Stability conditions

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In 2007 Bridgeland introduced the notion of *stability conditions* on triangulated categories, as a way of generalizing different notions of stability appearing throughout algebraic geometry and crucial for the construction of well-behaved moduli spaces.

In this seminar, we will first see through an explicit example how stability plays a role in the construction of moduli spaces. Then, we will move on to the more general theory of Bridgeland stability.

In the first part of the seminar, we are going to see the construction of the moduli space of (semi)stable vector bundles on a smooth projective curve. This is done through *geometric invariant theory* methods. However, we are not going to dwell on GIT: we are going to use only the tools necessary to the construction of the moduli space and focus instead on the role of stability. The main feature of the notion of *slope-stability* for vector bundles on a curve is the *Harder-Narasimhan filtration*.

Theorem 0.1. *[HN75] Let $\mathcal{E}$ be a vector bundle on a smooth projective curve over an algebraically closed field. Then there exists a canonical filtration*

$$0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_s = \mathcal{E}$$

such that for all i

(a) $\mu(\mathcal{E}_i) := \frac{\deg(\mathcal{E}_i)}{\mathrm{rk}(E_i)} > \mu(\mathcal{E}_{i+1})$;

(b) $\mathcal{E}_{i+1}/\mathcal{E}_i$ *is a semistable vector bundle.*

A nice reference for the construction of the moduli space is [Hos23]. At the cost of a few more technicalities, one could also see directly the construction of the moduli space of semistable sheaves on a projective scheme, as is neatly explained in [HL10]. Of course, the original papers by Seshadri, Newstead or Simpson remain to this day worthy of mention. For the aims of the seminar, the material contained in [Hos23] suffices, so that will be the main reference for this section.

The second part of the seminar is the main one, both from a conceptual point of view and for the amount of time we are going to spend on it. Here, we will see the general theory of Bridgeland stability conditions in abelian categories and triangulated categories, together with some geometric applications.

As mentioned, the main feature of slope-stability for vector bundles is the existence of the Harder-Narasimhan filtration. Therefore, the key idea involved in the definition of stability conditions in an abelian category is that objects should admit a canonical filtration with stable quotients. This idea is further generalized to a triangulated category $\mathcal{D}$ via the notion of *slicing*. This essentially amounts to choosing a family of abelian categories sitting inside $\mathcal{D}$ (obtained as hearts of a family of t-structures), where to look for stable quotients.

The main theorem concerning stability conditions on a triangulated category is the following, found in the original paper by Bridgeland.

Theorem 0.2. *[Bri07, Corollary 1.3.] The set of stability conditions $Stab(D)$ on a triangulated category D can be given naturally the structure of a complex manifold.*

One may hope to use Bridgeland stability conditions to attack some geometric problems. Here are a couple of relevant examples.

- The choice of a stability condition picks a class of stable objects, for which one can hope to construct a well-behaved moduli space.
- Bondal-Orlov reconstruction theorem tells us that a smooth projective variety with ample (or anti-ample) canonical bundle is determined by its derived category $D^b(X)$. Therefore, understanding the triangulated category $D^b(X)$ is vital in the classification of smooth projective varieties. One way one can hope to get some control over this category is by studying the associated manifold of stability conditions. In many interesting cases (e.g. for Calabi-Yau 3-folds) it is still unknown whether the space of stability conditions is nonempty.

In the seminar we will first go through the definition of stability conditions in abelian categories. Then, after introducing the necessary machinery of t-structures and slicings, move on to the case of triangulated categories. The main result we are going to prove is Bridgeland's Theorem 0.2.

Over the last few years, Bridgeland stability has become a classical topic in algebraic geometry and there several valid lecture notes or survey papers that explain the theory in detail. Two in particular that are worth mentioning and that we could follow for the seminar are [MS17] and [Huy14].

If time allows, we could devote the last part of the seminar to complementary results or examples. There are several interesting results also coming from other areas, for example in the case of quiver representations. In the seminar, we could devote the remaining time to the study of $D^b(X)$, where X is a surface or in particular a smooth projective $K3$-surface. The following are two interesting results proven in this case.

Theorem 0.3. *[Tod08] Let X be a $K3$-surface and fix a stability condition on $D^b(X)$. Then the moduli stack of semistable objects is an Artin stack.*

Theorem 0.4. *[BM14] The coarse moduli space of semistable objects exists and is a normal projective irreducible variety.*

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