Intersection theory

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This seminar aims to be a gentle introduction to *intersection theory*, following [EH16].

Intersection theory is one of the central branches of algebraic geometry. In its most basic form, it deals with understanding how the intersection of subvarieties changes with the variations of the parameters defining the subvarieties. In modern algebraic geometry, these are the kind of questions studied in enumerative geometry. Nevertheless, the tools and techniques coming from intersection theory turn out to be useful in a surprising wide range of different applications. For example, computations using Chern classes are one of the main tools used in the computation of cohomology rings of different moduli spaces of vector bundles.

In the seminar, we will follow the 2016 book [EH16] for the general theory, while referring to the classical [Ful98] for the proofs or the technicalities not present in [EH16].

The first part of the seminar will focus on the construction of the Chow ring and its properties under pullback and pushforward. The Chow ring of a variety is essentially defined as a free abelian group generated by irreducible subvarieties, with product given by intersection. In order to make this idea precise (in particular in order for the multiplicity of the intersection to be well defined), it is necessary to mod out by an equivalence relation, namely *rational equivalence*. This is basically an algebraic version of having a homotopy deforming one subvariety into the other. One nice aspect of studying this topic in the babyseminar is that we could have time to make several examples, e.g. computing Chow rings of projective spaces, projective surfaces, or compatibility with blow-ups.

As a central example, we will study the intersection rings of the Grassmannians and learn the basics of Schubert calculus. This is a central topic in enumerative geometry, because several enumerative problems are solved by using the Grassmannians as a parameter space and then employing Schubert calculus.

Finally, we will come to the central topic of Chern classes. Throughout algebraic geometry, there are several different ways of defining Chern classes. The one presented in [EH16] is as vanishing loci of general sections. In the seminar we will see in detail the definition and the formulas needed to make computations with Chern classes.

As a conclusion of the seminar, we could compare this with other definitions of Chern classes, or see some applications. One interesting application is an argument presented by Beauville in [Bea95], using Porteus' formula to show that the Künneth components of the Chern classes of the universal bundle on $Bun_n^{d,st}(C) \times C$ generate the cohomology ring of $Bun_n^{d,st}(C)$. There are several different and interesting applications of computations using Chern classes, and depending on the interests of the final speakers they could also focus on some other topic.

References

[Bea95] Arnaud BEAUVILLE. "Sur la cohomologie de certains espaces de modules de fibrés vectoriels". In: *Geometry and analysis (Bombay, 1992)*. Tata Inst. Fund. Res., Bombay, 1995, pp. 37–40. ISBN: 0-19-563740-2.

[EH16] David EISENBUD and Joe HARRIS. *3264 and All That: A Second Course in Algebraic Geometry*. Cambridge University Press, 2016.

[Ful98] William FULTON. *Intersection theory*. Second. Vol. 2. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 1998, pp. xiv+470. ISBN: 3-540-62046-X; 0-387-98549-2. DOI: 10.1007/978-1-4612-1700-8. URL: https://doi.org/10.1007/978-1-4612-1700-8.