

Problem sheet 2

Due date: Nov. 4, 2025.

Problem 5 Let k be a field and $n \in \mathbb{N}$, $n \geq 1$. Consider a polynomial

$$f = \sum_{i=0}^n c_i x^i \in k[x]$$

of degree n , i.e. $c_n \neq 0$. The goal of this exercise is to show that there exists a polynomial expression in the c_i called the *discriminant* of f , denoted $\text{disc}(f)$ such that: $\text{disc}(f) = 0$ if and only if f has a multiple root.

(1) Denote

$$k[x]_{\leq d} := \{h \in k[x] : \deg(h) \leq d\}.$$

For a polynomial $g \in k[x]_{\leq m}$ we define the *Sylvester matrix* $M(f, g)$ as the matrix of the linear map

$$\varphi: k[x]_{\leq m-1} \oplus k[x]_{\leq n-1} \rightarrow k[x]_{\leq m+n-1}, (a, b) \mapsto af + bg.$$

(we equip these spaces with the obvious bases consisting of powers of x).

Define the *resultant* as $\text{res}(f, g) = \det(M(f, g))$. Show that $\text{res}(f, g) = 0$ if and only if $\gcd(f, g) \neq 1$.

Hint: When is φ injective?

(2) Define¹ $\text{disc}(f) = \text{res}(f, f')$ (for $m = n - 1$) and observe that this is in fact a polynomial expression in the coefficients of f . Show that $\text{disc}(f) = 0$ if and only if f has a multiple root.

¹Depending on the order of the bases chosen above, there is a sign ambiguity and for the usual definition of *the* resultant of two polynomials and *the* discriminant of a polynomial a specific ordering and hence a specific sign are chosen, and then $\text{disc}(f) = (-1)^{n(n-1)/2} \text{res}(f, f')$ but this is irrelevant for our purposes and we ignore this issue here.

Problem 6

Let k be an algebraically closed field. Let $n \geq 1$. We identify the space $M := \text{Mat}_{n \times n}(k)$ of $(n \times n)$ -matrices with entries in k with k^{n^2} and equip it with the Zariski topology. By Problem 2 (2), it is irreducible.

- (1) Prove that the subset of M consisting of matrices A such that $\text{charpol}_A(A) = 0$ is closed in M (without using the Theorem of Cayley-Hamilton).
- (2) Use Problem 5 to prove that the subset of diagonalizable matrices with n different eigenvalues in k is open in M .
- (3) Prove the Theorem of Cayley-Hamilton, i.e., prove that the subset defined in (1) equals all of M .

Problem 7 The goal of this exercise is to prove the analogue of the abc-conjecture for the polynomial ring $\mathbb{C}[x]$:

Theorem 0.1. *Let $a, b, c \in \mathbb{C}[x]$ be such that $a + b = c$. We assume that the triple (a, b, c) is not of the form $(\alpha f, \beta f, \gamma f)$ for $\alpha, \beta, \gamma \in \mathbb{C}$ and $f \in \mathbb{C}[x]$. Then*

$$\max(\deg(a), \deg(b), \deg(c)) < |\{\alpha \in \mathbb{C} : (abc)(\alpha) = 0\}|$$

- (1) Reduce to the case where a, b, c are pairwise coprime.
- (2) Define $\Delta(x) := \det \begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$. Show that $\Delta \neq 0$ and

$$\Delta(x) = \det \begin{pmatrix} a & c \\ a' & c' \end{pmatrix} = \det \begin{pmatrix} c & b \\ c' & b' \end{pmatrix}$$

- (3) Show that

$$\deg(\Delta) \leq \min(\deg(a) + \deg(b) - 1, \deg(a) + \deg(c) - 1, \deg(c) + \deg(b) - 1).$$

(4) Use the expression $\Delta = ab' - ba'$ to show that a divides

$$\Delta(x) \prod_{\alpha \in \mathbb{C}: a(\alpha)=0} (x - \alpha)$$

and show an analogous statement for b, c .

(5) Deduce that abc divides

$$\Delta(x) \prod_{\alpha \in \mathbb{C}: (abc)(\alpha)=0} (x - \alpha)$$

(6) Combine (3) and (5) to prove Theorem 0.1.