Baby seminar proposals

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1 Real and étale cohomology

The proposal is to study the content of the book "Real and étale cohomology" by Claus Scheiderer [Sch94].

Étale cohomology is a cohomology theory for schemes with properties similar to singular cohomology, which is a cohomology theory for topological spaces. In case of algebraic varieties over $\mathbf{C}$, they are also related by a comparison isomorphism. This is established in Artin's comparison theorem (see for example [Mil13, §I.21]).

Theorem 1.1 (Artin's comparison theorem)**.** *Let X be an algebraic variety over $\mathbf{C}$. For any M finite abelian group and any integer $n \geq 0$, there exists a canonical isomorphism*

$$H^n(X_{\acute{e}t}, M) \cong H^n(X(\mathbf{C}); M)$$

between étale cohomology of X and singular cohomology of the complex points of X, with coefficients in M.

One of the final result in Scheiderer's book establishes a real counterpart to Artin's comparison theorem. In this case one has to take in account the action of the Galois group $G \coloneqq \mathrm{Gal}(\mathbf{C}/\mathbf{R})$ on the complex points, which is given by the complex conjugation. The precise statement is the following (see [Sch94, Corollary 15.3.1], where in fact it is formulated more generally for algebraic varieties over a real closed field).

Theorem 1.2. *Let X be an algebraic variety over $\mathbf{R}$. For any $\mathcal{A}$ abelian torsion sheaf and any integer $n \geq 0$, there exists a canonical isomorphism*

$$H^n(X_{\acute{e}t}, \mathcal{A}) \cong H^n_G(X(\mathbf{C}), \psi^*\mathcal{A})$$

between étale cohomology of X and G-equivariant cohomology of the complex points of X, with coefficients in $\mathcal{A}$.

This result is proved in the third part of Scheiderer's book. It follows without effort from some results obtained in previous parts, where is investigated the connection between étale cohomology of an algebraic variety (over any field!) X and the cohomology of the real spectrum $R(X)$. The real spectrum is a topological space constructed out of an algebraic variety, considering the orderings on the residue fields. For example, for an affine algebraic variety $X = \mathrm{Spec}(A)$, the associated real spectrum is

$$R(X) \coloneqq \mathrm{Sper}(A) \coloneqq \{(x, \leq) | x \in \mathrm{Spec}(A),\ \leq \text{ is an ordering on the residue field at } x\}.$$

The key to relate the étale site $X_{\text{ét}}$ and the site associated to $R(X)$ as a topological space is to introduce another Grothendieck topology on X, called the b-topology, whose associated site is denoted by X_b.

Then, the connection between the three sites $X_{\text{ét}}$, $R(X)$ and X_b is well understood under a G-equivariant framework, where G is the absolute Galois group of the base field. This is done systematically in the second part of Scheiderer's book, where, given a G-equivariant topos E, for G a finite group, he defines the notions of "fixed-topos" and "quotient-topos". These constructions, applied to the G-equivariant topos associated to the site $X'_{\text{ét}}$, where $X' \coloneqq X \times_{\mathbf{Z}} \mathbf{Z}[\sqrt{-1}]$, give respectively the sites $R(X)$ and X_b. So the real spectrum must be thought as the "fixed-objects" of the action of G on the étale site $X'_{\text{ét}}$.

This abstract topos-theoretic point of view, even if not strictly necessary, allows to recognise a common principle at work in different situations, since it can also be applied to the case of a topological space endowed with an action of G.

Analogously to what happens in topology, one of the main results of the book establishes the existence of a long exact sequence relating the (G-equivariant) cohomology of $X_{\text{ét}}$, $R(X)$ and X_b, which is induced by some morphisms between the sites (see [Sch94, Theorem]).

Theorem 1.3. *Let X be a scheme such that $\frac{1}{2} \in \mathcal{O}(X)$. For any $\mathcal{A}$ abelian sheaf on $X'_{\acute{e}t}$ there exists a canonical long exact sequence*

$$\cdots \to H^n(X_b; j_!\mathcal{A}) \to H^n(X_{\acute{e}t}; \mathcal{A}) \to H^n_G(R(X); \nu^*\mathcal{A}) \to \dots.$$

This long exact sequence, together with vanishing results of b-cohomology, allows to conclude that étale cohomology stabilizes against singular cohomology of the real spectrum. This generalizes some previous work of Cox [Cox79], who proved the same statement over $\mathbf{R}$ but using some complicated machinery, and also a theorem of Arason in [AEJ84], where is proved the same statement for fields using theory of quadratic forms (the proof of this would also be interesting to look at).

2 Topics in Algebraic K-theory

The proposal is to study some topics in algebraic K-theory starting from the very basic definitions and without assuming any particular background. The talks should not be necessarily all related one to the other, but of course there are some initial topics that can be useful for all the others talks.

Algebraic K-theory is a kind of cohomology theory that appears in several results and conjectures between intersection theory, algebraic topology and number theory. It was first introduced by Grothendieck in his studies in intersection theory and Chow group. He defined the K-group of a scheme (now called the K_0-group), as the group completion of the monoid given by isomorphism classes of algebraic vector bundles over the scheme with respect to the direct sum. Grothendieck related this group to the Chow group to obtain a generalization of the Riemann-Roch theorem. Afterwards, the analogous notion of K-theory was studied in algebraic topology, considering topological vector bundles. There, a notion of higher K-groups appeared, and so the question naturally arose of what the analogous was for algebraic K-theory. Different possible definitions of higher algebraic K-theory where proposed. The one that was finally accepted was Quillen's approach.

I don't already have a full list of topics to propose. The main references I would like to follow are [Wei13] and [Bas68]. Besides these, there are also a lot of course or seminar notes available on-line, where to take inspiration from.

Some of the possible topics are:

- Definition of Grothendieck's K_0-group [Wei13, §II].

- Topological K-theory and connections with the algebraic one.
- The isomorphism of K_0-group and the Chow group with rational coefficients. This justifies the complicated Serre's intersection formula of algebraic cycles.
- Lambda and Adam operations [Wei13, §II.4].
- At least one of the equivalent constructions of Quillen's higher K-theory groups [Wei13, §IV].
- Some properties and computations.

Further suggestions on any other topic related to algebraic K-theory that anybody would like to explore are well accepted.

References

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