Algebraic Geometry II Prof. Dr. Ulrich Görtz
SS 2026 Dr. Andreas Pieper

Problem sheet 5

Due date: May 26th, 2026.

Problem 13

Let k be a field. Let $\mathscr{O}(d) := \mathscr{O}_{\mathbb{P}^1_k}(d)$ denote the invertible sheaf on $\mathbb{P}^1_k$ introduced in Problem 12. Show that there is an isomorphism

$$\Gamma(\mathbb{P}^1_k, \mathscr{O}(d)) \cong k[X_0, X_1]_d,$$

where X_0, X_1 are the homogeneous coordinates on $\mathbb{P}^1_k$ and where $k[X_0, X_1]_d$ denotes the k-vector space of homogeneous polynomials of degree d (which has dimension $d+1$ for $d \geqslant 0$, and is $= 0$ for $d < 0$).

Problem 14

If $\mathscr{L}$ is a line bundle on a scheme X and $s \in \Gamma(X, \mathscr{L})$, we say that s vanishes at a point $x \in X$, if the image $s(x) \in \mathscr{L} \otimes_{\mathscr{O}_{X,x}} \kappa(x)$ is $= 0$.

Now let k be a field and $X = \mathbb{P}^1_k$ as in Problem 13. Show that every section $s \in \Gamma(\mathbb{P}^1_k, \mathscr{O}(1)) \smallsetminus \{0\}$ vanishes at a unique point of $\mathbb{P}^1_k$ with residue class field k. Show that conversely for every such point there exists a non-zero global section of $\mathscr{O}(1)$ vanishing there and that this section is unique up to multiplication by a scalar in $k^\times$.

Problem 15

Let X be a ringed space, and let $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite type, i.e., every point $x \in X$ has an open neighborhood U such that there exists $n \geqslant 0$ and an exact sequence

$$\mathscr{O}_U^n \to \mathscr{F} \to 0$$

of $\mathscr{O}_U$-modules. Show that

$$\operatorname{Supp}(\mathscr{F}) := \{x \in X;\ \mathscr{F}_x \neq 0\},$$

the support of $\mathscr{F}$, is a closed subset of X.