Kleine AG: Serre-Tate theory

Organizers: Thiago Solovera e Nery*

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It is now some sixty years since Serre-Tate discovered that over a ring in which a prime number p is nilpotent, the infinitesimal deformation theory of abelian varieties is completely controlled by, and is indeed equivalent to, the infinitesimal deformation theory of their p-divisible groups. Our goal in this Kleine AG is to understand Drinfel'd's simplification of the original proof.

Now we give an overview of the plan. For concreteness sake, fix an abelian variety A over a field k of characteristic $p > 0$. Consider its associated p-divisible group $A[p^\infty] = \operatorname{colim} A[p^n]$. For $R \in \mathtt{Art}_k$ an Artinian local ring with residue k, we define

$$\begin{cases} \mathtt{Def}_A(R) = \left\langle (\widetilde{A}, \epsilon) \mid \epsilon \colon \widetilde{A} \otimes_R k \xrightarrow{\sim} A \right\rangle, \\ \mathtt{Def}_{A[p^\infty]}(R) = \left\langle (\widetilde{X}, \epsilon) \mid \epsilon \colon \widetilde{X} \otimes_R k \xrightarrow{\sim} A[p^\infty] \right\rangle \end{cases}$$

the categories of deformations of A (resp. $A[p^\infty]$) over R.

Theorem (Serre-Tate, Drinfel'd)**.** If $R \in \mathtt{Art}_k$, then the natural map

$$\mathtt{Def}_A(R) \xrightarrow{\sim} \mathtt{Def}_{A[p^\infty]}(R),$$

taking an abelian variety $\widetilde{A}$ over R to the p-divisible group $\widetilde{A}[p^\infty]$ is a (functorial) equivalence of categories.

The theorem can be understood in two parts: first an older result saying that the deformation theory of abelian varieties is unobstructed, that is, a lift exists; and a second part which deforms the lift into one with the desired property.

This result has many aplications (eg. p-adic uniformization of Shimura curves and some other Shimura varieties, lifting of special points, Mantovan's product formula) and we won't have time to go through all of them. Following Katz, we then specialize on the case of an *ordniary* elliptic curve over a field and construct an explicit isomorphism $\mathtt{Def}_A \cong \widehat{\mathbf{G}}_m^{2g}$.

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*https://www.esaga.uni-due.de/thiago.solovera-e-nery/

Talk 1: Deformation Theory (60 min.)

Introduce $\mathtt{Art}_k$ and explain the main ideas of deformation theory (eg. define a deformation of schemes along nilpotent thickenings). Mention the Kodaira-Spencer map and class. Talk about Illusie's Theorem (eg. [3, Cor. 10.3]) and obstruction classes in the setting of deformation of smooth morphisms, and argue that the deformation obstruction vanishes for abelian varieties using the argument in [7, Pg. 14, proof of 2.2.1] (If there is not enough time, assume odd characteristic).

Talk 2: p-divisible and formal groups (45 min.)

Introduce p-divisible groups over a scheme S and define the p-divisible group $A[p^\infty]$ associated with an abelian variety. Then introduce formal Lie groups as a sheaves on R-Alg, the completion at identity $\widehat{G}$ and the formula $\widehat{G} = \ker(G(A) \to G(A_{\mathtt{red}}))$, and mention the examples of $\widehat{\mathbf{G}}_m$, $\widehat{A}$ and $\widehat{A[p^\infty]}$. Possible references are [8], [5] or the lecture notes [1, §3.2] and blog post [9].

Talk 3: The proof of the Main Theorem (45 min.)

Following Katz [4, Thm. 1.2.1], or the original article by Drinfel'd [2, Appendix], prove the theorem of Serre-Tate. (The speaker is advised to pick the former.) Another helpful reference is [6].

Talk 4: Serre-Tate local moduli (60 min.)

Explain how, following [4, §2], in the case of A ordinary, the deformation theory is then equivalent to a choice of pairing $q\colon T_pA(k) \times T_pA^t(k) \to \widehat{\mathbf{G}}_m$ and conclude that $\mathtt{Def}_A \cong \widehat{\mathbf{G}}_m^{2g}$. If possible, you go a bit more in depth in defining the pairing, eg. following [4, §5.2] and surrounding topics.

Talk 5: The fundamental compatibility (60 min.)

Present (without proof) the Main Theorem in [4, §3] on the compatibility between the Kodaira-Spencer class and the group structure on $\mathtt{Def}_A$. Another possibility is to present this section in a succinct manor and also go into some of the reduction steps in [4, §4] and talk about the role of the Gauß-Manin connection in the Theorem.

References

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