

Problem sheet 1

Due date: Oct. 28, 2025.

Problem 1

A non-empty topological space X is called *irreducible*, if it is not equal to the union of two proper closed subsets.

- (1) Determine all topological spaces which are Hausdorff and irreducible. (Recall that a topological space X is called *Hausdorff*, if for any two points $u, v \in X$, $u \neq v$, there exist disjoint open subsets $U, V \subseteq X$ with $u \in U$, $v \in V$.)
- (2) Let X be a non-empty topological space. Prove that the following properties are equivalent:
 - (i) The space X is irreducible.
 - (ii) Every non-empty open subset $U \subseteq X$ is dense in X (i.e., the smallest closed subset of X containing U is X).
 - (iii) Every non-empty open subset $U \subseteq X$ is connected. (A topological space is called *connected*, if it is non-empty and cannot be written as the union of two disjoint proper closed subsets.)
 - (iv) Any two non-empty open subsets of X have non-empty intersection.

Problem 2

Let k be an infinite field.

- (1) Let $n \geq 1$ and let $f \in k[T_1, \dots, T_n]$ be a polynomial such that $f(t_1, \dots, t_n) = 0$ for all $(t_1, \dots, t_n) \in k^n$. Prove that $f = 0$.

Hint. You can use induction on n .

- (2) Prove that k^n (with the Zariski topology) is irreducible.

Hint. First show that if $Z \subsetneq k^n$ is a closed subset strictly contained in k^n , then there exists an $f \in k[T_1, \dots, T_n]$ such that $Z \subseteq V(f) \subsetneq k^n$.

Problem 3

Let k be a field, let $n, m \geq 0$, and let $f_1, \dots, f_n \in k[T_1, \dots, T_m]$ be polynomials. We consider k^m and k^n as topological spaces with respect to the Zariski topology. Prove that the map

$$F: k^m \longrightarrow k^n, \quad (t_1, \dots, t_m) \mapsto (f_1(t_1, \dots, t_m), \dots, f_n(t_1, \dots, t_m)),$$

is continuous, i.e., for every closed subset $Z \subseteq k^n$, the inverse image $F^{-1}(Z)$ is closed.

Problem 4 Let k be an infinite field and equip k^n with the Zariski topology. Prove that the topology on k^2 is not the same as the product topology on $k^1 \times k^1$.