Algebraic Geometry I Prof. Dr. Ulrich Görtz
WS 2025/26 Dr. Andreas Pieper

Problem sheet 10

Due date: Jan. 14, 2025.

Problem 32 (Frobenius) Let X be a scheme of characteristic p, i.e., for every $U \subset X$ open one has $p \cdot 1 = 0$ in $\mathscr{O}_X(U)$. Show that there exists a unique morphism $(F, F^\flat)\colon X \to X$ of schemes such that on topological spaces, $F = \mathrm{id}_X$, and for an open $U \subseteq X$, $F^\flat$ is given by $\mathscr{O}_X(U) \to \mathscr{O}_X(U)$, $s \mapsto s^p$. This morphism is called the *absolute Frobenius morphism* of X.

Give an example where F is not an isomorphism.

Problem 33 Let X be a scheme. We say that X is *reduced* if the ring $\mathscr{O}_X(U)$ is reduced for every open $U \subset X$.

(1) Show that the following are equivalent:
   (i) X is reduced.
   (ii) For every affine open subscheme $U \subseteq X$, the ring $\mathscr{O}_X(U)$ is reduced.
   (iii) For every $x \in X$, the stalk $\mathscr{O}_{X,x}$ is reduced.

(2) Now let k be a field and let $X = V(xy, y^2) \subset \mathbb{A}^2_k$. Show that the topological space of X equals the topological space of $V(y) \cong \mathbb{A}^1_k$. In particular, X is irreducible. Show that $X \cap D(x)$ is a dense open subscheme of X which is reduced. Show that X is not reduced.

Problem 34 Let k be an algebraically closed field. An *affine conic* is a closed subscheme of the form

$$C = V(g) \subset \mathbb{A}^2_k$$

where $g(x, y) = ax^2 + by^2 + cxy + dx + ey + f$ with $a, \dots, f \in k$, and g has degree 2.

(1) Find affine conics C and C' such that $C \cong \mathbb{A}^1_k$ and $C' \cong \mathbb{A}^1_k \smallsetminus \{0\}$.

(2) Now assume that the characteristic of k is $\neq 2$. Let $g = x^2 + y^2 - 1$ and $C = V(g)$. We identify the sets of closed points of $\mathbb{A}^1_k$ with $\mathbb{A}^1(k) = k$ and of C with $C(k) := \{(x, y) \in k^2;\ x^2 + y^2 = 1\}$, respectively. Consider the map

$$\mathbb{A}^1(k) \smallsetminus \left\{t \in k; t^2 = -1\right\} \longrightarrow C(k) \smallsetminus \{(0, -1)\}, \quad t \mapsto \left(-\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right).$$

Show that this map is bijective.

Remark. Geometrically, the image of t is the second intersection point of C with the line of slope t through $(0, 1)$. To find the inverse map, it may be helpful to compute $\frac{1-t^2}{1+t^2} - 1$ and $\frac{1-t^2}{1+t^2} + 1$.

(3) With notation as in (2), show that there is an isomorphism

$$f\colon \mathbb{A}^1_k \smallsetminus V(t^2 + 1) \longrightarrow C \smallsetminus \{(0, -1)\}$$

of schemes which induces the above map on closed points.