem 1.2 and show the rigidity statement for the extremal gaps situation using the thermodynamic formalism.

Let $\rho : \pi_1(X) \to \mathrm{SL}(d,\mathbb{R})$ be a Hitchin representation. Remark that we have the following relationships between Lyapunov exponents:

$$\lambda_{i+1}(\rho, X) - \lambda_i(\rho, X) = \lambda_2(\wedge^i \rho, X) - \lambda_1(\wedge^i \rho, X),$$

for every $i = 1, \dots, d-1$. Since by Theorem 4.6 we can apply Theorem 1.2 to every wedge power $\wedge^i \rho$ of a Hitchin representation, for $i = 1, \dots, d-1$, we get the inequality part of Theorem 1.1.

We assume now that ρ is a Hitchin representation and that we are in the extremal gap case, i.e. for every $i = 1, \dots, d-1$:

$$\lambda_i(\rho, X) - \lambda_{i+1}(\rho, X) = 1.$$

We will show that ρ is conjugated to the image of the Fuchsian representation by the irreducible representation $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(d,\mathbb{R})$.

Let $i \in \{1, \dots, d-1\}$ and let $\rho_i := \wedge^i \rho$. Since by assumption and by Proposition 5.8 we have that

$$1 = \lambda_1(\rho_i, X) - \lambda_2(\rho_i, X) = J(\mathbf{1}, f_{\rho_i}),$$

the equality case of Proposition 3.1 implies that f_{ρ_i} is Livsic cohomologous to 1. In particular this means that the flows $\Psi_t^{f_{\rho_i}}$ and Ψ_t have the same periods, which by Remark 5.6 means that for every $\gamma \in \pi_1(X)$ it holds

$$\frac{\lambda_1(j_X(\gamma))}{\lambda_2(j_X(\gamma))} = \frac{\lambda_1(\rho_i(\gamma))}{\lambda_2(\rho_i(\gamma))} = \frac{\lambda_i(\rho(\gamma))}{\lambda_{i+1}(\rho(\gamma))},$$

where $j_X : \pi_1(X) \to \mathrm{SL}(2,\mathbb{R})$ is the Fuchsian uniformizing representation of X. Since the product of the eigenvalues of any $\rho(\gamma) \in \mathrm{SL}(d,\mathbb{R})$ is one, this implies by an elementary calculation that for every $\gamma \in \pi_1(X)$ we have

$$\lambda_1(\rho(\gamma)) = \left(\frac{\lambda_1(j_X(\gamma))}{\lambda_2(j_X(\gamma))} \right)^{(d-1)/2},$$

and it is well-known that

$$\lambda_1(j_X^d(\gamma)) = \left(\frac{\lambda_1(j_X(\gamma))}{\lambda_2(j_X(\gamma))} \right)^{(d-1)/2},$$

where $j_X^d : \pi_1(X) \to \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(d,\mathbb{R})$ is the image of j_X by the irreducible representation $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(d,\mathbb{R})$.

To conclude, we apply the "Hitchin rigidity" result of [7, Cor. 5.19]. This result states that if two Hitchin representations ρ_1 and ρ_2 satisfy

$$\lambda_1(\rho_1(\gamma)) = \lambda_1(\rho_2(\gamma))$$

for every $\gamma \in \pi_1(X)$, then ρ_1 is conjugated to ρ_2. Applying this result to ρ and j_X^d and using the last two equalities, we get that ρ and j_X^d are conjugated. This concludes the proof of Theorem 1.1.

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