

Problem sheet 2

Due date: May 5th, 2026.

Problem 4 (Closed subschemes of \mathbb{P}^n)

Let R be a ring, and let Z be a closed subscheme of \mathbb{P}_R^n . Show that there exists a homogeneous ideal I of $R[X_0, \dots, X_n]$ such that $Z = \mathcal{V}_+(I)$.

Hint: Define $I = (f \in R[X_0, \dots, X_n] \text{ homogeneous} : \forall i, \frac{f}{X_i^{\deg(f)}} \in \mathfrak{a}_i)$, where $\mathfrak{a}_i \subset R[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}]$ is the ideal defining the closed subscheme $Z \cap D_+(X_i)$ of $D_+(X_i)$.

Problem 5 (Segre embedding) Let R be a ring. Show that there is a closed immersion

$$\mathbb{P}_R^n \times_{\text{Spec } R} \mathbb{P}_R^m \hookrightarrow \mathbb{P}_R^{(n+1)(m+1)-1}$$

which on k -valued points is given by

$$((x_0 : \dots, x_n), (y_0 : \dots, y_m)) \mapsto (x_0 y_0 : \dots : x_i y_j : \dots : x_n y_m)$$

for any R -algebra k which is a field.

Problem 6 (Matrix rank)

Let k be an algebraically closed field, let $n, m, d \geq 1$ be integers, and let $V = \text{Mat}_{n,m}(k)$ be the k -vector space of n by m matrices. Let M_1, \dots, M_{mn} be the standard basis of V . Let $P_{n,m} := \mathbb{P}_k^{mn-1}$, and identify the points of $P_{n,m}(k)$ with the elements of $(V \setminus \{0\})/k^\times$ by sending $(a_1 : \dots : a_{mn})$ to the class of $\sum_i a_i M_i$. Show that the set of matrices of rank $\leq d$ is closed in $P_{n,m}$.

Hint: Consider the morphism $P_{n,d} \times_{\text{Spec } k} P_{d,m} \rightarrow P_{n,m}$ given by matrix multiplication.