Algebraic Geometry II Prof. Dr. Ulrich Görtz
SS 2026 Dr. Andreas Pieper

Problem sheet 13

Due date: July 15th, 2026.

Problem 37 Let k be a field, $d \geqslant 1$. Show that $\dim_k(H^1(\mathbb{P}^1_k, \mathscr{O}_{\mathbb{P}^1_k}(-d))) = d-1$.
Hint: Use Theorem 6.41.

Problem 38 Let $X = \operatorname{Spec}(A)$ be an affine scheme. Consider an exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

of sheaves of $\mathscr{O}_X$-modules. Assume that $\mathcal{F}_1, \mathcal{F}_3$ are quasi-coherent. The goal of this exercise is to show that $\mathcal{F}_2$ is quasi-coherent.

i) Show that the sequence

$$0 \longrightarrow \Gamma(X, \mathcal{F}_1) \longrightarrow \Gamma(X, \mathcal{F}_2) \longrightarrow \Gamma(X, \mathcal{F}_3) \longrightarrow 0$$

is exact.

ii) Show that there is a commutative diagram of sheaves of $\mathscr{O}_X$-modules with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{\Gamma(X,\mathcal{F}_1)} & \longrightarrow & \widetilde{\Gamma(X,\mathcal{F}_2)} & \longrightarrow & \widetilde{\Gamma(X,\mathcal{F}_3)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F}_2 & \longrightarrow & \mathcal{F}_3 & \longrightarrow & 0 \end{array}$$

iii) Deduce that $\widetilde{\Gamma(X, \mathcal{F}_2)} \to \mathcal{F}_2$ is an isomorphism.

Problem 39

Let $R \to R'$ be a faithfully flat ring homomorphism, i.e., for every complex

$$\cdots \to A^i \to A^{i+1} \to A^{i+2} \to \cdots,$$

the complex $(A^i)_i$ is exact if and only if the complex $(A^i \otimes_R R')_i$ is exact. Let M be an R-module.

For $n \geqslant 0$ let $R^{(n)}$ denote the n-fold tensor product $R' \otimes_R \cdots \otimes_R R'$ (where $R^{(0)} := R, R^{(1)} := R'$). For $i = 0, \dots, n$, define maps $f_i \colon M \otimes_R R^{(n)} \to M \otimes_R R^{(n+1)}$ by $f_i(m \otimes r_1 \otimes \cdots \otimes r_n) = m \otimes r_1 \otimes \cdots \otimes r_i \otimes 1 \otimes r_{i+1} \otimes \cdots \otimes r_n$

and

$$d^n\colon M \otimes_R R^{(n)} \to M \otimes_R R^{(n+1)}, \qquad x \mapsto \sum_{i=0}^{n} (-1)^i f_i(x).$$

Prove that we obtain a complex

$$0 \to M \to M \otimes R' \to M \otimes R^{(2)} \to \cdots .$$

Prove further that this complex is exact.
Hint: Since $R \to R'$ is faithfully flat, it is enough to show the exactness after tensoring the above sequence with R'. Show that for the resulting complex

$$0 \to M \otimes R' \to M \otimes R^{(2)} \to M \otimes R^{(3)} \to \cdots$$

the maps

$$\begin{aligned} M \otimes R^{(n+1)} &\to M \otimes R^{(n)}, \\ m \otimes x_0 \otimes \cdots \otimes x_n &\mapsto \sum_{i=0}^{n-1} (-1)^i m \otimes x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n, \end{aligned}$$

are a homotopy of id and 0.