On canonical models of Shimura varieties

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Abelian type Shimura data; algebraic models of Shimura varieties; canonical models and their connected version; proofs of existence; pro-Galois descent;

1 Shimura varieties of Abelian type

Definition 1. *A pair (G,X) with G a reductive $\mathbf{Q}$-group and X a $G(\mathbf{R})$-conjugacy class of homomorphisms*

$$h\colon \mathbf{S} \to G_{\mathbf{R}}$$

such that the following condition holds.

(SV1) For some h representing X (equivalently any) the induced weights of

$$\mathrm{ad}\circ h\colon \mathbf{S} \to \mathfrak{g}$$

lie in $\{(-1,1),(0,0),(1,-1)\}$.

(SV2) For some (resp. any) representative h of X the involution $h(i)$ is a Cartan involution of $G^{\mathrm{ad}}_{\mathbf{R}}$, meaning

$$G^*(\mathbf{R}) = \{g in G(\mathbf{C}) \mid \mathrm{ad}(h(i))g = \overline{g}\}$$

is a compact real Lie group.

(SV3) G^{ad} has no $\mathbf{Q}$-simple factor G' on which projection of h is trivial. Equivalently such that $G'(\mathbf{R})$ is compact.

This data defines for us, for each given compact open subgroup $K \subset G(\mathbf{A}_f)$ a variety

$$M(G,X)_{\mathbf{C}} = G(\mathbf{Q})\backslash X \times G(\mathbf{A}_f)/K$$

which one knows (cf. Baily-Borel) to be a quasi-projective variety.

The limit

$$M(G,X)_{\mathbf{C}} = \lim_{K} M_K(G,X)_{\mathbf{C}}$$

is a quasi-compact and separated scheme over $\mathbf{C}$, and is called the Shimura "variety" associated to (G,X). There is a canonical continuous action of $G(\mathbf{A}_f)$ on $M(G,X)_{\mathbf{C}}$.

Let now $G(\mathbf{R})^+$ be the connected component in the real topology of this group. Let also $G(\mathbf{Q})^+$ be the intersection of $G(\mathbf{Q})$ with this group.

Lemma 2. *For every connected component X^+ of X there is an isomorphism*

$$G(\mathbf{Q})^+\backslash X^+ \times G(\mathbf{A}_f)/K \xrightarrow{\sim} G(\mathbf{Q})\backslash X \times G(\mathbf{A}_f)/K$$

Definition 3. *A Shimura datum (G,X) is said to be of* abelian type *if there is another Shimura datum (G_2,X_2), an embedding $(G_2,X_2) \hookrightarrow (\mathrm{GSp},\mathfrak{H})$ and an central isogeny $G^{\mathrm{der}} \twoheadrightarrow G_2^{\mathrm{der}}$ inducing*

$$(G^{\mathrm{ad}},X^{\mathrm{ad}}) \xrightarrow{\sim} (G_2^{\mathrm{ad}},X_2^{\mathrm{ad}}).$$

To make sure that this is not a frivolous definition, let us mention an example of Shimura datum of abelian type which is not of Hodge type.

Let (V,q) be a quadratic space over $\mathbf{Q}$ which is of signature $(2,n)$ for $n > 0$. One associates as usual a connected reductive group $\mathrm{SO}(V)$, the connected component of the group of transformations preserving q. One defines the associated *Clifford algebra $C(V)$* via

$$C(V) = \bigotimes V/\langle v \otimes v - q(v)\rangle$$

which is a natural C_2-graded $\mathbf{Q}$-algebra $C(V) = C(V)^+ \oplus C(V)^-$. Then the group $\mathrm{GSpin}(V)$ is the reductive $\mathbf{Q}$-group whose rational points are defined as

$$\mathrm{GSpin}(V) = \{x \in (C(V)^+)^\times \mid xVx^{-1} = V\}.$$

One checks that the conjugation action on V is by isometries and hence defines a central extension $\mathrm{GSpin}(V) \twoheadrightarrow \mathrm{SO}(V)$ with kernel $\mathbf{G}_m$. The algebra $C(V)$ has furthermore an involution $x \mapsto x^*$ which gives us a norm map $\nu\colon \mathrm{GSpin}(V) \twoheadrightarrow \mathbf{G}_m$ mapping $x \mapsto xx^*$.

We can relate $\mathrm{GSpin}(V)$ to $\mathrm{SO}(V)$ via the following diagram:

$$\begin{tikzcd} & \mathrm{Spin}(V) \arrow[d, hook] \arrow[dr, "2:1"] & \\ \mathbf{G}_m \arrow[r, hook] \arrow[dr, "2:1"'] & \mathrm{GSpin}(V) \arrow[r, two heads] \arrow[d, two heads, "\nu"] & \mathrm{SO}(V) \\ & \mathbf{G}_m & \end{tikzcd}$$

We then have an isogeny from the derived group $\mathrm{Spin}(V)$ of $\mathrm{GSpin}(V)$ to $\mathrm{SO}(V)$ which induces an isomorphism on the adjoint groups $\mathrm{GSpin}(V)^{\mathrm{ad}} \cong \mathrm{SO}(V)^{\mathrm{ad}}$.

Up to details not made precise here (cf. [Mad16, §1.6]) one can use the (reduced) trace of $C(V)$ to construct a (non-canonical) symplectic pairing on $C(V)$ which induces an embedding

$$\mathrm{GSpin}(V) \hookrightarrow \mathrm{GSp}(C(V)).$$

In order to see that we have a Shimura datum of abelian type, we now need only to construct the associated Hermitian symmetric domains. (This is where our assumption on the signature comes in.) Briefly, this will be X: the space of *oriented* 2-planes L of $V_{\mathbf{R}}$ for which $(L_{\mathbf{R}}, q)$ is negative definite, we have a Shimura data $(\mathrm{SO}(V), X)$ of abelian type. Further details ommited. (For a beautifully written version of the above terse exposition, cf. [Shi78]).

1.1 Classification (Deligne)

Let (G, X) be a Shimura datum and $T \subset G$ a maximal torus and $B \supset T$ a Borel. The conjugacy class X determines a cocharacter

$$\mu \colon \mathbf{G}_{m,\mathbf{C}} \to T_{\mathbf{C}} \subset G_{\mathbf{C}},$$

uniquely so if required to be positive with respect to B.

Then one can show that a $G(\mathbf{C})$-conjugacy class of cocharacters corresponds to a Shimura datum if, and only if, when you decompose

$$\mu = \sum n(\alpha)\alpha,$$

REST OF SECTION IS §2.3.7 - §2.3.13 *ibid*. READ MORE CAREFULLY AND COME BACK HERE.

2 Canonical models of Shimura varieties

Definition 4. *Let (G,X) be a Shimura datum. A* model *(or* form*) of $M_{\mathbf{C}}(G,X)$ over some finite extension $E/\mathbf{Q}$ is an E-scheme $M(G,X)$ endowed with a $G(\mathbf{A}_f)$-action together with an equivariant isomorphism $M(G,X)\otimes_E \mathbf{C} \xrightarrow{\sim} M_{\mathbf{C}}(G,X)$.*

A model is said to be weakly canonical *if the set of special points is algebraic and the action of* $\mathrm{Gal}(\overline{\mathbf{Q}}/E)\cap\mathrm{Gal}(\overline{\mathbf{Q}}/E(\tau))$ *on the special points of type τ agrees with the canonical action of* $\mathrm{Gal}(\overline{\mathbf{Q}}/E(\tau))$.

A model is said to be canonical *if it is weakly canonical and defined over the reflex field $E(G,X)$.*

Deligne has showed in his foundational works that special points are dense in $M_{\mathbf{C}}(G,X)$ and hence weakly canonical are unique over a given base, if they exist.

The case of existence is as good as possible.

Theorem 5 (Deligne, Borovoi, Milne-Shih, Milne, Moonen)**.** *Let (G,X) be a Shimura datum. Then canonical models exist uniquely.*

Given the importance of the theorem above, we go into the three proofs available (with their varying degrees of generality).

2.1 The historical one: Deligne's original proof for abelian type Shimura varieties

To explain the extent of Deligne's proof and method, we need some preliminaries. The strategy of this proof is to reduce the construction to the simple case of Hodge type. To do this, we define a connected version of the notion of (weakly) canonical models.

2.1.1 Recollection on connected components

The exposition belows mimics [Moo98], which in turn is just a re-explanation of Deligne's original ideas in [Del79].

Let us recall how the connected components work. Let (G^{ad},G',X^+) be a triple with G^{ad} an adjoint group over $\mathbf{Q}$, $G'\to G^{\mathrm{ad}}$ a covering and X^+ a $G(\mathbf{R})^+$-conjugacy class

$$X^+ \subset \mathrm{Hom}(\mathbf{S},G^{\mathrm{ad}}_{\mathbf{R}})$$

defining a Shimura variety (cf. [Del79, §2.1.8] or [Moo98, §1.6.5]).

Then we can define a connected Shimura variety the following way: define a topology $\tau(G')$ on $G^{\mathrm{ad}}(\mathbf{Q})$ using the images of congruence subgroups $K \subset$ "$G'(\mathbf{Q}) \cap G'(\mathbf{Z})$" of G' via the cover map $G' \to G^{\mathrm{ad}}$. Then

$$M^0(G^{\mathrm{ad}}, G', X^+)_{\mathbf{C}} = \lim_{\Gamma} \Gamma \backslash X^+$$

is the limit indexed over all compact open subgroups of $G^{\mathrm{ad}}(\mathbf{Q})$. Naturally $G^{\mathrm{ad}}(\mathbf{Q})$ acts on this space, and this action is continuous for the topology τ by design, hence descends to an action of $G^{\mathrm{ad}}(\mathbf{Q})^{+\wedge}$

Deligne then shows that for a Shimura data (G,X) and $X^+ \subset X$ a connected component then $M^0(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^+)$ is a connected component of $M(G,X)_{\mathbf{C}}$ corresponding to image of X^+ via the natural quotient, and depends only on G^{ad}, G^{der} and of course on X^+ ([Del79, §2.1.8]).

The question of "how many" components a Shimura variety has is of course delicate as it is not of finite type, and hence we expect $\pi_0(M(G,X)_{\mathbf{C}})$ to be profinite. At finite level, one has ([Del79, §2.1.3])

$$\pi_0 M_K(G,X)_{\mathbf{C}} = G(Q)_+ \backslash G(\mathbf{A}_f)/K$$

and at infinite level $\pi_0 M(G,X)$ is a torsor under $G(\mathbf{A}_f)/\overline{G(\mathbf{Q})^+}$ [Del79, Prop. 2.1.14].

We expand a bit more on the actions at hand. (Taken from [Moo98, §2.7] which comes from [Del79, § 2.1].) The closed center[1] $\overline{Z(\mathbf{Q})} \subset G(\mathbf{A}_f)$ acts trivially on $M(G,X)_{\mathbf{C}}$ by design, and hence extends to an action of $G(\mathbf{A}_f)/\overline{Z(\mathbf{Q})}$ on the Shimura variety at infinite level. Another action comes from the adjoint action of G^{ad} on G by conjugation, which, by functoriality, induces an action of

$$G^{\mathrm{ad}}(\mathbf{Q})_1 = G^{\mathrm{ad}}(\mathbf{Q}) \cap G^{\mathrm{ad}}(\mathbf{R})_1, \quad G^{\mathrm{ad}}(\mathbf{R})_1 = \mathrm{Im}\left(G(\mathbf{R}) \to G^{\mathrm{ad}}(\mathbf{R})\right),$$

on the Shimura variety also. (We remind the reader that the quotient map $G \to G^{\mathrm{ad}}$ is not surjective on k-valued points necessarily.)

Now the important relaization ([Del79, §2.1.13]) is that the action of $g \in G^{\mathrm{ad}}(\mathbf{Q})_1$ on the left agrees with the action of g^{-1} via $G(\mathbf{A}_f)$ on the right. Hence it induces an action by the group

$$\Delta = \left(G(\mathbf{A}_f)/\overline{Z(\mathbf{Q})}\right) *_{G(\mathbf{Q}/Z(\mathbf{Q})} G^{\mathrm{ad}}(\mathbf{Q})_1 = \left(G(\mathbf{A}_f)/\overline{Z(\mathbf{Q})}\right) *_{G(\mathbf{Q}^+/Z(\mathbf{Q})} G^{\mathrm{ad}}(\mathbf{Q})^+$$

on the Shimura variety.

[1]Perhaps a pedantic point, but quite often $Z(\mathbf{Q})$ is already closed for the adelic topology. This is true for example when the weight homomorphism is rational and $h(i)$ is a Cartan involution of $G/w(G)$ (Cf. [Del79, Cor. 2.1.11]).

Now in view of the the simple transitive action of $G(\mathbf{A}_f)/\overline{G(\mathbf{Q})^+}$ on $\pi_0(M(G,X)_{\mathbf{C}})$ we conclude that the stabilizers fixing a given component is identified with the group

$$G^{\mathrm{ad}}(\mathbf{Q})^{+\wedge} \cong \left(\overline{G(\mathbf{Q})^+}/\overline{Z(\mathbf{Q})}\right) *_{G(\mathbf{Q}^+/Z(\mathbf{Q})} G^{\mathrm{ad}}(\mathbf{Q})^+$$

where we complete, again, using the topology from congruence subgroups of G^{der}.

2.1.2 Connected canonical models and Galois descent

Now suppose that we have a model $M(G,X)_E$ over E of $M(G,X)_{\mathbf{C}}$. Then we have a Galois semi-linear action by $\Gamma = \mathrm{Gal}(\overline{Q}/E)$ on $M(G,X)_{\mathbf{C}}$. Hence a subgroup $\mathscr{E}(G^{\mathrm{der}},X^+) \subset \Delta \times \Gamma$ with

$$1 \to G^{\mathrm{ad}}(\mathbf{Q})^{+\wedge} \to \mathscr{E}_E(G^{\mathrm{der}},X^+) \to \Gamma \to 1$$

stabilizing a component $M^0(G^{\mathrm{ad}},G^{\mathrm{der}},X^+)$ of the Shimura variety.

We can now replicate Deligne's definition of canonical models for connected Shimura varieties [Del79, §2.7.10].

Definition 6. *Let (G,G',X^+) be a connected Shimura datum. Let $E/\mathbf{Q}$ be a number field containing $E(G,X^+)$. a* weakly canonical model *for $M^0(G,G',X^+)_{\mathbf{C}}$ is a $\bar{\mathbf{Q}}$-model $M^0(G,G',X^+)$ together with an identification*

$$i : M^0(G,G',X^+) \otimes_{\bar{\mathbf{Q}}} \mathbf{C} \xrightarrow{\sim} M^0(G,G',X^+)_{\mathbf{C}},$$

a continuous left action by $\mathscr{E}_E(G',X^+)$ which is Γ-semilinear via the natural quotient with i being such that it is equivariant for the action of the subgroup $G^{\mathrm{ad}}(\mathbf{Q})^{+\wedge}$, and finally that special points are algebraic and the action is compatible with the above in a suitable sense.

A weakly canonoical model is said to be a canonical model *if furthermore $E = E(G,X)$.*

To justify the definition a bit more, we need to unravel the Galois descent that goes into the main result of [Del79]. We remark that to use these techniques one really needs to be in an affine situation, or, as present, in one where one has an ample invertible sheaf available.

Proposition 7. *A scheme (cf. remark above) S over $\overline{\mathbf{Q}}$ with a semi-linear $\mathscr{E}$-action amounts to the same as a scheme S_E over E with a $G^{\mathrm{ad}}(\mathbf{Q})^{+\wedge}$-action together with an equivariant map*

$$(S_E)_{\overline{\mathbf{Q}}} \to \pi = \pi_0(M(G,X)_{\mathbf{C}}),$$

where we see the right hand side as a profinite sets embedded into $\overline{\mathbf{Q}}$-schemes in the usual way.

Proof. This is [Del79, §2.7.9]. Essentially one splits both actions and use Galois descent. □

This proposition justifies the following Theorem, which is done with ε more work.

Theorem 8. *Let (G,X) be a Shimura datum. Then the equivalence of categories in the previous proposition induces an equivalence between* weakly *canonical models $M(G,X)$ and* weakly *canonical models $M^0(G^{\mathrm{ad}},G^{\mathrm{der}},X^+)$.*

To finish the main result, we need only fix the problems with the reflex fields. Of course, this is the most crucial part of the proof. For clarity, even if you have a canonical model for $M^0(G^{\mathrm{ad}},G^{\mathrm{der}},X^+)$ it could be that $E(G,X) \supsetneq E(G^{\mathrm{ad}},X^+)$ is a proper field extension!

2.1.3 The last clutch: toric nudging of $E(G_1,X_1)$s

To recall, we have a Shimura datum (G,X) and a Hodge-type Shimura datum (G_1,X_1) with an isogeny $G_1 \to G$ inducing an iso on the connected $(G_1^{\mathrm{ad}},X_1^{\mathrm{ad}}) \xrightarrow{\sim} (G^{\mathrm{ad}},X^{\mathrm{ad}})$ Shimura data.

Proposition 9. *Suppose that for all finite extensions F of E there is a finite extension F' of E in $\overline{\mathbf{Q}}$ which is linearly disjoint to F and a weakly canonical model of $M^0(G,G',X^+)$ over F'. Then there is a weakly canonical model of $M^0(G,G',X^+)$ over E.*

Proof. This is [Del74, §5.10]. □

So even if $E(G_1^{\mathrm{ad}},X^{\mathrm{ad}})$ is too small we just need to guarantee that it is big enough often enough for the desired theorem to hold. Finally, Deligne finishes the proof of the theorem "by hand" for explicit group types. (I don't really undestand whether this is suppose to cover all abelian types. Moonen seems skeptical, see remark after Theorem 2.13.)

Theorem 10. *Let (G,X) be a Shimura datum, and suppose that*

$$(G^{\mathrm{ad}},X^{\mathrm{ad}}) \cong (G_1,X_1) \times \cdots \times (G_n,X_n)$$

with each G_i simple. Then if each (G_i,X_i) is of type $A,B,C,D^{\mathbf{R}},D^{\mathbf{H}}$—where in type $D^{\mathbf{H}}$ one further assume that G^{der} is a quotient of a prescribed double cover (an inner form of $\mathrm{SO}(2n)$*)—then the canonical model exists.*

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