

Problem sheet 1

Due date: Apr. 28, 2026.

Problem 1 (Quasi-compact morphisms) Let $f : X \rightarrow Y$ be a morphism of schemes. Let $Y = \bigcup_i V_i$ be a cover by affine open subschemes such that $f^{-1}(V_i)$ is quasi-compact for all i . Show that f is a quasi-compact morphism, i.e., $f^{-1}(V)$ is quasi-compact for every quasi-compact open $V \subseteq Y$.

Problem 2 (Properties of morphisms) Let \mathbf{P} be a property of morphisms of schemes that is stable under composition and base change. Moreover, we assume that every closed immersion possesses \mathbf{P} . The goal of this exercise is to prove the following:

Lemma 0.1. *Consider a commutative diagram of schemes*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow u & \swarrow v \\ & & S \end{array}$$

where u possesses \mathbf{P} and v is separated. Then f possesses \mathbf{P} , too.

- 1) Show that, under the assumptions of the lemma, the graph $\Gamma_f : X \rightarrow X \times_S Y$ is a closed immersion.
- 2) Use 1) to prove the lemma.
- 3) Explain why properness is an example of such a property \mathbf{P} .

Problem 3 (Group schemes)

- 1) Let S be a scheme and G_1, G_2 be S -group schemes. Define the notions of a *homomorphism of S -group schemes* $\varphi : G_1 \rightarrow G_2$ and *kernel of a homomorphism of group schemes*.
- 2) Consider now, as an example, $S = \text{Spec}(k)$ and the $\text{Spec}(k)$ -group scheme \mathbb{G}_m . Show that for any $n \in \mathbb{N}$ the ring homomorphism $k[x, x^{-1}] \rightarrow k[x, x^{-1}]$, $x \mapsto x^n$ induces a homomorphism of group schemes $\varphi : \mathbb{G}_m \rightarrow \mathbb{G}_m$.
- 3) Compute the kernel $\ker(\varphi)$. When is $\ker(\varphi)$ reduced?