Geometric Unramified Class Field Theory,

(according to Deligne)

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These are notes for the Research Seminar Essen SS2025. They are loosely based around Bhatt's notes on geometric class field theory, expanded at the points where I felt that I needed to expand, and which themselves are based around Deligne's proof. Thanks to Sebastian Bartling for answering innumerous questions on the subject and stimulating conversation and Jochen Heinloth for many insightful remarks during the talk (which have since been added).

1 Recollections on Class Field Theory

Let k be a finite field and X a geometrically connected curve over k. Then $F = K(X)$ is a global field in characteristic p. If $x \in X$ is a closed point we denote by $\mathscr{O}_x$ the completion of $\mathscr{O}_X$ at x, and by F_x it's fraction field. Let $\mathbf{A} = \mathbf{A}_F = \prod' F_x$ and $\mathbf{O} = \prod \mathscr{O}_x$.

Recall global class field theory for F. In here we fix a closure F^{sep} of F and a maximal abelian extension $F^{\mathsf{ab}} \subset F^{\mathsf{sep}}$. Let $\Gamma_F^{\mathsf{ab}} = \mathrm{Gal}(F^{\mathsf{ab}}/F)$. Then we've constructed an Artin reciprocity map $\mathrm{Art} = \mathrm{Art}_F$

$$\mathrm{Art}\colon F^\times \backslash \mathbf{A}_F^\times \hookrightarrow \Gamma_F^{\mathsf{ab}}$$

which is the unique homomorphism sending uniformizers to Frobenii. The Artin map is injective and induces an isomorphism $\widehat{\mathrm{Art}}\colon \widehat{F^\times \backslash \mathbf{A}_F^\times} \xrightarrow{\sim} \Gamma_F^{\mathsf{ab}}$.

Our goal is to relate the above theorem to the arithmetic geometry of the curve X/k. We start by stating an unramified variant of the above. Let $F^{\mathrm{nr},\mathsf{ab}} \subset F^{\mathsf{sep}}$ be the maximal unramified abelian extension of F and $G^{\mathrm{nr},\mathsf{ab}}$ be it's Galois group.

Theorem 1.1 (Unramified Class Field Theory)**.** *There is an (injective) Artin reciprocity homomorphism*

$$\mathrm{Art}\colon F^\times \backslash \mathbf{A}_F^\times / \mathbf{O}^\times \hookrightarrow \Gamma_F^{nr,\mathsf{ab}}$$

taking uniformizer to Frobenii. It induces an equivalence on profinite completions.

To see the geometry in this, we prove a baby case of a deep statement about bundles on curves.

Lemma 1.2. *The quotient $\mathbf{A}_F^\times/\mathbf{O}_F^\times$ is canonically isomorphic with the group of divisors on X. The quotient $F^\times\backslash\mathbf{A}_F^\times/\mathbf{O}_F^\times$ is canonically isomorphic to* $\mathrm{Pic}(X)$.

Proof. Just send $(f_x)_x$ to $v_x(f_x)[x]$. □

Lemma 1.3. *The étale fundamental group $\pi_1^{\text{ét}}(X,\eta)$ with coefficients in the generic point identifies canonically with Γ_F^{nr}. Hence, the abelianized fundamental group, $\pi_1^{\text{ét}}(X,\eta)^{\mathrm{ab}} = \pi_1^{\text{ét}}(X,\eta)/\overline{[\pi_1,\pi_1]}$ identifies with $\Gamma_F^{nr,\mathrm{ab}}$.*

Here is a reformulation of Theorem 1 which is equivalent to it but stated in a way which is closer to the Langlands phisolophy and geometry. This implies the theorem above by a Pontrjagin duality style statement, which we sketch below.

Theorem 1.4. *Let ℓ be a prime number, possibly equal to p. There is a bijection between characters*

$$\mathrm{Hom}(\mathrm{Pic}(X),\bar{\mathbf{Z}}_\ell^\times) \leftrightarrow \mathrm{Hom}(\pi_1(X),\bar{\mathbf{Z}}_\ell^\times)$$

such that, if χ corresponds to ρ then $\chi(\mathcal{O}([x])) = \rho(\mathrm{Frob}_x)$.

Remark 1.5. The Picard group $\mathrm{Pic}(X)$ is finitely generated as we will see below, hence it follows that

$$\mathrm{Hom}(\mathrm{Pic}(X),\bar{\mathbf{Z}}_\ell^\times) = \operatorname*{colim}_{E/\mathbf{Z}_\ell}\mathrm{Hom}(\mathrm{Pic}(X),\mathcal{O}_E^\times) = \operatorname*{colim}_{E/\mathbf{Z}_\ell}\mathrm{Hom}(\widehat{\mathrm{Pic}}(X),\mathcal{O}_E^\times) = \mathrm{Hom}(\widehat{\mathrm{Pic}}(X),\bar{\mathbf{Z}}_\ell^\times).$$

Also, if $\pi_1(X,\eta)^{\mathbf{Z},\mathrm{ab}}$ denotes the inverse image of $\mathrm{Frob}^{\mathbf{Z}} \subset \pi_1(\mathbf{F}_q)$ via the map $\pi_1(X)^{\mathrm{ab}} \to \pi_1(\mathbf{F}_q)$, then the image $\mathrm{Art}(\mathrm{Div}) = \pi_1(X)^{\mathbf{Z},\mathrm{ab}}$ and in fact this is a dense subgroup of $\pi_1(X)^{\mathrm{ab}}$ as the argument below implies. One also has $\mathrm{Hom}(\pi_1(X)^{\mathbf{Z},\mathrm{ab}},\bar{\mathbf{Z}}_\ell^\times) = \mathrm{Hom}(\pi_1(X),\bar{\mathbf{Z}}_\ell^\times)$. Hence one can really write a diagram

$$\begin{tikzcd}
\mathrm{Hom}(\mathrm{Pic}(X),\bar{\mathbf{Z}}_\ell^\times) \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathrm{Hom}(\pi_1(X)^{\mathbf{Z}},\bar{\mathbf{Z}}_\ell^\times) \\
\mathrm{Hom}(\widehat{\mathrm{Pic}}(X),\bar{\mathbf{Z}}_\ell^\times) \arrow[r, "\cong"] & \mathrm{Hom}(\pi_1(X),\bar{\mathbf{Z}}_\ell^\times) \arrow[u, hook]
\end{tikzcd}$$

Remark 1.6. There is a group structure on both sides since $\bar{\mathbf{Z}}_\ell{}^\times$ is abelian. A fortiori, this is an isomorphism of groups: the Picard group is generated by principal divisors so we can compare $\chi_{\rho_1\rho_2}$ with $\chi_{\rho_1}\chi_{\rho_2}$ on those, where it follows from the stated condition.

Remark 1.7. The finite generation of $\mathrm{Pic}(X)$ also implies that $\mathrm{Pic}(X) \hookrightarrow \widehat{\mathrm{Pic}(X)}$, and hence the Abel-Jacobi map is already injective before profinite completion.

Lemma 1.8. *Let G be a profinite abelian group and Λ a topological group containing $\mathbf{Q}/\mathbf{Z}$ abstractly. Then if $g \in G$ is different from the identity e, we can find a continuous homomorphism $\chi\colon G \to \Lambda$ with $\chi(g) \neq e$.*

Proof. Since G is Hausdorff we can find an open subgroup U with $g \notin U$. Then G/U is a finite group and we can find a continuous character $\chi'\colon G/U \to \mathbf{Z}/N\mathbf{Z}$ with $\chi'(g) \neq 0$. But by assumption we can embbed $\mathbf{Z}/N\mathbf{Z} \hookrightarrow \Lambda$. □

Proposition 1.9. *Theorem* 1.4 *implies Theorem* 1.1.

Proof. To show that this statement implies the previous one, note that there is a map $\mathrm{Art}\colon \mathrm{Div}_X \to \pi_1(X)^{\mathrm{ab}}$ sending $[x] \mapsto \mathrm{Frob}_x$. The statement can be rephrased by saying that for all χ there is a ρ, and for all ρ there is a χ making

$$\begin{tikzcd} \mathrm{Div}_X \arrow[r, "\mathrm{Art}"] \arrow[d] & \pi_1^{\mathrm{ab}}(X) \arrow[d, "\chi"] \\ \mathrm{Pic}(X) \arrow[r, "\rho"] & \bar{\mathbf{Z}}_\ell^\times \end{tikzcd}$$

commute. Suppose that D is a divisor and that $D = \operatorname{div} f$ for $f \in F$. Then all characters $\rho\colon \mathrm{Pic}(X) \to \bar{\mathbf{Z}}_\ell^\times$ satisfy $\rho(\mathscr{O}(D)) = \rho(\mathscr{O}) = 1$, which implies that for all characters $\chi\colon \pi_1^{\mathrm{ab}}(X) \to \bar{\mathbf{Z}}_\ell^\times$ we have $\chi(\mathrm{Art}(D)) = 1$. But this implies $\mathrm{Art}(D) = 1$ by the Lemma above.

We get then a map $\mathrm{Art}\colon \mathrm{Pic}(X) \to \pi_1(X)^{\mathrm{ab}}$, and if $\mathscr{L}$ is sent to zero, then it has to be sent to zero in $\widehat{\mathrm{Pic}}(X)$ since every cocharacter ρ will kill it and the Lemma above. Hence the map is injective.

Now let $H \subset \pi_1(X)^{\mathrm{ab}}$ be the closure of the image of $\widehat{\mathrm{Pic}}$. The quotient $\pi_1(X)^{\mathrm{ab}}/H$ is a non-zero profinite group and hence admits a non-trivial character by the Lemma above. Lifting this we obtain a cocharacter of $\pi_1(X)^{\mathrm{ab}}$ which is trivial on all Frobenii, but the corresponding ρ would have to be the zero cocharacter. Contradiction. □

As a corollary, we get a Frobenius density statement for global function fields.

Corollary 1.10. *The group $\pi_1(X)^{\mathrm{ab}}$ is topologically generated by the Frobenius elements. The same holds for $\pi_1(X)$ if one replaces Frobenius elements with Frobenius conjugacy classes.*

1.1 How does the Picard group look like?

Here is a small section to help with the intuition behind the results of geometric class field theory. Here is a result we will prove later on:

Proposition 1.11. *If X is a geometrically connected, smooth curve over a finite field then* $\mathrm{Pic}^0(X)$ *is finite.*

Proof. Later. $\square$

So the group $\mathrm{Pic}^0(X)$ works as the finite part of $\mathrm{Cl}(F)$, and so behaves a bit more like its number field counterpart.

Example 1.12 (The rational case)**.** This is the trivial case of the whole theory. Recall that $X/\mathbf{F}_q$ is of genus 0 if and only if it is a form of $\mathbf{P}^1_{\bar{\mathbf{F}}_q}$. Hence, we get a sequence

$$1 \to \cancel{\pi_1(\mathbf{P}^1_{\bar{\mathbf{F}}_q})} \to \pi_1(X) \to \Gamma_{\mathbf{F}_q} \to 1$$

which implies that $\pi_1(X) = \widehat{\mathbf{Z}}$, and that the map $\mathrm{Pic}(X) \to \pi_1(X)$ sending $\mathscr{O}_X(1)$ to the generator (which is the frobenius at every point) has the desired property.

Note that if $\Gamma_k \neq \widehat{\mathbf{Z}}$ then the statement $\widehat{\mathrm{Pic}}(X) \cong \pi_1(X)^{\mathrm{ab}}$, which makes sense for every field k and geometrically integral smooth curve X/k, cannot be true.

Example 1.13 (Elliptic curves)**.** Let (E,e) be an elliptic curve over a finite field. One can show that if $\bar{E}$ is the base change to the algebraic closure then

$$\pi_1(E,e) \cong \prod_{\ell \neq p} \mathbf{Z}_\ell^2 \times Z_p,$$

where Z_p is either $\mathbf{Z}_p$ (ordinary) or trivial (supersingular).

One then has a big diagram with short exact (and canonically split) rows as below. Here, $\pi_1^{\mathrm{ab},0}(E)$ denotes the automorphisms of the maximal étale cover which is defined over k.

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Pic}^0(E) \arrow[r] \arrow[d, "\sim"'] & \mathrm{Pic}(E) \arrow[r] \arrow[d, hook] & \mathbf{Z} \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \mathrm{Pic}^0(E) \arrow[r] \arrow[d, "\sim"', "\mathrm{Art}"] & \widehat{\mathrm{Pic}}(E) \arrow[r] \arrow[d, "\sim"', "\mathrm{Art}"] & \widehat{\mathbf{Z}} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1^{\mathrm{ab},0}(E) \arrow[r] & \pi_1^{\mathrm{ab}}(E) \arrow[r] & \Gamma_k \arrow[r] & 1 \\
1 \arrow[r] & \pi_1(\bar{E}) \arrow[r] \arrow[u, two heads] & \pi_1(E) \arrow[r] \arrow[u, two heads] & \Gamma_k \arrow[r] \arrow[u, equal] & 1
\end{tikzcd}$$

In particular we learn something funny: that the fundamental group of $\pi_1(E)$ cannot be abelian, as otherwise $\pi_1(\bar{E})$, which is abelian, would be isomorphic to the *finite* group $\mathrm{Pic}^0(E)$. As a matter of fact, $\pi_1^{\mathrm{ab},0}(E)$ is the finite group $E(k)$.

Example 1.14. For a general X, the finite part $\mathrm{Pic}^0(X)$ of $\mathrm{Cl}(F)$ is computed as the finite group of k rational points on the Picard variety $\mathrm{Pic}^0_{X/\mathbf{F}_q}$. More details in the next section.

This is a well studied group, but I believe not completely understood: which groups arise from this construction? What if you fix q? We know that unless you allow some ramification then not every subgroup can occur. A result of Stichtenoth [Sti79] says that if E is the exponent of $\mathrm{Pic}^0(X)$ and n is its order, then

$$n < E^{2(48n/e)^4}$$

where e is Euler's constant. In particular $(\mathbf{Z}/2\mathbf{Z})^{10^7}$ cannot occur.

2 A digression on the Picard scheme

Let k be an arbitrary field and X/k a smooth, projective, geometrically connected curve. All products are defined over k. The group $\mathrm{Pic}(X)$ has a algebro-geometric incarnation as the k-rational points of the Picard variety.

Definition 2.1 (The Picard functor)**.** *The Picard group sheaf* $\mathrm{Pic}_{X/k}$ *is defined to be the fppf-sheafification of the functor* $T \mapsto \mathrm{Pic}(X \times T)$.

The sheafification is indeed necessary since there are non-trivial line bundles on $X \times T$ which are trivial T-locally. When X admits a k-rational point, one can show that

$$\mathrm{Pic}_{X/k}(T) = \mathrm{Pic}(X \times T)/\mathrm{Pic}(T)$$

and furthermore, one has an embedding $\mathrm{Pic}_{X/k}(T) \hookrightarrow \mathrm{Pic}(X \times T)$ consisting on those bundles which are trivial along the section $x \times T$.

If there is no k-rational section, not every object in $\mathrm{Pic}_{X/k}(T)$ needs to come from a line bundle on X_T. More precisely there is an extension

$$0 \to \mathrm{Pic}(X_T)/\mathrm{Pic}(T) \to \mathrm{Pic}_{X/k}(T) \to \ker(\mathrm{Br}(T) \to \mathrm{Br}(X \times T)) \to 0.$$

In particular, it is in general not true that $\mathrm{Pic}_{X/k}(k) = \mathrm{Pic}(X)$, unless you can also control the Brauer group. For $X/\mathbf{F}_q$, it is a celebrated fact that $\mathrm{Br}(\mathbf{F}_q) = 0$, hence $\mathrm{Pic}_{X/\mathbf{F}_q}(\mathbf{F}_q) = \mathrm{Pic}(\mathbf{F}_q)$.

Here is a beautiful Theorem which we will state in the smallest generality as needed for our purposes.

Theorem 2.2. *The functor* $\mathrm{Pic}_{X/k}$ *is represented by a disjoint union of proper varieties over* k. *More precisely, it is isomorphic to* $\mathrm{Pic}^0_{X/k} \times \underline{\mathbf{Z}}$ *where* $\mathrm{Pic}^0_{X/k}$ *is an abelian variety.*

To prove this, we compare to another functor, one which we'll be able to describe more easily. Recall that if $X \to S$ is a morphism of schemes, then an effective Cartier divisor $D \subset X$ is said to be a *relative Cartier divisor* (to S) if the induced map $D \to S$ is flat.

Relative Cartier divisors are preserved via pullback, hence can be thought as continuous families of divisors on the fibers. They are also clearly local on S (on X-even).

Definition 2.3. *Let* X/k *be a curve as before. We define the functor of effective divisors*

$$\mathrm{Div}_{X/k} : S \mapsto \{D \subset X_S/S\}$$

sending S *to the set of relative effective Cartier divisors over* S. *This is an fppf-sheaf on schemes over* k.

One can show that the notion of degree works well in families, and hence that one has a decomposition into clopen subschemes

$$\mathrm{Div} = \bigsqcup_{n \geq 0} \mathrm{Div}^n, \quad \mathrm{Pic}_{X/k} = \bigsqcup_{n \in \mathbf{Z}} \mathrm{Pic}^n_{X/k}$$

where $\mathrm{Div}^n(S)$ consists of divisors which are of degree n in every fiber of $s \to S$.

Definition 2.4 (Abel-Jacobi map)**.** *We define the Abel-Jacobi map to be the functor*

$$\mathrm{AJ} : \mathrm{Div}_{X/k} \to \mathrm{Pic}_{X/k} \quad (\mathrm{AJ}_d : \mathrm{Div}^d_{X/k} \to \mathrm{Pic}^d_{X/k})$$

given by sending a divisor $D \subset X_T$ *to* $\mathscr{O}(D) \in \mathrm{Pic}(X_T) \to \mathrm{Pic}_{X/k}(T)$.

Remark 2.5. Traditionally, only AJ_1 is called the Abel-Jacobi map, after identifying $\mathrm{Div}^1 \cong X$ (cf. below).

It is of surprising elegance that this definition works without any assumptions on S, as any relative effective Cartier divisor is an effective Cartier divisor in the usual sense, and these always have associated line bundles.

Now to prove the representability of $\mathrm{Pic}_{X/k}$ it is enough by the group scheme structure to show that $\mathrm{Pic}^d_{X/k}$ for high enough d. Recall the classical theorem that when $\mathscr{L}$ is effective, then the set $|\mathscr{L}|$ of divisors linearly equivalent to $\mathscr{L}$ is given by

$$|\mathscr{L}| \xrightarrow{\sim} \mathrm{H}^0(X, \mathscr{L}) \setminus \{0\}/k^* = \mathbf{P}^m(k).$$

The Abel-Jacobi map allow us to understand the situation as follows:

Proposition 2.6. *Let $d > \max\{2g-1, 0\}$. Then the Abel-Jacobi map* $\mathrm{AJ}\colon \mathrm{Div}^d \to \mathrm{Pic}^d$ *is a fiber bundle with fibers $\mathbf{P}^{d-g}$, étale-locally trivial (even Zariski locally if it admits a section). In particular, if* Div *is representable then so is* Pic.

Proof. By [BLR90, §8.2 Prop. 7], one even has that AJ^d is a Zariski locally trivial projective bundle if one assumes that X has a rational point. The bundle is given by the pushforward of the universal line bundle trivialized at the given section. One needs the degree asumption to use the Riemann-Roch Theorem to prove the relative representability here.

In general, one reduces to this situation since étale locally X has a rational point. □

Remark 2.7. Here is a way to see from this that if k is a finite field then $\mathrm{Pic}^0(X)$ is finite. It is enough to show that $\mathrm{Pic}^d(X)$ is finite for some d, but this admits a surjection from $\mathrm{Div}^d(k)$, for which one can easily give an upper bound.

Note that this would follow from the reprsentability of Pic if you assume the existence of a rational point or if you assume Weddeburn's Theorem so that you know that $\mathrm{Pic}^0(X) = \mathrm{Pic}^0_{X/k}(k)$.

Example 2.8. Exercise: Use the above to give a relation between the rational points of Pic^0 and of X, as explicitly as you can.

Now, we turn to the representability of Div. Intuitively, a point in Div corresponds to a sum of points in X, but the order does not matter. To make this idea precise, we construct a morphism

$$\Sigma\colon X^d \to \mathrm{Div}^d$$

which takes sections $f_i\colon T \to X$ to the divisor $\Sigma_{i=1}^d \Gamma_{f_i}$, which indeed is a relative effective Cartier divisor since X is smooth. On a:

Proposition 2.9. *The map Σ above induces an isomorphism*

$$\Sigma\colon X^d/S_d \xrightarrow{\sim} \mathrm{Div}^d$$

where X^d/S_d is the fppf-sheafification of the naive presheaf quotient of X^d by the S_d-action permuting the components.

Proof. We've seen that Div^d is a sheaf. Clearly the induced map on the pre-sheaf quotient

$$\Sigma^{\mathrm{pre}}\colon (X^d/S_d)^{\mathrm{pre}} \hookrightarrow \mathrm{Div}^d$$

is a monomorphism. Call a section in the image of Σ^{pre} *horizontal*. To show that it becomes an isomorphism after sheafification, we have to show that any section $T \to \mathrm{Div}^d$ is fppf-locally on T horizontal.

But $D \to T$ is an fppf-cover for $d > 0$. (The reader is invited to deal with this case on their own.) Base changing along itself we get a section, that is, the diagonal, hence D_D has at least one horizontal component. Throwing away all horizontal components, we finish by inducion on d. $\square$

Finally, we have the last breath of the proof of the main Theorem of the section. Namely that X^d/S_d is a scheme and can be locally computed by the invariant sections of the S_d action on the associated rings. The miracle, in dimension 1, is that this quotient is even smooth.

Proposition 2.10. *The sheaf X^d/S_d is represented by a scheme. If $\operatorname{Spec} A \subset X$ is an affine open, then $\operatorname{Spec}(A^{\otimes d})^{S_d}$ is an affine open of X^d/S_d and these cover X^d/S_d. Furthermore, X^d/S_d is smooth proper variety.*

Proof. I will not mention much about the first part, which is well known. For the smoothness, note that this is an étale local statement, which can be reduced to $X = \mathbf{A}^1$ and the result is the celebrated theorem on symmetric polynomials, valid over any field and characteristic:

$$k[T_1,\dots,T_n]^{S_n} \cong k[T'_1,\dots,T'_n].$$

The properness follows from the surjection $X^d \to X^d/S_d$. $\square$

Remark 2.11. In fact, Pic is even projective but this is a much harder and celebrated Theorem due to Weil. It uses the moduli description to cook up an ample line bundle using the universal one.

We leave the proof that this implies the Main Theorem of this section as an exercise to the reader.

3 The geometrization: character local systems

In this section we let $\Lambda = \bar{\mathbf{Z}}_\ell$, where ℓ is an arbitrary prime number, k be an arbitrary field and k^{sep} a fixed *separable* closure.

If X is an algebraic variety over k, it follows from Grothendieck's reinterpretation of Galois theory that $\operatorname{Hom}(\pi_1(X), \Lambda^\times)$ can be seen as isomorphism classes of rank 1 local systems on X, canonically once a base point $x = \operatorname{Spec} k^{\text{sep}} \to X$ has been chosen.

Definition 3.1. *For any X locally noetherian scheme we denote by $\operatorname{Loc}_1(X,\Lambda) = \operatorname{Loc}_1(X)$ the category of rank 1 local systems of Λ-modules on X, that is, the subcategory of étale Λ-modules on X which are étale-locally isomorphic to Λ.*

If X is connected, then the set of isomorphism classes in $\operatorname{Loc}_1(X)$ is identified with $\operatorname{Hom}(\pi_1(X), \Lambda)$.

Concretely, given a local system L on X, trivial over some Galois cover Y/X, we can pick a point $y \to Y$ lifting x and then $\pi_1^{\text{ét}}(X,x)$ will act on Λ linearly by sending automorphism ϕ to

$$\Lambda = L_y \xrightarrow{\sim} L_{\phi(y)} = \Lambda.$$

where equality is used to emphasize the trivialization. The converse is given by descent and over the cover corresponding to the kernel of $\pi_1^{\text{ét}}(X,x) \to \Lambda^\times$ (or the universal cover).

Much more interesting is how to categorify the morphisms $\mathrm{Pic}(X) \to \Lambda^\times$. We start with a definition. If G is a smooth algebraic group over k we denote by $m\colon G \times G \to G$ its multiplication and by $\pi_i\colon G \times G \to G$ its natural projections for $i = 1,2$.

Definition 3.2. *Let G be a smooth* commutative *algebraic group over k. A* character local system *on G is a local system of Λ-modules L on G such that there exists an isomorphism*

$$m^*L \cong L \boxtimes L := \pi_1^*L \otimes \pi_2^*L.$$

We denote the group of isomorphism classes of such, where a morphism is just a morphism of local systems, as $\mathrm{CharLoc}(G,\Lambda) = \mathrm{CharLoc}(G)$*. Note that this implies that the rank of L is* 1.

One can show that if G is connected, then the isomorphism above has a canonical representative satisfying some sort of cocycle condition. More precisely there is an equivalence of groupoids between the category of character local sytems on G and $\underline{\mathtt{Hom}}_{\mathsf{Grp}}(G, B\Lambda)$, the mapping space in the $(2,1)$-category of group stacks.

Example 3.3. Here is an example/exercise: let G be the group $\mathbf{Z}$ seen as a constant group scheme over k. Show that a rank n character local system in the sense above is the same as a rank n local system on $\operatorname{Spec} k$, ie. to a continuous representation $\pi_1(k) \to \mathrm{GL}(\Lambda, n)$.

You can generalize this to other free abelian groups. More crucial however is the observation that if G is a finite (or torsion) constant group, then all character local systems on G are trivial.

We can now state the main theorem of the section: a categorification result for the homomorphism space.

Theorem 3.4. *Let G be an extension of a connected commutative algebraic group over k and a constant group scheme of the form $\mathbf{Z}^n$. Then there is a canonical bijection*

$$\mathrm{Hom}(G(k), \Lambda^\times) \cong \mathrm{CharLoc}(G)/\cong$$

between characters of $G(k)$ and isomorphism classes of character local systems on G.

Proof. We begin by reducing to the case of G connected. Let $0 \to G_0 \to G \to \mathbf{Z}^n \to 0$ be a short exact sequence of commutative k-group schemes with G_0 connected then

$$0 \to \mathrm{CharLoc}(\mathbf{Z}) \to \mathrm{CharLoc}(G) \to \mathrm{CharLoc}(H) \to 0$$

is exact. A cheating argument is to use that G has a rational point (the identity) and so the original sequence is in fact split and CharLoc is an additive functor. Now it since comparision isomorphism is canonical and natural (cf. below) we now assume that $G = G_0$ is connected by the five lemma.

The comparison isomorphism is given by the "Sheaf-Function correspondence". Namely we consider

$$\begin{aligned} \mathtt{SF}\colon \mathrm{CharLoc}_1(G) &\to \mathrm{Hom}(G(k), \Lambda^\times) \\ L &\mapsto (x \mapsto \mathtt{tr}(\mathrm{Frob}_x | L_x)). \end{aligned}$$

More precisely if $x\colon \operatorname{Spec} k \to G$ is a rational point then the Frobenius arise via the pullback

$$x^* : \mathrm{Loc}_1(G) \to \mathrm{Loc}_1(\operatorname{Spec} \mathbf{F}_q) = \Lambda\text{-}\mathsf{Mod}^{\widehat{\mathbf{Z}}}_{\mathtt{rk}=1}$$

and we can take the trace in this category. This is indeed a function with values in $\Lambda^\times$, since the Frobenius is an invertible operator on a 1 dimensional vector space. To see that it is a morphism of groups, use that $m^*L \cong L \boxtimes L$ and the fact that

$$M_{x+y} = (x,y)^* m^* M \cong (x,y)^* M \boxtimes M = M_x \otimes M_y$$

in $\mathrm{Loc}_1(\operatorname{Spec} \mathbf{F}_q)$. (This also implies that the trace is invertible.)

The converse starts with a map $f\colon G(\mathbf{F}_q) \to \Lambda^\times$. The Lang isogeny is defined for G connected as follows

$$\begin{aligned} L_G\colon G &\to G \\ g &\mapsto \mathrm{Frob}(g) g^{-1} \end{aligned}$$

It is a surjective group homomorphism (isogeny) which is even a Galois étale cover with fiber $G(\mathbf{F}_q)$.

(Sketch: compute the differential at identity which will be multiplication by -1 since the Frobenius differential vanishies. This implies surjectivity and smoothness by connectedness. The kernel (which is then smooth) can be seen to be finite by computing the closed points $\ker L_G(\bar{k}) = G(\bar{k})^{\widehat{\mathbf{Z}}} = G(k)$.)

Now by descent we get a local system N on G for which $L_G^* N$ is trivial. This is a character local system since $L_G \times L_G = L_{G\times G}$ and so we can descend the trivial isomorphism $m^*\Lambda \cong \Lambda \boxtimes \Lambda$ to N.

We have to check that these constructions are inverse to each other. Note that if $x \in G$ and Frob_x is its Frobenius conjugacy class (recall that $\pi_1(G)$ could be non-abelian) then Frob_x gets sent to $x \in G(k)$ via the Lang isogeny. Hence the trace of Frobenius of N at x is just $f(x)$.

A similar argument yields that if you know that (M,ψ) is a character sheaf and $L_G^* M \cong \Lambda$ then $M \cong N$. This is by descent along the Lang isogeny, which amounts to putting a $G(\mathbf{F}_q)$-structure on Λ. To see that this is true, we use that $\mathrm{Frob}^* M \cong M$ to see that

$$L_G^* M = (\mathrm{Frob}^*, \iota)^* m^* M = \mathrm{Frob}^* M \otimes \iota^* M \cong M \otimes \iota^* M = (1_G, \iota)^* m^* M \Lambda$$

since $m \circ (g, g^{-1})$ factors through the identity section, and $e^* M \cong \Lambda$ by the character sheaf property. □

Corollary 3.5. *The group* $\mathrm{Hom}(\mathrm{Pic}(X), \Lambda^\times)$ *is isomorphic to* $\mathrm{CharLoc}(G)$.

4 Geometric Categorified Unramified Class Field Theory

Let k be an arbitrary field, X a geometrically connected smooth curve. As before $\Lambda = \bar{\mathbf{Z}}_\ell$ where ℓ is a prime, possibly equal to the characteristic of the field. We can now state the Main Theorem of these notes.

Theorem 4.1 (Main Theorem)**.** *Pullback along the Abel-Jacobi map induce an equivalence of categories*

$$\mathrm{AJ}_1^* : \mathrm{Loc}_1(X,\Lambda) \xrightarrow{\sim} \mathrm{CharLoc}(\mathrm{Pic}_{X/k}, \Lambda).$$

Lemma 4.2. *Projective spaces* $\mathbf{P}^d_{\bar{k}}$ *are simply connected.*

Proof. The Riemann-Hurwitz formula tells us that $\mathbf{P}^1_{\bar{k}}$ is simply connected. For $n \geqq 2$, we can use induction and the fact that on connected smooth variety of dimension $\geqq 2$ over an algebraically closed field effective ample divisors are connected. This implies that if H is a hyperplane of $\mathbf{P}^n_{\bar{k}}$ then the pre-image over a connected étale cover will be connected, hence map isomorphically to H by induction. But this implies the degree is 1.

Alternatively, one can use that product of simply connected varieties over algebraically closed fields to conclude that $\mathbf{P}^1 \times \cdots \times \mathbf{P}^1$ is simply connected. Then one uses that π_1 is a birational invariant to conclude the result. □

Proof (of Theorem). Consider the sequence of maps

$$X^d \to [X^d/S_d] \to X^d/S_d = \mathrm{Div}^d \to \mathrm{Pic}^d_{X/k}$$

Choose d high enough so that $d > \max\{2g-1, 0\}$ and we have that $f : \mathrm{Div}^d \to \mathrm{Pic}^d$ is a projective bundle. SGA1.X.1.7 (which we can use because this is a proper morphism with geometrically reduced, connected fibers), we have a sequence

$$\pi_1(\mathbf{P}^1_{\bar{k}}) \to \pi_1(\mathrm{Div}^d) \to \pi_1(\mathrm{Pic}^d) \to 0$$

which by the lemma above we use to conclude that the middle arrow is an isomorphism. More crucially, pulling back induces an iso

$$\mathrm{Loc}_1(\mathrm{Pic}^d) \xrightarrow{\sim} \mathrm{Loc}_1(\mathrm{Div}^d).$$

Now start with a local system L on X. The box product

$$L \boxtimes \cdots \boxtimes L \in \mathrm{Loc}_1(X^d)$$

admits a canonical S_d-action, which for $d = 2$ looks like

$$\sigma^* L \boxtimes L = \sigma^*(p_1^* L \otimes p_2^* L) = p_2^* L \otimes p_1^* L \xrightarrow{\sim} p_1^* L \otimes p_2^* L,$$

which hence defines a line bundle on $[X^d/S_d]$. In fact the category of local systems on $\mathrm{Div}^d = X^d/S_d$ sits fully faithfully inside $\mathrm{Loc}_1([X^d/S_d])$ via the pullback map and is identified with those local systems with a trivial inertial action. This can be formalized, for example, using Noohi's computation of the étale fundamental group of coarse moduli spaces associated to a stack [Noo04, Thm. 7.2].

Crucially, because we are in the rank 1 case, the inertial action of $L \boxtimes \cdots \boxtimes L$ is trivial, eg. for $d = 2$ in the locus $x_1 = x_2$ say, we have that the canonical "switch" map

$$L \otimes L = \Delta^*(\sigma^* L \boxtimes L) \to \Delta^*(L \boxtimes L) = L \otimes L$$

is the identity.

Now we can define a composite functor

$$(\_)_d : \mathrm{Loc}_1(X) \to \mathrm{Loc}_1(X^d)^{S_d}_{\mathrm{trv.in}} \cong \mathrm{Loc}_1(\mathrm{Div}_d) \xleftarrow{\sim} \mathrm{Loc}_1(\mathrm{Pic}_{X/k}).$$

and we must show that in fact we can extend these to $\mathrm{CharLoc}(\mathrm{Pic}_{X/k})$. To argue this we first note that

$$L_d \boxtimes L_e \cong L_{d+e}. \tag{1}$$

whenever we have defined all terms in the above equation. This begs us to define $L_d = L_{-d}^{-1}$ when $-d$ has been defined, and then

$$L_d = L_{n+d} \boxtimes L_{-d}$$

whenever $n+d$ and $-d$ have been defined. Equation (1) now implies that this is a well defined local system.

It is clear that these constructions are indeed inverse to each other. $\square$

As mentioned before, the startling thing about this formulation is that it makes sense over every field. For $k = \mathbf{C}$, it boils down to the identification of $\pi_1^{\mathrm{ab}}(X) \xrightarrow{\sim} \pi_1(\mathrm{Pic}^0_{X/k})$ for a curve X. Using the analytic topology, one can even use $\mathbf{Z}$-coefficients.

Restated in the language of Stacks, the Main Theorem says that

$$\mathrm{AJ}_1^* : \underline{\mathrm{Hom}}(\mathrm{Pic}_{X/k}, B\Lambda^\times) \xrightarrow{\sim} \underline{\mathrm{Hom}}(X, B\Lambda^\times)$$

where $B\Lambda^\times$ is the classifying stack of Λ-local systems. Hence, $X \to \mathrm{Pic}_{X/k}$ behaves as a sort of commutative group-stackification of X. This cannot be true on the nose, since in fact one also has a stacky version of the Abel-Jacobi map

$$\mathrm{AJ}_1 : X \to \mathscr{P}ic_{X/k}$$

mapping X to the Picard stack $T \mapsto B\mathbf{G}_m(X_T)$. However, this $\mathscr{P}ic_{X/k}$ is still not exactly the commutative group-stackification of X, it only has a universal property for *reflexive* group stacks (A confusion arises from the distiction between the Jacobian and the Albanese variety, which is the Jacobian of the Jacobian).

For an alternative proof of the categorical unramified geometric class field theory pursuing these ideas, see the notes on "Unramified geometric class field theory and Cartier duality" by Justin Campbell.

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