

Problem sheet 4

Due date: May 19th, 2026.

Problem 10 Give an example of a scheme X and an \mathcal{O}_X -module \mathcal{F} which is not quasi-coherent.

Hint. You could take $X = \text{Spec}(R)$ where R is a discrete valuation ring.

Problem 11 Prove the following proposition:

Proposition 0.1. *Let $\varphi : A \rightarrow B$ be a homomorphism between rings A, B ; and denote by $f : \text{Spec } B \rightarrow \text{Spec } A$ the corresponding morphism of affine schemes. Let M be a B -module and $\mathcal{F} = M^\sim$ its associated sheaf of $\mathcal{O}_{\text{Spec } B}$ -modules. Then:*

$$f_*\mathcal{F} = (\varphi_*M)^\sim$$

where φ_*M is the restriction of scalars of M , i.e., the A -module with underlying abelian group M and scalar multiplication given by $a \cdot m = \varphi(a)m$.

Problem 12 (Invertible sheaves on \mathbb{P}_k^1) Let k be a field. We cover \mathbb{P}_k^1 by the standard charts $U_i := D_+(X_i)$, $i = 0, 1$, as usual, and write $U_{01} := U_0 \cap U_1$. For $d \in \mathbb{Z}$, multiplication by the element $(X_0/X_1)^d \in \Gamma(U_{01}, \mathcal{O}_{\mathbb{P}_k^1}^\times)$ defines an isomorphism $\mathcal{O}_{U_0|U_{01}} \xrightarrow{\sim} \mathcal{O}_{U_1|U_{01}}$ which gives rise to a gluing datum of the \mathcal{O}_{U_i} -modules $\mathcal{O}_{U_i}(d)$. By gluing of sheaves, we obtain an invertible sheaf $\mathcal{O}_{\mathbb{P}_k^1}(d)$.

- 1) Show that $\mathcal{O}_{\mathbb{P}_k^1}(d_1) \otimes_{\mathcal{O}_{\mathbb{P}_k^1}} \mathcal{O}_{\mathbb{P}_k^1}(d_2) \cong \mathcal{O}_{\mathbb{P}_k^1}(d_1 + d_2)$ for any $d_1, d_2 \in \mathbb{Z}$.
- 2) Prove that for every invertible $\mathcal{O}_{\mathbb{P}_k^1}$ -module \mathcal{L} there exists $d \in \mathbb{Z}$ such that $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}_k^1}(d)$.

Hint: Show first that every invertible sheaf on \mathbb{A}_k^1 is isomorphic to $\mathcal{O}_{\mathbb{A}_k^1}$.

Remark: We will show later that d is uniquely determined by \mathcal{L} .