Algebraic Geometry II Prof. Dr. Ulrich Görtz
SS 2026 Dr. Andreas Pieper

Problem sheet 7

Due date: June 9nd, 2026.

Problem 19 Let k be a field and let $A = \frac{k[x,y,z]}{xy-z^2}$. Consider the normal affine scheme $X = \operatorname{Spec}(A)$ and its closed subscheme $D = \mathcal{V}(x,z) \subset X$.

1. Show that D is a Weil divisor.
2. Let $\mathfrak{m} = (x,y,z) \subset A$. Show that the ideal generated by x and z inside the local ring $A_{\mathfrak{m}}$ is not principal and conclude that D is not a Cartier divisor.

Problem 20 Let X be an integral scheme such that $\Gamma(X, \mathcal{O}_X)$ is a field. Let $\mathcal{L}$ be an invertible sheaf. Show that the following are equivalent:

i) $\mathcal{L}$ is isomorphic to $\mathcal{O}_X$.

ii) $\Gamma(X, \mathcal{L}) \neq 0$ and $\Gamma(X, \mathcal{L}^{-1}) \neq 0$.

Hint. Assuming ii), construct a sequence $\mathcal{O}_X \to \mathcal{L} \to \mathcal{O}_X$ and note that the composition "is" an element of $\Gamma(X, \mathcal{O}_X)$.

Problem 21 Let k be a field. Consider the Dedekind scheme $\mathbb{P}^1_k$. Define the divisor $K = -2[x_0]$ where $x_0 \in \mathbb{P}^1_k$ is any k-valued point. (This divisor depends on the choice of the point x_0, but according to Problem 17 its rational equivalence class does not.) Show that $\mathbb{P}^1_k$ satisfies the *Riemann-Roch theorem* with $g = 0$, i.e., for any divisor D on $\mathbb{P}^1_k$ one has

$$l(D) + l(K - D) = \deg(D) + 1.$$

Remark. You may assume that k is algebraically closed.