HENSELIAN LOCAL RINGS

Notation and remarks.

In this document we give an overview of basic and fundamental properties of (strictly) henselian local rings and henselian pairs, from an algebraic, geometric and topological point of view. These objects arise as the local rings in the étale topology of schemes and, crucially, can be thought as being contractible from a tubular neighoubourhoods which are homotopy retracts of the closed locus over which they are henselian.

Here we adopt the slightly unusual notation where we denote the local rings by $(\mathscr{O}, \mathfrak{m}, k)$, to alude to their everyday use.

Definition 1. A local ring $\mathscr{O}$ is said to be *henselian* when, for all monic polynomials $f \in \mathscr{O}[t]$ and all factorizations

$$\overline{f} = \overline{g}\overline{h} \tag{1}$$

where $\overline{g}, \overline{h}$ are *coprime* monic polynomials in $k[t]$, we can lift this to a factorization to $f = gh$, where g and h reduce to $\overline{g}$ and $\overline{h}$.

If k is furthermore separably closed, we say that $\mathscr{O}$ is *strictly henselian*.

Remark. We do not need to ask for both $\overline{g}$ and $\overline{h}$ to be monic, only one suffices.

Example (Hensel)**.** Let $(\mathscr{O}, \mathfrak{m}, k)$ be a *complete* local ring. Then $\mathscr{O}$ is henselian.

Example. Quotients of henselian rings are henselian.

Example. Let X be a scheme, $\xi\colon \operatorname{Spec} k \to X$ a point of X and $\overline{\xi}\colon \operatorname{Spec} \Omega \to \operatorname{Spec} k \to X$ where $k \subset \Omega$ is an extension of k with Ω

separably closed[1]. An *étale neighborhood* of ξ is an étale X-scheme U/X together with a lift

$$\begin{tikzcd} & U \arrow[d] \\ \operatorname{Spec} k \arrow[ur, dashed] \arrow[r, "\xi"] & X \end{tikzcd}$$

The colimits

$$\mathcal{O}^{\mathrm{h}}_{X,\xi} = \operatorname*{colim}_{\xi \to U} \Gamma(U, \mathcal{O}_U), \quad \mathcal{O}^{\mathrm{hs}}_{X,\xi} = \mathcal{O}_{X,\bar{\xi}} = \operatorname*{colim}_{\bar{\xi} \to U} \Gamma(U, \mathcal{O}_U),$$

are henselian local rings (Corollary 3). They exists due to the local description of standard étale maps, which reduce the above colimit to one indexed by a small set.

Remark. The étale topos of a field $\operatorname{Spec} k$ is identified with the topos of Γ_k sets for the absolute galois group Γ_k. Therefore from the étale point of view it is $\operatorname{Spec} \mathcal{O}^{\mathrm{hs}}_{X,\xi}$ that is the true local ring of X at the point ξ. Nonetheless the conept of $\mathcal{O}^{\mathrm{h}}_{X,\xi}$ is still very useful.

The proof of the fact that these rings are henselian will follow from our discussion on "henselianization". For now it suffices to note that from the example above, strictly henselian local rings are the stalks of the structure sheaf on the étale site, hence its fundamental importance.

Lemma 1. If $\mathcal{O}$ is henselian, and g, h are lifts as in 1, then they are *strictly coprime* in the sense that they generate $\mathcal{O}[t]$.

Proof. Consider the $\mathcal{O}$-module

$$\mathcal{O}[t]/(g,h);$$

this is finitely generated, since one of these lifts is monic, and the special fiber is 0, NAK implies the result. □

Corollary 1. Under the same hypothesis, g, h are unique.

There are many different characterizations of henselian rings; first lets explore the similar variants.

[1] One may usually always assume that $\Omega = \bar{k}$. We do not do this for two reasons. First it will take a while to show that étale cohomology is independent of choice of Ω (need locally acyclic base change) and also for convenience: eg. when $k = \mathbf{Q}$ it is sometimes clearer to choose $\mathbf{Q} = \mathbf{C}$ instead of $\overline{\mathbf{Q}}$.

Theorem 1. *Let $X = \operatorname{Spec}\mathcal{O}$ with closed point $x \in X$; then the following are equivalent:*

1. *The local ring $\mathcal{O}$ is henselian;*
2. *All finite X-schemes are a disjoint union of local schemes.*
3. *All quasi-finite and separated X-schemes are a disjoint union of finite local schemes and something not lying over the closed point.*
4. *For all (affine) étale X-schemes Y, the map $Y(\mathcal{O}) \twoheadrightarrow Y(k)$ is surjective.*
5. *For all (affine) smooth X-schemes Y the map $Y(\mathcal{O}) \twoheadrightarrow Y(k)$ is surjective.*
6. *For all $f \in \mathcal{O}[t]$ and a simple root $\alpha \in k$ of $\overline{f}$, there is a lift $a \in \mathcal{O}$ reducing to α;*

Proof. **For** 1 **implies** 2: Any finite X-scheme is affine (because X is); let A be its global sections. By the going-up theorem, any maximal ideal $\mathfrak{n}$ of A must lie over $\mathfrak{m}$; letting

$$A_k = A \otimes_{\mathcal{O}} k = A/\mathfrak{m}A,$$

we have that $\operatorname{Spec}A_k$ is identified with the maximal spectrum of A.

We first assume that $A = \mathcal{O}[a] = \mathcal{O}[x]/f$ is a primitive extension, where f is monic. Then the maximal spectrum of $A_k = k[x]/\overline{f}$ corresponds to a factorization $\overline{f} = \overline{f}_1 \ldots \overline{f}_n$; by induction we can lift this to a factorization $f = f_1 \ldots f_n$ where the f_i are in pairs strictly coprime by lemma 1, which implies that we can use the CRT and get our desired result.

If the result is true for A then it is true for A/I for any ideal I; in particular we have proved the result whenever A is generated by one element. In the general case, if A is not already zero, then we can find an element $a \in A$ and the morphism $\mathcal{O}[a] \subset A$ will be finite; the map $\operatorname{Spec}A \twoheadrightarrow \operatorname{Spec}\mathcal{O}[a]$ is surjective (finite extensions of rings induce surjective morphisms on spectra) and so the decomposition of the latter implies a decomposition of the first.

For 2 **implies** 3: If Y/X is quasi-finite and separated, we obtain by Zariski's "main theorem" a factorization

$$Y \hookrightarrow \coprod Y_i \twoheadrightarrow X$$

with each Y_i local with closed point y_i. Therefore one has a factorization

$$Y = Y_0 \coprod \coprod_{y_i \in Y} Y_i$$

and $(Y_0)_k = 0$.

For 3 **implies** 4: We can suppose that Y is affine (hence separated), and finitely presented over $\mathcal{O}$. The fiber over any (geometric) point is now étale and finitely presented over a field, whence discrete and Y/X is quasi-finite. Applying the hypothesis, we can even suppose that Y/X is finite étale of degree 1, hence an isomorphism.

For 4 **implies** 5: By adding more equations and passing to the non-zero locus of the Jacobian, we can suppose Y/X is smooth of dimension 0, that is, it is standard étale. The result now follows by hypothesis.

For 5 **implies** 1: Let $f \in \mathcal{O}[t]$, and write

$$f = \sum_{i=0}^{n} f_i t^i,\ g = \sum_{j=0}^{r} g_j t^j,\ h = \sum_{k=0}^{s} h_k t^k,$$

with f_n, h_s non-zero and $g_r = 1$ (so g is monic) and $\overline{f} = \overline{g}\overline{h}$. Then the equation $f = gh$ is a solution $(g_0, \dots, g_{r-1}, h_1, \dots, h_s)$ of the polynomials

$$B : G_0 H_0 = f_0,\ G_0 H_1 + H_0 G_1 = f_1,\ \dots, H_s = f_n.$$

The rank of the the jacobian of the above equations, when reduced to k, is maximal if $\overline{f}$ and $\overline{g}$ are coprime by general properties of the resolvent; therefore this is smooth at a neighborhood of $(\overline{f}, \overline{g})$ and we may lift this solution by assumption. □

Corollary 2. All finite local $\mathcal{O}$-algebras are henselian.

Corollary 3. Let X be a scheme and $\overline{\xi}, \xi \to X$ be a geometric point and a point. Then $\mathcal{O}^{\mathrm{h}}_{X,\xi}$ and $\mathcal{O}^{\mathrm{hs}}_{X,\xi}$ are henselian.

Proof. Indeed, let $\mathcal{O}$ be either one of the above rings and take an affine étale $U \to \operatorname{Spec}\mathcal{O}$. Then since $\mathcal{O} = \lim U$ for the diagram of all $\xi, \overline{\xi} \to U$, any étale map such as $U \to \operatorname{Spec}\mathcal{O}$ comes via base change from some étale $\widetilde{V} \to V$. Now if $\xi, \overline{\xi} \to U$ is a section, then this defines a point on $\widetilde{V} \to V$ which implies that it lies in the diagram category. Hence, from the definition of limit, there is a map $\operatorname{Spec}\mathcal{O} \to \widetilde{V}$ which determines a section $\mathcal{O} \to U$ via pullback. Diagramatically we have:

$$\begin{tikzcd} & U \arrow[r] \arrow[d] & \widetilde{V} \arrow[d] \\ & & V \arrow[d] \\ \xi, \overline{\xi} \arrow[r] \arrow[uur] & \operatorname{Spec}\mathcal{O} \arrow[r] \arrow[uur, dotted] & X \end{tikzcd}$$

□

Corollary 4. A local $\mathcal{O}$ is henselian if and only if its its reduction $\mathcal{O}_{\mathtt{red}}$ is. A (local) ring with only one prime ideal is henselian.

Proof. The first part implies the second since the reduction is a field, which is clearly henselian. The first part follows from the fact that $\operatorname{Spec}R = \operatorname{Spec}R_{\mathtt{red}}$ as topological spaces for any ring R, and the fact that for any finite $\mathcal{O}$ algebra A, the reduction $A_{\mathtt{red}}$ splits a product of local $R_{\mathtt{red}}$-algebras by assumption. □

Proposition 1. If X is the spectrum of a henselian ring $\mathcal{O}$ and $\operatorname{Spec}k \to \operatorname{Spec}\mathcal{O}$ is the closed point, $\Gamma = \operatorname{Gal}(k^{\mathsf{sep}}/k)$ the absolute Galois group, then the pullback functor

$$\mathsf{Fet}(X) \xrightarrow{\sim} \mathsf{Fet}(k) \cong G\text{-}\mathsf{Set}$$

is an equivalence of categories.

In particular, if $\overline{x} = \operatorname{Spec}k^{\mathsf{sep}}$, then $\pi_1^{\mathsf{\acute{e}t}}(X, \overline{x}) = \pi_1^{\mathsf{\acute{e}t}}(x, \overline{x})$, and therefore we have a canonical isomorphism

$$\Gamma_k = \operatorname{Gal}(k^{\mathsf{sep}}/k) \cong \operatorname{Gal}(K^{\mathrm{nr}}/K),$$

or equivalently a short exact sequence

$$0 \longrightarrow I \longrightarrow \operatorname{Gal}(\overline{K}/K) \longrightarrow \operatorname{Gal}(k^{\mathsf{sep}}/k) \longrightarrow 0$$

where I is the subgroup corresponding to the Galois extension K^{nr}/K (the *inertia subgroup* of x.)

Proof. For essential surjectivity: We know that finite étale algebras over k are just products of separable finite field extensions. Therefore it suffices to show that any such extension l/k comes from a finite étale (local) $\mathscr{O}$-algebra. This is now follows from the primitive element theorem, the criterion 6 above, and the fact that this will be now standard étale.

For fully faithfulness: We let A, A' be two finite étale $\mathscr{O}$-algebras and lets show that

$$\hom_{\mathscr{O}}(A, A') \to \hom_k(A \otimes_{\mathscr{O}} k, A' \otimes_{\mathscr{O}} k)$$

is a bijection. We can suppose that A, A' are connected and therefore $A = \operatorname{Spec}\mathscr{O}[x]/f(x)$ for some f separable and the result follows by Hensel's lemma. $\square$

By this last result that étale covers of $\mathscr{O}$ don't change upon completion for henselian local rings. The following result can be interpreted as extending this to possibly branched covers as well.

Proposition 2. Let $\mathscr{O}$ be a henselian local ring with fraction field K, $\mathscr{O}^\wedge$ the completion at the maximal ideal and $K^\wedge$ the fraction field of the completion. Then the pullback defines an equivalence of categories

$$\mathsf{Fet}(K) \xrightarrow{\sim} \mathsf{Fet}(K^\wedge),$$

and therefore an isomorphism of absolute Galois groups $\Gamma_K \cong \Gamma_{K^\wedge}$.

Proof (SGAIV.X.2.2.1). We consider the obvious functor

$$(\_)^\wedge : \mathsf{Fet}(K) \to \mathsf{Fet}(K^\wedge).$$

To see essential surjectivity we just note that $K^\wedge$ is still henselian and hence it is enough to show this for monogenic finite étale algebras, on which case it is of the form $K^\wedge[T]/f$ for f monic with $\mathscr{O}^\wedge$-coefficients. Then we note that

$$(K[T]/f_0)^\wedge \cong K^\wedge[T]/f$$

for f_0 a polynomial in $\mathscr{O}$ which is congruent to f modulo $\mathfrak{m}^N$ for N big.

For the fully faithfulness (the faithfulness being clear) $\square$

On left adjoints Here we construct, and give geometric interpretations, on the processes of making local rings henselian, and henselian rings strict. The construction is a variant on the construction of (separable) closure, and can be seen as a stalkwise comparison between two topologies.

Proposition 3 (Strictification)**.** The inclusion of the category of strictly henselian local rings into the category of henselian ones has a left adjoint

$$\mathscr{O} \mapsto \mathscr{O}^{\mathrm{s}}$$

called the *strictfication* of $\mathscr{O}$.

Furthermore, $\mathscr{O}^{\mathrm{s}}$ is a filtered colimit of étale $\mathscr{O}$-algebras.

Proof. We start by noting that for any extension l/k gives us a unique finite étale and extension $\mathscr{O} \subset \mathscr{O}_l$ given by the equivalence between $\mathsf{Fet}(X)$ and $\mathsf{Fet}(x)$ in Proposition 1.

$$\begin{tikzcd}
\mathscr{O} \arrow[r] \arrow[d] & \mathscr{O}_l \arrow[d] \\
k \arrow[r] & l
\end{tikzcd}$$

Concretely it is given by adjoining a separable element lifing the one generating the extension $k \subsetneq l$.

The rings $\mathscr{O}_l$ are indeed henselian, being finite over $\mathscr{O}$. Since henselian local rings are closed under filtered colimits, we see that

$$\mathscr{O}^{\mathrm{s}} = \operatorname*{colim}_{\mathsf{Fet}(x)} \mathscr{O}_l$$

satisfies the adjoint universal property we are looking for. □

Remark. Functoriality follows immediately from the universal property. We note that the counit

$$\mathscr{O} \xrightarrow{\sim} \mathscr{O}^{\mathrm{s}}$$

is an isomorphism for all strictly henselian local rings $\mathscr{O}$. Also the residue of $\mathscr{O}^{\mathrm{s}}$ is clearly k^{sep}.

An equivalent way of describing this ring is as the integral closure of $\mathscr{O}$ inside of the extension K^{nr}/K. In particular $\mathscr{O} \to \mathscr{O}^{\mathrm{s}}$ is integral.

Example. Let k be a field and consider the complete (henselian) local ring $A = k[[T]]$. The strictification of A can be computed as the colimit

$$A^{\mathrm{s}} = \bigcup_{[l:k]<\infty} l[[T]].$$

A similar result follows in more variables.

We can also construct henselization functors and strict henselization functor directly from the category of local rings. To do this we consider the following construction.

Proposition 4 (Henselizations and strict henselizations)**.** The inclusion of the category of henselian local rings (resp. strictly henselian local rings) into the category of local rings has a left adjoint

$$\mathscr{O} \mapsto \mathscr{O}^{\mathrm{h}}, \quad \mathscr{O} \mapsto \mathscr{O}^{\mathrm{hs}}$$

called the *henselization* and *strict henselization* of $\mathscr{O}$ respectively.

Furthermore, $\mathscr{O}^{\mathrm{h}}$ and $\mathscr{O}^{\mathrm{hs}}$ are filtered colimits of étale $\mathscr{O}$-algebras and by adjunction properties, there is a canonical isomorphism of functors $\mathscr{O}^{\mathrm{hs}} \cong (\mathscr{O}^{\mathrm{h}})^{\mathrm{s}}$.

Proof. We consider the category of pairs (U,u) of schemes with $U \to \operatorname{Spec}\mathscr{O}$ a (connected) étale scheme over $\operatorname{Spec}\mathscr{O}$ and u a point mapping to the closed point of $\operatorname{Spec}\mathscr{O}$ and such that the induced extension $k = k(u)$ is trivial. A morphism in this category is just a pointed morphism. Then the colimit

$$\mathscr{O}^{\mathrm{h}} = \operatorname*{colim}_{u\to U} \Gamma(U,\mathscr{O}_U)$$

is a henselian local ring. (The proof: identical to Corollary 3.)

To see the universal property note that by the Theorem on henselian rings, if $\mathscr{O}$ is already henselian, then $\mathscr{O}^{\mathrm{h}} = \mathscr{O}$. Indeed, in that case then each connected étale $U \to \operatorname{Spec}\mathscr{O}$ with a point over the closed point of $\mathscr{O}$ is finite étale and hence an isomorphism since it induces one on residue fields.

The proof of strict henselization is analogous by asking $k \subset k(u)$ to be finite separable instead. □

Remark. It is clear from the construction that $\mathfrak{m}^{\mathrm{h}}$ is just $\mathfrak{m}\mathscr{O}^{\mathrm{h}}$ and similarly for the strict henselization.

Example. As it is clear from the proof, since limits commute with limits and the étale site commutes with colimits of rings, if X is a scheme and x is a point, then $\mathcal{O}^{\mathrm{hs}}_{X,x}$ and $\mathcal{O}^{\mathrm{h}}_{X,x}$ agree with our definition given in the first part of these notes.

Example. Let k be a (separably closed) field and A the localization of $k[t_1,\dots,t_n]$ at the maximal ideal $(t_1,\dots,t_n)$; then

$$A^h = \left\{ f \in \overline{k}[[t_1,\dots,t_n]] \;\mid f \text{ is algebraic over } A \right\}.$$

Example. Let K be a number field, $\mathcal{O}_K$ the rings of integers and $\mathfrak{p}$ a maximal ideal with local ring $A = \mathcal{O}_{K,\mathfrak{p}}$. Let K^{sep} be the separable closure of K, $\mathcal{O}_{K^{\mathrm{sep}}}$ its rings of integers and fix a lift $\overline{\mathfrak{p}}$ of $\mathfrak{p}$ in this ring.

In algebraic number theory one defines a decomposition group

$$D_{\overline{\mathfrak{p}}/\mathfrak{p}} = \{\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K) = G_K | \overline{\mathfrak{p}}^{\sigma} = \overline{\mathfrak{p}}\}.$$

The henselization of A can be computed as the localization of

$$\mathcal{O}^{D_{\overline{\mathfrak{p}}/\mathfrak{p}}}_{K^{\mathrm{sep}},\overline{\mathfrak{p}}}$$

at the image of $\overline{\mathfrak{p}}$ in this (elements in $\overline{\mathfrak{p}}$ fixed by D).

For the proofs of both examples, we follow [Raynaud, 1970: Ch. X §2 Thm. 2]:

Theorem 2. *Let A be a normal local ring, $\mathfrak{p}$ the maximal ideal, K its fraction field. Let K^{sep} be a fixed separable closure of K, and G the absolute Galois group acting on the integral closure $\overline{A}$ of A in K^{sep}. Let $\overline{\mathfrak{p}}$ be a maximal ideal over $\mathfrak{p}$ and $D = D_{\overline{\mathfrak{p}}/\mathfrak{p}}$ as above.*

The inertia subgroup $I = I_{\mathfrak{P}}$ is the kernel of the homomorphism

$$1 \longrightarrow I \longrightarrow D \longrightarrow \mathrm{Gal}(k\mathfrak{P}/k\mathfrak{p}) \longrightarrow 1$$

Let B and B' be the fixed subrings of $\overline{A}$ by D and I respectively, $\mathfrak{q}$ and $\mathfrak{q}'$ the induced prime ideals. Then

$$B_{\mathfrak{q}} = A^h, \quad B'_{\mathfrak{q}'} = A^{hs}.$$

Applications to the étale topology We start by using the characterization of henselian rings to understand the stalks of sheaves on the étale site.

Definition 2. If X is a scheme, $\mathscr{F}$ a sheaf for the étale topology and $\overline{x} \to X$ a geometric point. Then the stalk of $\mathscr{F}$ at $\overline{x}$ is defined to be the pullback $\overline{x}^*\mathscr{F}$, or, unravelling the definition, the colimit

$$\operatorname*{colim}_{(U,u)} \mathscr{F}(U)$$

where (U,u) are elements in the cofiltered category of étale neighborhoods of $\overline{x}$ in X, the slice category of $X_{\text{ét}}$ *under* $\overline{x} \to X$. An object in this category is therefore a pair (U,u) with u being a lift of of $\overline{x} \to X$:

$$\begin{tikzcd} & U \arrow[d] \\ \overline{x} \arrow[ur, "u"] \arrow[r] & X \end{tikzcd}$$

a morphism of

Remark. It follows without much trouble that the functor $\mathscr{F} \mapsto \mathscr{F}_{\overline{x}}$ is exact, commutes with colimits and that sheafification preserves stalks. Also, the definition of stalk only depends on the equivalence class of geometric point, that is, any commutative diagram of the form

$$\begin{tikzcd} & \overline{x}' \arrow[d] \\ \overline{x} \arrow[ur] \arrow[r] & X \end{tikzcd}$$

induces a canonical isomorphism $\mathscr{F}_{\overline{x}} \xrightarrow{\sim} \mathscr{F}_{\overline{x}'}$.

Proposition 5. Let $X = \operatorname{Spec}\mathscr{O}$ be the spectrum of a strictly henselian local ring. Then any $Y \twoheadrightarrow X$ étale cover of X admits a section and the functor sending an étale sheaf $\mathscr{F}$ to its global sections is naturally isomorphic

$$\Gamma(X_{\text{ét}}, \mathscr{F}) = \mathscr{F}_x$$

to the functor taking the stalk at the closed point $x \to X$.

Proof. The first part follows immediately from the characterization 3 of Theorem 1. It also follows that we need only consider finite étale covers of X to compute the global sections. But $\mathsf{Fet}(X) = \mathsf{Fet}(x)$ is trivial by strictness, which proves exactness of global sections directly, and also that it equals the stalk at the closed point. $\square$

We can now prove a very important result computing the stalks of the (derived) pushfoward of étale sheaves.

Proposition 6. Let $f : Y \to X$ be a qcqs morphism of schemes, $x \to X$ geometric point, and $K \in D(X_{\text{ét}})$ a derived étale sheaf Let $X_x = \operatorname{Spec} \mathcal{O}^{\text{hs}}_{X,x}$ and consider the base change

$$\begin{array}{ccc} Y_x & \xrightarrow{g} & Y \\ \big\downarrow{\scriptstyle f_x} & & \big\downarrow{\scriptstyle f} \\ X_x & \longrightarrow & X \end{array}$$

then the stalk of the (derived) pushfoward can be computed as

$$(f_*K)_x = \Gamma(Y_x, g^*K), \quad (Rf_*K)_x = R\Gamma(Y_x, g^*K).$$

Proof. Since X_x is the limit of étale neighborhoods of x in X, Y_x is the limit of all étale neighborhoods of $f^{-1}(x)$ in Y. Since global sections commute with filtered colimits of qcqs schemes, the result follows from their definition. $\square$