Introduction

Periods:

1) A period is a complex number defined as an integral $\int_\gamma \omega$ where ω is some algebraic differential form over some algebraic variety (defined over $\mathbb{Q}$) and γ is a certain integration domain.

2) Towards Hodge structures.

k a subfield of $\mathbb{C}$. A $(\mathbb{Q}, k)$-vector space is a triple $(H_B, H_{dR}, comp_{B,dR})$ where

·) H_B finite dim. $\mathbb{Q}$-vector space;

·) H_{dR} finite dim. k-vector space;

·) $comp_{B,dR}: H_{dR} \otimes_k \mathbb{C} \to H_B \otimes_{\mathbb{Q}} \mathbb{C}$ isomorphism of $\mathbb{C}$-vector spaces.

Let X be a smooth affine variety over k. Then there is a $(\mathbb{Q}, k)$-vector space $(H^n_{sing}(X, \mathbb{Q}), H^n_{dR}(X, k), comp_{B,dR})$ where $comp_{B,dR}: H^n_{dR}(X,k) \otimes_k \mathbb{C} \to H^n_{sing}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$ is induced by the perfect pairing

$$\begin{aligned} H^n_{dR}(X,k) \times H_n^{sing}(X,\mathbb{Q}) &\longrightarrow \mathbb{C} \\ (\omega, \gamma) &\longmapsto \int_\gamma \omega \end{aligned}$$

In general, $comp_{B,dR}$ does not respect the rational structures

<u>Example</u>: $X = \mathbb{G}_m$ over $\mathbb{Q}$. $H^1_{dR}(\mathbb{G}_m, \mathbb{Q}) = \mathbb{Q} \cdot \frac{dz}{z}$, $H_1^{sing}(\mathbb{G}_m(\mathbb{C}), \mathbb{Q}) = \mathbb{Q} \cdot \gamma$ with γ the unit circle. Then $\int_\gamma \omega = 2\pi i \notin \mathbb{Q}$.

"Periods" are the numbers appearing as coefficients of the matrix of $comp_{B,dR}$ with respect to rational bases for any choice of $X/\mathbb{Q}$.

3) Motivic interpretation.

<u>Goal</u>: Construct a category of "motives" $MM_{Nori}(\mathbb{Q})$, which should enjoy the following properties:

·) it is abelian

·) universal cohomology theory (compatible with singular cohomology);

·) $MM_{Nori}(\mathbb{Q})$ is Tannakian (f.d. repr.'s of an algebraic group scheme);

·) two fiber functors (Betti and de-Rham realization) $\omega_B, \omega_{dR}: MM_{Nori}(\mathbb{Q}) \to \mathbb{Q}\text{-Vec}$.

There is a torsor of isomorphisms between these two functors, and a complex point of this yields the comparison isomorphism as above.

The entries of these isomorphism are "periods".

Let us expand on the last points.

Let G be an affine group scheme over a subfield k of $\mathbb{C}$. Consider the category $\mathcal{C} = \mathrm{Rep}_k(G)$ of finite dimensional representations of G on k-vector spaces.

Some properties of this category:

·) $\mathcal{C}$ is abelian;

·) We can make the tensor product of two representations ($\mathcal{C}$ is a tensor category with identity object);

·) We can make the dual of a representation, which is "very well-behaved" ($\mathcal{C}$ is rigid);

·) There is an exact k-linear tensor functor $\mathcal{C} \to k\text{-VectorSpaces}$, given by forgetting the action of the group G ($\mathcal{C}$ has a fibre functor $\mathcal{C} \to k\text{-VectorSpaces}$).

Actually, these properties characterize the category of representations of an affine group scheme.

Def: A k-linear category $\mathcal{C}$ is a "Tannakian category" if

·) $\mathcal{C}$ is abelian;

·) $\mathcal{C}$ is a tensor category with identity object;

·) $\mathcal{C}$ is rigid;

·) there is an exact k-linear tensor functor $\omega: \mathcal{C} \to k\text{-Vector spaces}$.

Every (neutral) Tannakian category is equivalent to $\mathrm{Rep}_k(G)$ for some affine group scheme G, which is called the "fundamental group" of $\mathcal{C}$.

In the seminar, we will do much more than this: we will mimic the Tannakian formalism for categories of the form $\mathrm{Rep}_R G$, where R is either a field or a Dedekind domain. This is because singular cohomology naturally produces abelian groups before $\mathbb{Q}$-vector spaces, so we want to have a fiber functor of the form $\mathcal{MM}_{Nori}(k) \to \mathbb{Z}\text{-Mod}$.

Thus, our category of Nori motives will fit in a diagram like this:

$$\begin{tikzcd} Sch/k \arrow[d, "H^*_{Nori}"'] \arrow[dr, "H^*_{sing}"] & \\ MM_{Nori}(k) \arrow[r, "\omega_B"'] & \mathbb{Z}\text{-Mod} \end{tikzcd}$$

So, singular cohomology factors through Nori motives, and $\omega_B : MM_{Nori}(k) \to \mathbb{Z}\text{-Mod}$ is a fiber functor. Thus, $MM_{Nori}(k) \cong Rep_{\mathbb{Z}} G$ for some affine group scheme G, the "motivic fundamental group".

We actually have another diagram like the one above:

$$\begin{tikzcd} Sch/k \arrow[d, "H^*_{Nori}"'] \arrow[dr, "H^*_{dR}"] & \\ MM_{Nori}(k) \arrow[r, "\omega_{dR}"'] & k\text{- Vector spaces.} \end{tikzcd}$$

Up to some change of coefficients, ω_{dR} is also a fibre functor.
By the existence of the comparison isomorphism, ω_B and ω_{dR} become isomorphic when changing coefficients to $\mathbb{C}$.

$$\begin{tikzcd} & \mathbb{C}\text{- Vector spaces} \\ MM_{Nori}(k) \arrow[ur, "\omega_B"] \arrow[dr, "\omega_{dR}"'] & \\ & \mathbb{C}\text{- Vector spaces} \arrow[uu, Rightarrow, "comp_{B,dR}"'] \end{tikzcd}$$

We can represent the isomorphisms between ω_B and ω_{dR} by some affine group scheme $X = \mathrm{Spec}\, A$ over $\mathbb{Z}$.
X is a torsor under G in the fpqc-topology

For a faithfully flat $\mathbb{Z}$-algebra T, the T points of X are
$X(T) = \mathrm{Iso}\,(\omega_{dR} \otimes_k T, \omega_B \otimes_{\mathbb{Z}} T)$
The comparison isomorphism $comp_{B,dR}$ is a complex point of X, so $comp_{B,dR} \in X(\mathbb{C})$.
We therefore obtain an evaluation map of regular functions at $comp_{B,dR}$, say $ev: A \longrightarrow \mathbb{C}$, $f \mapsto f(comp_{B,dR})$
Def: A "period" is a complex number which lies in the image of this evaluation map.
Period conjecture: $ev: A \to \mathbb{C}$ is injective.

In the context of Nori motives, it is possible to give a very explicit description of the k-algebra A. From an arithmetic point of view, the period conjecture tells us that all algebraic relations among periods are described by elementary operations with integrals.

List of the talks:

1) <u>Algebraic and holomorphic de Rham cohomology</u> 17/04

2) <u>Mixed Hodge structures</u> 24/04

Comparison isomorphism, definition of pure and mixed Hodge structures with some basic properties.

3) <u>Nori's diagram category</u> (Clémentine) 8/05

D a diagram (connected graph with identities)

$T : D \to R\text{-Mod}$ a representation

Diagram category: $\mathcal{C}(D,T) = \mathrm{End}(T)\text{-Mod}$ (D finite)

$$\mathcal{C}(D,T) = \underset{F \subseteq D \text{ finite}}{2\text{-colim}} \mathrm{End}(T|_F)\text{-Mod}$$

For $T : D \to R\text{-Proj}$, $A(D,T) = \underset{F \subseteq D \text{ finite}}{\mathrm{colim}} \mathrm{End}(T|_F)^{\vee}$ is a coalgebra.

Then $\mathcal{C}(D,T) = A(D,T)\text{-Comod}$

4) <u>Universal property of the diagram category</u> 15/05

$$\begin{tikzcd}
 & \mathcal{C}(D,T) \arrow[dr, "f_T"] \arrow[dd, dashed, "\exists!"] & \\
D \arrow[ur, "\tilde{T}"] \arrow[rr, "T"] \arrow[dr, "F"'] & & R\text{-Mod} \\
 & \mathcal{A} \arrow[ur, "f"'] &
\end{tikzcd}$$

f_T faithful, exact, R-linear

$\mathcal{A}$ R-linear abelian

f faithful, exact, R-linear

F representation

5) <u>Nori's rigidity criterion</u> (Niklas) 22/05

- Tensor structure on the diagram category (coming from products of diagrams)
- Given a product structure on D, we see that $A(D,T)$ is a bialgebra.

We can set $M = \mathrm{Spec}\, A(D,T)$ (a monoid scheme). Then

$\mathcal{C}(D,T) \cong \mathrm{Rep}_R(M)$.

- When is M a group scheme? We need to check when $A(D,T)$ is a Hopf-algebra.

This is the content of Nori's rigidity criterion.

6) <u>Nori motives</u> (Thiago) 29/05

Define the diagram $\mathrm{Pairs}^{\mathrm{eff}}$ with vertices: (X, Y, i) with X a k-variety, Y a closed subvariety of X and i an integer.

$H^*: \mathrm{Pairs}^{\mathrm{eff}} \to \mathbb{Z}\text{-Mod}, \quad (X, Y, i) \mapsto H^i_{sing}(X, Y)$

(Effective) Nori motives: $MM^{\mathrm{eff}}_{\mathrm{Nori}}(k) = \mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}, H^*)$

\+ Basic lemma

7) <u>Rigidity of Nori motives</u> (Lukas) 5/06

Apply Nori's rigidity criterion in order to prove that $MM_{\mathrm{Nori}}(k)$ is rigid.

8) <u>Equivalence of the definitions of periods</u> (Giulio) 12/06

The whole story about periods in the introduction. At the end, we construct the evaluation map $ev: A \to \mathbb{C}$ and state the period conjecture.

9) <u>1-Motives</u> (Guillermo) 19/06

We want to prove the period conjecture "for curves".

$d_1 MM_{\mathrm{Nori}}(k)$: subcategory of $MM_{\mathrm{Nori}}(k)$ generated by motives of curves.

$d_1 MM^{\mathrm{eff}}_{\mathrm{Nori}}(k)$ is the diagram category associated with the subdiagram $D \subseteq \mathrm{Pairs}^{\mathrm{eff}}$ given by vertices of the form (X, Y, i) with $\dim X = 0, 1$.

We can formulate the period conjecture for this smaller category.

Explicit description of $d_1 MM_{\mathrm{Nori}}(k)$ via semi-abelian varieties.

10) The analytic subgroup theorem (Paolo) 26/06

Main tool to prove the period conjecture for curves

11) The period conjecture for 1-motives 3/07

Glue together the results of the previous talks to prove the period conjecture for curves

12) Ideas? 10/07

Possible topics:

·) the proof of the analytic subgroup theorem;

·) the anti-equivalence between 1-motives and Nori 1-motives;

·) comparison between categories of motives;

·) motivic multiple zeta values.