

Problem sheet 8

Due date: June 16th, 2026.

Problem 22

Let k be an algebraically closed field and let X be a proper connected k -scheme. Show that $\Gamma(X, \mathcal{O}_X) = k$.

Hint. Let $s \in \Gamma(X, \mathcal{O}_X) = \text{Hom}_k(X, \mathbb{A}_k^1)$. Apply Problem 7 to the following commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{s} & \mathbb{A}_k^1 & \xleftarrow{\quad} & \mathbb{P}_k^1 \\ & \searrow & & & \swarrow \\ & & \text{Spec } k & & \end{array}$$

Problem 23

Definition 0.1. Let k be an (algebraically closed) field. An *elliptic curve* over k is a pair $(E, 0_E)$ where E is an irreducible smooth projective curve over k of genus 1 and $0_E \in E(k)$ is a k -point.

Let $(E, 0_E)$ be an elliptic curve over an algebraically closed field k .

- (1) Define $\text{DivCL}(E)^0$ to be the kernel of the map $\text{deg} : \text{DivCL}(E) \rightarrow \mathbb{Z}$. Show that the map

$$E(k) \rightarrow \text{DivCL}(E)^0, \quad P \mapsto [P] - [0_E],$$

is bijective.

Hint: For a divisor D of degree 0, apply the Riemann–Roch theorem to $D + [0_E]$. Recall that $\text{deg}(K) = 0$ in this case.

- (2) Use (1) to show that $E(k)$ can be equipped with a commutative group structure with neutral element 0_E .

Problem 24 Let C be an irreducible smooth projective curve over an algebraically closed field k , D a Cartier divisor on C , and let s_0, \dots, s_n be a basis of $\Gamma(C, \mathcal{O}_C(D))$. Show that the following are equivalent:

- i) The map $\mathcal{O}_C^{n+1} \rightarrow \mathcal{O}_C(D)$ associated to s_0, \dots, s_n is surjective.
- ii) For every point $P \in C(k)$ one has $\dim \Gamma(C, \mathcal{O}_C(D - P)) < \dim \Gamma(C, \mathcal{O}_C(D))$.