Babyseminar: Eigenvarieties

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Summer Semester 2025

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1 The topic

In this seminar, we want to introduce the topic of eigenvarieties. We do this by developing a general theory that allows to construct eigenvarieties, and then see a few examples of this theory in action. An eigenvariety is roughly the following: suppose that you have a family of Banach modules over a complete non-archimedean field K (for example $\mathbb{C}_p$) parametrized by a p-adic analytic (i.e. rigid analytic or, more generally, adic) space $\mathscr{P}$ over K. Suppose for simplicity that K is algebraically closed. In particular, for every point x of $\mathscr{P}$, you have a complete Banach module M_x over K (which come from Banach modules over affinoid algebras covering your $\mathscr{P}$ satisfying some gluing conditions). Suppose you have a commutative algebra H that acts compatibly on all these modules. Then an eigenvariety attached to this datum is a rigid analytic (or adic) variety $\mathscr{E}$ with a locally finite map $\mathscr{E} \to \mathscr{P}$ that parametrizes system of eigenvalues of H acting on the modules M_x, i.e. the fiber over every point x of $\mathscr{P}$ along the map $\mathscr{E} \to \mathscr{P}$ is in natural bijection with systems of eigenvalues for the action of H on the module M_x.

Historically, this was first done with the M_x's being some modules of (p-adic) modular forms. In this case, you can think of the resulting eigenvariety (the Coleman-Mazur eigencurve) as a moduli space of p-adic modular forms. I've tried to motivate the topic by writing a more extended introduction about this example starting from section 1.1 below. If you know nothing about modular form, you can skip to section 2, we will learn them in the seminar.

One can look for generalizations beyond modular forms. In fact, there is a construction that attaches to a modular form an automorphic form for the group GL_2. A p-adic family of modular forms can be interpreted as a p-adic variation of automorphic forms for GL_2. One could then try to formulate similar definitions of p-adic automorphic form for other algebraic groups and construct eigenvarieties for them. We will not go into the theory of p-adic automorphic forms, but we will see an example in the end.

1.1 A bit of history of p-adic families of modular forms

The theory of modular forms has been at the bottom of several of the most important developments in number theory in the last decades, and the study of their p-adic variation has played an important role. Classically, in the theory over $\mathbb{C}$, a modular form is a section of a line bundle on a modular curve, that is the quotient of the complex upper half plane $\mathbb{H}$ by a congruence subgroup Γ (a certain kind of finite index subgroup of $SL_2(\mathbb{Z})$). The definition is usually given by pulling back the section along $\mathbb{H} \to \Gamma \setminus \mathbb{H}$:

Definition 1. A modular form of level Γ and weight $k \in \mathbb{Z}$ is an holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that

$$f(\gamma z) = (cz + d)^k f(z)$$

for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and which satisfies a certain growth condition (which has to do with the fact that the curve $\Gamma \setminus \mathbb{H}$ is not compact, and one really wants to define modular forms as sections of line bundles on the compactification).

In the seminar, we restrict mainly to modular forms for the groups

$$\Gamma := \Gamma_1(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0, a \equiv 1, c \equiv 1 \mod N \right\}.$$

If one applies the definition with matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, one finds that modular forms have a Fourier expansion. Thus, one can thus think of a modular form as a converging sum $f = \sum_{n\geq 0} a_n(f)q^n$, where $a_n := a_n(f) \in \mathbb{C}$ and $q := e^{2\pi i z}$ for $z \in \mathbb{H}$ (the fact that one can restrict to $n \geq 0$ is implied by the growth condition). One can then ask how to determine the a_n's in therm of f. The answer is: using the Hecke algebra. More precisely, there is a commutative algebra $\mathbb{T}$ (the Hecke algebra, spanned by operators T_ℓ and S_ℓ indexed by primes ℓ) that acts on the space of modular forms $M_k(\Gamma)$ of weight k and level Γ (also denoted $\mathbb{T}(N,k)$ to stress the dependencies on k and N). The subalgebra $\mathbb{T}_0 = \mathbb{T}_0(N,k)$ generated by operators T_ℓ and S_ℓ indexed by primes ℓ coprime with N acts semisimply on $M_k(\Gamma)$. Thus, one has a decomposition of it in eigenspaces for $\mathbb{T}_0$, with a basis of eigenforms. Each eigenform f gives rise to a ring homomorphisms $\lambda_f : \mathbb{T}_0 \to \mathbb{C}$ (i.e. to $\mathbb{C}$-valued points of $\mathbb{T}_0$), by sending $T \mapsto \lambda_f(T)$, where $\lambda_f(T)$ is the eigenvalue of T acting on f. If, for some particular character λ of $\mathbb{T}_0$, the corresponding eigenspace is one-dimensional, then one gets the striking property that for each prime ℓ (even those not coprime with N), $a_\ell(f) = \lambda(T_\ell)a_1(f)$ for any f in such eigenspace (i.e. the system of eigenvalues determines the modular form up to scalar). These eigenforms are called newforms of conductor N. We will see that one can easily describe each space $M_k(\Gamma_1(N))$ knowing all the newforms of conductor some factor of N. Thus, one can focus on describing newforms, and hence systems of eigenvalues of the Hecke algebras $\mathbb{T}_0(N)$ for varying N. Another important property of modular forms (that one can derive from the study of the Hecke algebras), is that Hecke eigenvalues are actually algebraic integers, so any system of eigenvalues can be written as $\lambda : \mathbb{T}_0(N) \to \overline{\mathbb{Z}}$ (the algebraic closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$). This allows one to study the theory p-adically.

1.1.1 p-adic interpolation of Eisenstein series

Fix N and a prime p coprime with N. One first naïve to a p-adic theory is just to tensor a system of eigenvalues of $\mathbb{T}_0(N,k)$ with $\mathbb{Z}_p$. This alone does not add much to the complex theory. It turns out that, the p-adic theory is much more interesting if one allows the weight k to vary. We view an example of this, which was noticed by Serre in the '70s. The space of modular forms $M_k(\Gamma)$ can be written as $E_k(\Gamma) \oplus S_k(\Gamma)$, where the direct summand $E_k(\Gamma)$ is called the module of Eisenstein series and it contains elements that one can explicitly write down. One of this is the Eisenstein series $G_k \in E_k(SL_2(\mathbb{Z}))$ of weight $k > 2$ and level $SL_2(\mathbb{Z})$ (so it can also be viewed as having level $\Gamma_1(N)$ for any N), whose Fourier expansion is given by

$$G_k(z) := -\tfrac{B_k}{2k} + \sum_{n\geq 1} \left(\sum_{d|n} d^{k-1} \right) q^n \tag{1}$$

(where B_k is the k-th Bernoulli number and $q = e^{2\pi i z}$, $z \in \mathbb{H}$). The interesting phenomenon is that it satisfies certain congruences modulo p.

For $f = \sum_{n\geq 0} a_n q^n$, denote $V_p f(q) = \sum_{n\geq 0} a_n q^{np}$ and $U_p f(q) = \sum_{n\geq 0} a_{np} q^n$ and define the p-depletion of f: $f^{(p)} = (1 - V_p U_p) f = \sum_{p\nmid n} a_n q^n$.

The above-mentioned congruence is the following: the p-depleted Eisenstein series

$$G_k(z)^{(p)} = G_k(z) - p^{k-1} G_k(pz) = (p^{k-1} - 1)\tfrac{B_k}{2k} + \sum_{n\geq 1} \left(\sum_{p\nmid d|n} d^{k-1} \right) q^n,$$

satisfies the following congruence:

$$\text{if } k \equiv k' \mod (p-1)p^n \text{ and } (p-1) \nmid k, \quad G_k(z)^{(p)} \equiv G_{k'}(z)^{(p)} \mod p^{n+1}. \tag{2}$$

The congruence between higher coefficients is easy to see:

$$d^a - d^{a+(p-1)p^n} = d^a(1 - d^{(p-1)p^n}) = d^a(1 - (1+px)^{p^n})$$

and one uses the congruence properties of the binomial coefficient (notice that one can write $d^{(p-1)} = 1 + px$ because $p \nmid d$, otherwise the congruences do not hold). Instead, the congruence for the constant term relies on the Kummer congruences between Bernoulli numbers.

1.2 Coleman-Mazur eigencurve

In order to detect congruences between forms of different weights, one defines $\mathbb{T}(N)^{(p)} := \varprojlim(\mathbb{Z}_p \otimes \mathbb{T}_0(N,k)^{(p)})$, where $\mathbb{T}_0(N,k)^{(p)}$ is the subalgebra of $\mathbb{T}_0(N,k)$ generated by the T_ℓ, S_ℓ's with $\ell \neq p$ and the transition maps are induced by $\bigoplus_{i=0}^{k} M_i(\Gamma) \subset \bigoplus_{i=0}^{k'} M_i(\Gamma)$ for $k' \geq k$. Then one has an embedding

$$\mathbb{T}(N)^{(p)} \hookrightarrow \prod_{k \in \mathbb{N}} \mathbb{T}_0(N,k).$$

One has the striking property that $\mathbb{T}(N)^{(p)}$ is the product of a finite number of p-adically complete algebras. This is precisely because there are only finitely many possibilities for the reduction mod p of systems of eigenvalues of $\mathbb{T}(N)^{(p)}$ with values in $\overline{\mathbb{Z}}_p$, and hence $\mathbb{T}(N)^{(p)}$ has finitely many maximal ideals (see [Eme09, Proposition 2.8]).

In fact, (2) is a manifestation of this: the systems of eigenvalues of $\mathbb{T}(N)^{(p)}$ with values in $\overline{\mathbb{F}}_p$ associated with $\{G_k^{(p)}\}$ are already contained in weights $\{4, \dots, p(p-1)+3\}$.

The scheme $Spec(\mathbb{T}(N)^{(p)})$ is conjecturally of dimension 3 over $Spec(\mathbb{Z}_p)$ and hence, by Noether normalization, it conjecturally admits a finite map to $\mathbb{A}^3_{\mathbb{Z}_p}$, which however is not canonical. On the other hand, one can construct a canonical map $Spec(\mathbb{T}(N)^{(p)}) \to Spec(\mathbb{Z}_p[[T]])$. Supposing $p \neq 2$ to simplify notations, it is given by composing the isomorphism $\mathbb{Z}_p[[T]] \cong \mathbb{Z}_p[[1+p\mathbb{Z}_p]]$ given by sending $T \mapsto [1+p] - 1$ (where $\mathbb{Z}_p[[1+p\mathbb{Z}_p]]$ is the completed group algebra $\varprojlim_n \mathbb{Z}_p[(1+p\mathbb{Z}_p)/(1+p^n\mathbb{Z}_p)]$) with the morphism that interpolates the one sending a prime ℓ in $1 + p\mathbb{Z} \subset 1 + p\mathbb{Z}_p$ to S_ℓ. If one has a system of eigenvalues λ attached to a modular form of weight k, then $\lambda(S_\ell) = \ell^{k-2}$ for each ℓ as above, and hence the map $Spec(\mathbb{T}(N)^{(p)}) \to Spec(\mathbb{Z}_p[[1+p\mathbb{Z}_p]])$ sends the $\overline{\mathbb{Z}}_p$-valued point of $Spec(\mathbb{T}(N)^{(p)})$ defined by $\lambda : \mathbb{T}(N)^{(p)} \to \overline{\mathbb{Z}}_p$ to the ring homomorphism $\mathbb{Z}_p[[1+p\mathbb{Z}_p]] \to \overline{\mathbb{Z}}_p$ induced by the continuous character of $1 + p\mathbb{Z}_p$ that sends $x \to x^{k-2}$, which is uniquely determined by the weight $k \in \mathbb{Z}$. So the map $Spec(\mathbb{T}(N)^{(p)}) \to Spec(\mathbb{Z}_p[[T]])$ can be thought of as sending each system of eigenvalues of $\mathbb{T}(N)^{(p)}$ arising from a modular form to its weight.

This discussion allows us to make precise what we mean by interpolating p-adically modular forms in terms of their weight: can we find a closed subscheme Z of $Spec(\mathbb{T}(N)^{(p)})$ such that the induced map $Z \to Spec(\mathbb{Z}_p[[T]])$ is finite and dominant and such that there is a Zariski dense subset of points corresponding to system of eigenvalues of classical modular forms?

The answer is no. But it is yes if we move to the world of rigid analytic spaces. In order to perform the construction we have to first identify the parameter space that we want to use. This will be the rigid space $W/\mathbb{Q}_p$ that takes values $\mathrm{Hom}_{cts,grp}(\mathbb{Z}_p^\times, K^\times)$ over a complete field extension K of $\mathbb{Q}_p$. When $p > 2$, it is isomorphic to copies of the open unit disk centered at 1 and indexed by $(\mathbb{Z}/(p-1)\mathbb{Z})^\times$ (In fact, $\mathbb{Z}_p^\times \cong (\mathbb{Z}/(p-1)\mathbb{Z})^\times \times (1+p\mathbb{Z}_p)$, $1+p\mathbb{Z}_p \cong \mathbb{Z}_p$ via the logarithm, so the image of a continuous function φ is uniquely determined by a topological generator and the continuity condition ensures that $\lim_{k\to\infty}(\varphi(1+p))^{p^k} = \lim_{k\to\infty}\varphi((1+p)^{p^k}) = 1$). It is the rigid analytic variant of $Spec(\mathbb{Z}_p[[1+p\mathbb{Z}_p]])$ considered above. We can still view an integral weight k there as the homomorphisms in $\mathrm{Hom}_{cts,grp}(\mathbb{Z}_p^\times, K^\times)$ that sends $x \to x^{k-2}$ (which determines a homomorphism of algebras $\mathrm{Hom}(\mathbb{Z}_p[[\mathbb{Z}_p^\times]], K)$).

Next we want to find appropriate modules parametrized by points of W on which the algebra $\mathbb{T}(N)^{(p)}$ acts, in such a way that over points of W that correspond to integral weight, the system of eigenvalues occurring in $M_k(\Gamma)$ appear. They will be the spaces of overconvergent modular forms.

Next we will adjoin the U_p operator to $\mathbb{T}(N)^{(p)}$ and show that this operator is compact (for some norm that one can define on overconvergent modular forms). We then develop some p-adic functional analysis that allows, in a affinoid neighborhood U of every point w of W, to find, for enough positive numbers $\nu \in \mathbb{R}$, a splitting of the module M_U into a finitely generated module $M^{\leq\nu}$ (finitely generated over the affinoid algebra corresponding to U) on which the norms of eigenvalues of the action of U_p are controlled by ν. We will then take the maximal spectrum of the faithful quotient $\mathbb{T}^*$ of $\mathbb{T}(N)^{(p)}[U_p]$ acting on $M^{\leq\nu}$ and finally glue all this maximal spectra. Doing so, one gets the Coleman-Mazur eigencurve:

Theorem 1. *There exists a rigid analytic curve $C \to W$ whose $\mathbb{C}_p$ points classify normalized overconvergent eigenforms f that are not in the kernel of the U_p operator.*

If one is also able to show that system of eigenvalues of classical modular forms appear in the $M^{\leq\nu}$ for enough ν's, one gets the desired family. This follows from Coleman's classicality theorem, but we will not go into it. Rather, we will concentrate on how to construct the eigencurve.

A nice feature is that its general construction is somewhat independent of modular forms, in the sense that it can be carried out in generality in presence of a family of Banach modules over a rigid analytic space $\mathscr{P}$ with action of a commutative algebra H (as in the beginning of the introduction) that contains a preferred operator that acts compactly (U_p in he case of overconvergent modular forms). The only point where modular forms appear is in choosing the right modules so that they see systems of eigenvalues of the Hecke algebra that come from classical modular forms.

The next section describes briefly the spaces of overconvergent modular forms, that are one of the possible spaces that do the job (it is not the only possibility).

1.2.1 Overconvergent modular forms

A first tentative approach to finding the right space of p-adic modular forms was done by Serre.

Definition 2. A Serre's p-adic modular form (of full level) is an element $f \in \mathbb{Q}_p[|q|]$ such that there exists a sequence $f_i \in M_{k_i}(SL_2(\mathbb{Z}))$, with weight k_i converging to a finite limit in the p-adic topology, approximating f p-adically: $v_p(f - f_i) \to \infty$ (where the valuation v_p of a series denotes the inf of the valuations of the coefficients).

The problem with the space of Serre's modular forms is that it is too big. In particular, the spectrum of the Hecke operator U_p is not discrete. To see this observe that for any $\lambda \in p\mathbb{Z}_p, f \in M_k(SL_2(\mathbb{Z}))$

$$f_\lambda := (1 + \lambda V_p + (\lambda V_p)^2 + (\lambda V_p)^3 + \dots)(1 - V_p U_p) f \text{ satisfies } U_p f_\lambda = \lambda f_\lambda.$$

This is because $U_p V_p = 1$: $U_p f_\lambda = (U_p + \lambda + \lambda^2 V_p + \lambda^3 V_p^2 + \dots)(1 - V_p U_p) f = (U_p - U_p V_p U_p) f + \lambda(1 + \lambda V_p + (\lambda V_p)^2 + (\lambda V_p)^3 + \dots)(1 - V_p U_p) f = \lambda f_\lambda$. This shows that the spectrum of U_p contains $p\mathbb{Z}_p$ and hence it is not discrete. It also follows that U_p is not compact (the set of non-zero eigenvalues of a compact operator is discrete). This is no good, as the compactness of U_p is central in the construction of the eigencurve.

A solution to the problem above was proposed by Katz by looking at geometry. For representability reasons, one should work with level $\Gamma_1(N)$ for $N \geq 5$.

Then one considers the moduli space of generalized elliptic curves $X/\mathbb{Z}_p$ (for $p \nmid N$), an algebraic model of the compact modular curve $X/\mathbb{C}$, which comes with (the canonical de-singularization of) the universal generalized elliptic curve $\pi : E \to X$. As we have seen in my talk in the research seminar this semester, over $\mathbb{C}$ one can view modular forms of weight k and level $\Gamma_1(N)$ as sections of the line bundle $\omega^{\otimes k}$, where $\omega := \pi_* \Omega^1_{E/X}(log(cusps))$. One also has a reduction map $X(\mathbb{C}_p) \to X_{\mathbb{F}_p}(\overline{\mathbb{F}}_p)$ (X is proper, so $X(\mathbb{C}_p) = X(\mathcal{O}_{\mathbb{C}_p})$) and one can define X^{ord} as the affinoid in X^{rig} (the rigid analytification of $X_{\mathbb{Q}_p}$) whose $\mathbb{C}_p$ points reduce to points in $X_{\mathbb{F}_p}(\overline{\mathbb{F}}_p)$ that parametrize elliptic curves that are not supersingular. It is isomorphic to the complement in X^{rig} of some rigid analytic open disks, indexed by supersingular points. One has

Theorem 2 (Katz)**.** *Serre's p-adic modular forms of weight k coincide with* $H^0(X^{ord}, \omega^{\otimes k})$.

Katz idea was to consider sections of $\omega^{\otimes k}$ over a bigger affinoid X_r: $X^{\mathrm{ord}} \subset X_r \subset X^{\mathrm{rig}}$ obtained by removing from X^{rig} smaller disks (of radius depending on $r \in [0, p/(p+1))$ and such that $X_0 = X^{\mathrm{ord}}$). One thus defines $M_k^\dagger(\Gamma_1(N))(r) := H^0(X_r, \omega^k)$ and the total space of overconvergent modular forms of weight k as $M_k^\dagger(\Gamma_1(N)) = \varinjlim_r M_k^\dagger(\Gamma_1(N))(r)$. Let me drop $\Gamma_1(N))$ from the notation.

One can define a norm on each $M_k^\dagger(r)$ and make it into an infinite-dimensional Banach space. The operator U_p extends to $M_k^\dagger(r)$ and it is moreover compact. Thus there exists a discrete spectrum of non-zero eigenvalues $|\lambda_1|_p > |\lambda_2|_p > |\lambda_3|_p > \dots$ with $|\lambda_n|_p \to 0$ whose inverses are the zeros of a well-defined characteristic series (the Fredholm's Determinant). Moreover, each element of the spaces has an asymptotic expansion in terms of generalized eigenvectors for U_p.

This property is crucial, for the construction of the eigencurve.

One has still the problem of defining overconvergent modular forms for non-integral weights, and this is originally due to Coleman [Col97] (as we will see).

1.2.2 p-adic L-functions and p-adic representations

There are several p-adic constructions that one can attach to classical eigenforms. For example, there is a p-adic L-function that interpolates special values of the algebraic part of its complex L-function and there is a p-adic Galois representation (of rank 2 for newforms) over the p-adic completion of the field generated by its Fourier coefficients.

One can attach to points of the eigencurve a Galois representation and a p-adic L-function as well, thus obtaining also p-adic families of p-adic L-functions and of p-adic representations, attached not only to classical eigenforms but more generally to overconvergent modular forms.

We won't probably go into it.

2 The seminar: Overview

The seminar is divided into three parts. The heart is the second part, which should contain most of the proofs. The first and third parts serve to put part two into context.

- In the first one (section 3.1), we see some background on modular forms. It mainly serves as a motivation for the whole seminar, although the third talk contains already important ingredients.
- The second part (section 3.2) is the heart of the seminar, in which we present the abstract construction of eigenvarieties and set the stage for the construction of the eigencurve of theorem 1. As already hinted, the key ingredient is the spectral theory of the Hecke operator U_p. In fact, the construction makes use of some easy p-adic functional analysis that we introduce in the first talks. In the last talks of this part, we give the definition and first properties of eigenvarieties. As eigenvarieties are rigid analytic spaces, in the middle (talk 6) there is a survey on rigid geometry, for people who have never seen it. There is a refined version of the construction using adic spaces, but I've decided to stick to rigid analytic spaces because the main reference [Bel21] does. This section is independent of modular forms and you can easily follow it even if you forgot everything from the first talks (except maybe talk 3).
- Finally in the last part (section 3.4 and section 3.5) we specialize the machinery to concrete examples (one small example should be already contained in the last talk of part two). The main example is the Coleman-Mazur eigencurve. Since it is nice to see more than one example having a whole theory available, I also propose to look at what happens for the p-adic automorphic forms on the group of units in a definite quaternion algebra over $\mathbb{Q}$, which is even easier to treat.

2.1 A comment on the references

The main reference will be [Bel21], a recent book on the subject that gives arguments and proofs in detail. He uses rigid analytic spaces (instead of adic). The constructions using adic spaces are given in [Lud24]. Both ultimately rely on [Buz07]. For background, see [DS05; Mil90; DI95] and [Bel21, §2] for modular forms, [Bos14] for rigid analytic spaces, [Bel21, §3] for the p-adic functional analysis that we need. The construction of the eigenvarieties of units of definite quaternion algebras is explained in [New24].

For the preparation of the introduction, I have used the notes [Von] and [Cal13],for the part on overconvergent modular forms and [Eme09] for the part on p-adic families. We will use [Von] and [Cal13] to introduce overconvergent modular forms with integral weights, and [Col97] to see how to extend the definition to non-integral ones.

3 Program of the seminar

3.0.1 Talk 0: Introduction (9/4)

I will sketch the content of the seminar and try to find speakers for all the empty talks.

3.1 The general theory of modular forms

In these three talks, we survey the general theory of modular forms.

3.1.1 Talk 1: Modular forms (16/4)

In this first talk, we want to get some confidence with modular forms. There are a lot of different sources that one could use, for example [DS05; Bel21; Mil90; Sil94; RS11]. You can use any other that you find. For the purpose of summarizing all the theory in 90 minutes [Mil90, §I.4] can be especially useful. Let me list what your talk should contain:

- the definition of a modular form over $\mathbb{C}$ for $SL_2(\mathbb{Z})$ and for the congruence subgroups $\Gamma \in \{\Gamma_0(N), \Gamma_1(N), \Gamma(N)\}$ as a holomorphic function on the upper half plane $\mathbb{H}$ with some properties. In particular say that modular forms have a Fourier expansion;
- the interpretation of the modular curves $\Gamma \setminus \mathbb{H}$ as moduli spaces of elliptic curves with additional structure (e.g. [Mil90, §II.8]), and modular forms as sections of line bundles on their compactifications. In particular as sections of the k-iterated tensor product of the cotangent bundle (e.g [Mil90, page 50]) or of the pushforward of the relative tangent bundle of the universal elliptic curve (when the moduli problem is representable, see [Mil90, page 58] or your notes of my talk in the last research seminar. You can also have a look at [DI95, §12.1]). After this, you can give some examples of dimensions of spaces of modular forms (they are computed using Riemann-Roch);
- The definition of the Petersson inner product and the decomposition of the space of modular forms in Eisenstein series and cusp forms. Give example of Eisenstein series (among which (1)). Say that the ring of modular forms of level $SL_2(\mathbb{Z})$ is the free algebra generated by E_4 and E_6.

3.1.2 Talk 2: Hecke operators (23/4)

We continue with the theory of modular forms and we introduce the key ingredient for their study: Hecke operators. Again there are several references that you could follow, for example [DS05, Chapter 5], [Bel21, §2.6], [Mil90, §I.5]. Your talk should include:

- The definition of Hecke operators (and diamond operators) as double coset operators, and as correspondences (using the moduli description). Give the action of T_ℓ (for a prime ℓ) on q-expansions and stress the fact that Hecke operators behave differently if ℓ divides or not the level of the modular form (as you can see conceptually from the moduli description). Present the decomposition of the spaces of modular forms of level $\Gamma_1(N)$ in eigenspaces for diamond operators. You should also stress the fact that the Hecke algebras of level $\Gamma_0(N)$ and $\Gamma_1(N)$ are commutative;

- The spectral theory of Hecke operators and the theory of old-forms and new-forms. The idea is the following: since Hecke operators coprime with the level of the modular form commute with their adjoints w.r.t. Petersson product, one may find a basis of eigenvectors. There are very special elements that generate eigenspaces of dimension 1, which are then eigenspaces also for the Hecke algebra containing operators T_ℓ for $\ell \mid N$ (often called U_ℓ to stress their different behavior). These are called newforms and their q-expansion is completely determined by the action of Hecke operators. For the spectral theory you should state [DS05, Thm 5.5.4] or equivalently [Bel21, Cor 2.6.14]. For the theory of new-forms state [Bel21, Thm 2.6.8] or [DS05, thm 5.8.2]. State also [DS05, thm 5.8.3];
- The duality between Hecke operators and modular forms. This is contained in [Bel21, Prop 2.6.16] and [Bel21, p. 2.6.19] (ignore the problems related to E_2);
- A brief final discussion on modular forms with coefficients in rings other than $\mathbb{C}$: give the (naïve) definition of modular forms with coefficients in a subring A of $\mathbb{C}$ [DI95, §12.3]. Then state the deep fact that the space of modular forms $M_k(\Gamma_1(N), \mathbb{C})$ has a basis of forms with Fourier coefficients in $\mathbb{Z}$ [DI95, Corollary 12.3.8] and that Hecke operators act on them [DI95, Proposition 12.4.1]. This follows for example by the duality of modular forms and the Hecke algebra and the fact that the Hecke algebra acts on homology with coefficients in some finitely generated abelian group. You can briefly explain this for $k = 2$ following [DS05, theorem 6.5.1]. See also [DI95, §12.4]. For $N = 1$ everything follows trivially from the description $M(SL_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6]$ and the explicit description of the Hecke action. From the discussion, it also follows another fact that it might be useful to keep in mind: Fourier coefficients of modular forms are actually algebraic integers ([DI95, Corollary 12.4.3, corollary 12.4.5]).

3.1.3 Talk 3: Spectra of eigenalgebras (30/4)

This talk is purely algebraic-geometric (with a bit of linear algebra) and contains the main idea of future constructions. The idea is the following: by the duality seen in the previous talk, we can interpret a normalized eigenform f with coefficients in a ring R as ring homomorphisms $\psi_f : \mathbb{T} \to R$ from the Hecke algebra to R, i.e. an R-valued point of $Spec(\mathbb{T})$ (even though, written like this, it is only true for example if R is an algebraically closed field. Already if R is any field, then the map a $\psi_f : \mathbb{T} \to R$ usually represents a Galois orbit of modular forms with coefficients in the algebraic closure). Thus, we want to study the geometry of $Spec(\mathbb{T})$. This can be seen as an algebraic version of the eigencurve.

We do this in generality: define what an eigenalgebra is, namely a commutative algebra $\mathcal{T}$ acting faithfully on a finitely generated flat R-module M ([Bel21, §2.2]). It is often convenient to see it as the image in $End_R(M)$ of a commutative algebra H acting R-linearly on M. This is because if M is a R-module of modular forms, we certainly have the abstract Hecke algebra $\mathbb{T}$ freely generated over $\mathbb{Z}$ by the symbols T_ℓ, S_ℓ for ℓ a prime, and the corresponding eigenalgebra would be the image of $\mathbb{T}$ in $End_R(M)$ (which thus depends on M). Show how the notion behaves under base change ([Bel21, §2.4]), then describe the structure of an eigenalgebra over a field [Bel21, §2.5.1], define systems of eigenvalues for H and $\mathcal{T}$ and relate them ([Bel21, theorem 2.5.9, corollary 2.5.10, corollary 2.5.12])

Then describe as much as you can of the geometry of $Spec(\mathcal{T})$ when R is a DVR ([Bel21, theorem 2.7.4]) and a complete DVR ([Bel21, corollary 2.7.6]). You can give the statement of [Bel21, exercise 2.7.7] for further ideas of geometric properties of $Spec(\mathcal{T})$. Finally, discuss the results of [Bel21, §2.9] that give criteria for checking when $Spec(\mathcal{T})$ is reduced and a tool to compare different modules with $\mathbb{T}$-action (this will be used to compare the construction of eigenvarieties made with different data).

3.2 The p-adic theory

We develop the necessary theory to construct eigenvarieties. We want to modify the previous example of an eigenalgebra as a ring homomorphism of rings $\psi : \mathbb{T} \to End_R(M)$, where $\mathbb{T}$ is commutative and M is a finitely generated flat R-module as follows: we want to replace R with a rigid analytic space W (but one can work more generally with adic spaces) and M with a sheaf of not necessarily finitely generated Banach modules. In order to handle the non-finiteness, we need to use some analysis, and in particular the theory of compact operators. We follow [Bel21], except for background on rigid geometry.

3.2.1 Talk 4: Banach modules, compact operators, Fredholm's Determinant (7/5)

The main aim of this talk is to introduce the Fredholm's Determinant of a compact operator on a Banach space. This is morally the characteristic power series of the operator, having as zeroes the inverse of non-zero eigenvalues. You should explain all the terms appearing in the title and describe the properties of Fredholm's Determinant. This is done in [Bel21, §3.1].

3.2.2 Talk 5: Riesz's theory (14/5)

We proceed with the study of compact operators. The main aim of this talk is to introduce an important property of compact operators: if a is a good zero of the Fredholm's Determinant P_ϕ of a compact operator ϕ acting on Banach R-module M, then there exists a decomposition $M = N \oplus F$, with N finite projective on which $1 - a\phi$ acts nilpotently and F a module on which $1 - a\phi$ is invertible. The decomposition comes from a factorization $P_\phi = (1 - aT)^s Q$, $Q \in R[[T]]$.

Follow [Bel21, §3.2]. Introduce ν-dominant polynomials as in [Bel21, §3.2.1] (they will be needed also in later talks), then follow the path you prefer in order to state and prove [Bel21, Proposition 3.2.18], which is the result sketched above, and [Bel21, Theorem 3.2.19], which is a stronger version allowing other polynomials than $(1 - aT)^s$ in the above factorization. You can also have a look at [Buz07, Theorem 3.3] for a slightly different statement of the latter.

3.2.3 Talk 6: Crash course on rigid geometry (21/5)

This talk will probably be proof-free. You will have to give a survey of rigid geometry. Having seen some rigid geometry before might help, although it is not strictly necessary, as the reference [Bos14] explains everything in detail. The guide below is intended to help you prepare the survey without getting lost in the material. You will probably not have the time to state every proposition and corollary that I've pointed to, but at least you know the way. For the sake of the seminar, what is important is to introduce affinoid algebras (because they are the kind of Banach algebras that are used later) and to talk a bit about the GAGA functor (we will need the rigid analytification of the modular curve in talk 10 in order to define overconvergent modular forms). The only other explicit rigid analytic space that will appear (again in talk 10) is the weight space, which is very simple to describe. A good way to approach the talk might be to introduce affinoid algebras, then try to explain how the rigidification of the affine space looks like, and introduce any concept than you need to explain the example by going backwards.

Here I've written a short guide to get quickly to the definition of a rigid analytic space: define Tate algebras over a complete non-Archimedean field K (§2.2, Definition 2) and interpret them as functions on the closed unit ball (§2.2, Proposition 1). State that they are complete under the sup norm (§2.2, Proposition 3). State Noether normalization (§2.2, Corollary 11) and two of its consequences: Corollary 12, that states that the quotient of a Tate algebra by any maximal ideal is a finite extension of K (and hence a unique extension of norm) and Corollary 13, that states that we can loosely interpret maximal ideals of a Tate algebra as the points of the closed unit ball. State that ideals in Tate algebras are finitely generated and closed (beginning of §2.3). Define affinoid K-algebras, the residue norm, and state Proposition 5 in §3.1. Next, define affinoid K spaces as in the beginning of §3.2: they are the maximal spectrum of an affinoid K algebras. The maximal spectrum of an affinoid algebra can be endowed with the Zariski topology, but one can view them also as closed subsets of the closed unit ball (quotiented by $Aut(\overline{K}/K)$) and thus one can introduce a topology induced by the non-archimedean topology on the ball. This is the canonical topology. Define it (§3.3, definition 1) and introduce the (open sub)spaces of definition 7. State lemma 8 and define what is an affinoid subdomain as in definition 9 and say that the subspaces of definition 7 are in fact affinoid subdomains (I suggest that you at least read the proof of that). Then you can state proposition 19 of §3.3, which says that affinoid subdomains behave as you would expect i.e. they are open and the induced topology is their canonical topology, and corollary 12 in §4.1, which gives the structure of affinoid subdomains. Finally, we are ready to head to the definition of a rigid space; informally it will be a locally ringed space that locally looks like an affinoid space with its canonical topology, however there are some subtleties to take care of. First, define the presheaf of affinoid functions on an affinoid space as in the beginning of §4.1 and state that their stalks are local rings (proposition 1), then move to the important "Tate's acyclicity theorem" (theorem 10 of §4.3), that says that the presheaf of affinoid functions on an affinoid in actually a sheaf if one restricts to finite coverings. The idea is then to consider a Grothendieck topology in the category of affinoid subdomains of an affinoid X, where the coverings are only the finite ones. Define such topology as in definition 3 of §5.1 and its strong version as in definition 4 (propositions 5,10,11 explain why definition 4 is needed, also corollary 5 of §5.2 says that we can extend the sheaf of affinoid functions to a sheaf on the strong topology). State also corollary 9: the strong topology is finer than the Zariski one. You are now ready to define a rigid analytic space (definition 4 of §5.3) and talk about the rigid analytic GAGA.

3.2.4 Talk 7: Adapted pairs and modules of bounded slope (28/5)

In this talk, we specialize what we have learned in talks 4 and 5 to affinoid algebras encountered in talk 6. With notations as in talk 5: we fix some $\nu \in \mathbb{R}$ and we want to find a factorization $P_\phi = QS$, with $Q \in R[T]$ and $S \in R[[T]]$, such that in the corresponding decomposition $M = N \oplus F$, N is the submodule of M on which ϕ acts with generalized eigenvalues of valuation $\leq \nu$. This is possible if R and ν are adapted, and, in this case, N is called the submodule of slope $\leq \nu$. The precise statement is [Bel21, Proposition 3.4.6], and the technical heart is [Bel21, Theorem 3.3.6].

In your talk go through [Bel21, §3.3.1] and then state [Bel21, Theorem 3.3.6] (as it is rather technical, you can choose how many details of the proof to give, if any). Then discuss the notion of adapted pairs in [Bel21, §3.3] and show that there are enough of them [Bel21, Proposition 3.3.12]. Then discuss the notion of a module of bounded slope and [Bel21, Proposition 3.4.2], [Bel21, Proposition 3.4.6]. In the last part talk about links ([Bel21, §3.5]), which are essentially the gluing condition for patching the Banach modules over different affinoid algebras.

3.3 Construction of eigenvarieties

3.3.1 Talk 8: Definition and construction of eigenvarieties (4/6)

This talk is the heart of the seminar. Define an eigenvariety datum [Bel21, §3.6.1]. Then define eigenvarieties [Bel21, Definition 3.6.2] and talk about their construction [Bel21, §3.6.2] with particular attention to [Bel21, Theorem 3.6.3], which states that every eigenvariety datum produces a unique eigenvariety. Then prove [Bel21, Theorem 3.7.1], which explains the name eigenvariety. You can use your remaining time stating some geometric properties of eigenvarieties such as [Bel21, Proposition 3.7.5, 3.7.6, 3.7.7].

3.3.2 Talk 9: Properties of eigenvarieties and a first example (11/6)

In this talk, we want to (further) survey some properties of eigenvarieties. In particular we give a criterion for when an eigenvariety is reduced. In the remaining time we look at a first example.

First of all, state any of [Bel21, Proposition 3.7.5, 3.7.6, 3.7.7] that wasn't stated in the previous talk. Then discuss the notion of classical structures [Bel21, §3.8.1]. They are the crucial tool in order to prove the reducedness criterion [Bel21, Theorem 3.8.8] You can spend some word on the proof. If you have time you can also state a comparison theorem for eigenvarieties [Bel21, Theorem 3.8.10].

In the remaining time construct the eigenvariety of p-adic overconvergent automorphic forms for the group $G = Res_{L/\mathbb{Q}}GL_1$, where L is a number field. This is explained in [Buz04, §2]. It does not need the full force of the theory that we have developed.

3.4 The Coleman-Mazur eigencurve

3.4.1 Talk 10: Overconvergent modular forms (18/6)

The aim of this talk is to define the space of overconvergent modular forms. We follow the exposition in [Von, §3] and refer to [Cal13] for more details. Define algebraic modular forms as in [Von, §3.2] and their q-expansion. State the q-expansion principle [Cal13, proposition 1.3.1], then define the Hasse invariant and state its main properties, namely that it has q-expansion 1 and it vanishes only at supersingular elliptic curves (a sketch of the first property is contained in [Cal13, §1.7]). State that the Eisenstein series E_{p-1} (that is G_{p-1} normalized to have the 0-th Fourier coefficient equal to 1) is a lift of the Hasse invariant in characteristic 0 when $p \geq 5$ (this follows from the Von-Staudt - Clausen congruences, that you might want to recall) and use this to define overconvergent modular forms as in [Von, §3.3]. Next, explain how to extend the definition of U_p to overconvergent modular forms following [Cal13, §3.1, §3.3], define a norm on the space of overconvergent modular forms as in [Von, §3.5] and show that U_p is compact with this norm [Cal13, theorem 3.5.5].

3.4.2 Talk 11: Coleman-Mazur eigencurve (25/6)

In this talk, we finally construct the Coleman-Mazur eigencurve. First, we must address the following problem: in the previous talk we have only defined overconvergent modular forms for integral weights

k, but recall that we want our k to vary p-adically. Thus, start with describing the weight space $\mathcal{W}$ and its $\mathbb{C}_p$ points as in [Bel21, §6.3.1] and [Bel21, lemma 6.3.7]. Define overconvergent modular forms of non-integral weight as in [Col97, §B.4] and define the action of the Hecke algebra following [Col97, §B.5]. See also [New24, §2.4]. Finally, give the eigenvariety datum for the Coleman-Mazur eigencurve following [Bel21, §7.2.1]. You can use your remaining time (if any) to do some of the following: you can prove some of the geometric properties of the eigencurve, for example that it is equidimensional of dimension 1 and reduce. To do this you will just have to apply results of talk 9 (cf. [Bel21, Proposition 3.7.5, theorem 3.8.8]). You can also briefly speak of the ordinary part of the eigencurve: although the geometry of the eigencurve is pretty difficult to understand, there is a subspace, the ordinary curve, that it easier, namely the subspace consisting of points on which the U_p operator acts with slope zero (i.e. with eigenvalues that are $\overline{\mathbb{Z}}_p$ units). This follows from the fact that one can parametrize such systems of eigenvalues by applying the ordinary projector $e^{\text{ord}} := \lim_{n\to\infty} U_p^{n!}$ to the algebra $\mathbb{T}^*$ of the introduction and then taking its spectrum. The result follows from theorems of Hida that state that this algebra is finite over $\mathbb{Z}_p[[T]]$ (under the weight map of the introduction). In particular the ordinary curve could be constructed using methods from talk 3. For statements about the ordinary curve you can consult [Eme09, theorem 2.20] (and also [Eme09, page 22]), [Von, §3.7] or [Cal13, §2.2].

3.5 Further topics

3.5.1 Talk 12: Eigencurve of units in definite quaternion algebras over $\mathbb{Q}$ (2/7)

In this talk, we apply the eigenvariety machinery to the group G of units in a definite quaternion algebra over $\mathbb{Q}$. This case is easier to treat than the one of overconvergent modular forms, because $G(\mathbb{R})$ is compact modulo the center. We follow [New24, §4]. You can also look at [Buz04], which is the original source. Define what a definite quaternion algebra over $\mathbb{Q}$ is and the space of p-adic automorphic forms over it (definition 4.1.1) and their structure (proposition 4.1.2). Define also the Hecke operators on them. In order to explain the reason for these definitions, it might be a good idea to see what happens for $G = GL_2$ and what this has to do with modular forms. For this cover briefly the material in [DI95, §11.1] that explains what modular forms have to do with automorphic forms for GL_2. Now state theorem 4.1.5, which explains the relation between automorphic forms for G and for GL_2. We now want to put such automorphic forms in p-adic families. As we have seen so far in the seminar, we need some Banach modules over affinoids covering the weight space and an algebra action on them with a distinguished compact operator. The Banach modules over integer points of the weight space are defined in [New24, definition 4.2.6] and the distinguished operator U_p right below that. Prove that it is compact ([New24, lemma 4.2.7]). Prove also [New24, corollary 4.2.11], which explains the relation between the Banach modules that you have introduced and the module of automorphic forms we have started with. Then define how to interpolate them in order to get a Banach module over affinoids of the weight space ([New24, definition 4.3.1]). You now have all the ingredients to define the eigencurve for G. If you still have time, you can compare it to the Coleman-Mazur eigencurve in view of theorem 4.1.5, that you have previously stated, and the comparison theorem [Bel21, Theorem 3.8.10].

3.5.2 Talk 13: ? (9/7)

For the moment I leave this talk empty. We will skip it if there is no volunteer. If someone has a topic that he/she wants to cover, we can use this spot. Possible topics could be families of Galois pseudo representations over the eigencurve or some link to Langlands seminar.

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